The choice of the symbol b_0, with the corresponding vector $B^\mu = b_0 dx^\mu/ds$, to denote the proper rate of energy production can serve to remind us that the proposed equations are an outcome of Bohr's suggestion as to the possibility of such energy creation.

3. *Conclusion.*—In conclusion it may be emphasized that the equations proposed above are to be regarded as macroscopic in character, and that the quantity b_0, which gives the rate of energy production in unit volume as measured by a local observer, is to be regarded as a function of the chemical constitution and physical state of the matter involved, to be determined by empirical methods.

It should also be noted that an acceptance of equations (4) and (5) as an expression of the energy-momentum principle would also involve some modification of the usual expression for the dependence of space-time metric on the distribution of matter and energy, since this is such as to make the divergence of the energy-momentum tensor necessarily equal to zero. Some remarks as to metric in a non-conservative mechanics will be made in a following note.

[1] Tolman, these PROCEEDINGS, 20, 379 (1934).

SUGGESTIONS AS TO METRIC IN A NON-CONSERVATIVE MECHANICS

BY RICHARD C. TOLMAN

NORMAN BRIDGE LABORATORY OF PHYSICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated May 24, 1934

1. *Introduction.*—The purpose of this note is to give tentative consideration to the dependence of space-time metric on the distribution of energy and momentum, in the case of a mechanics which abandons the principle of the conservation of energy. The question is a difficult one, and its final correct answer might necessitate some new conceptual apparatus. In the present note, however, we shall merely try to show the consequences of a direct and somewhat plausible method of attack, which can be made with familiar tools.

It has been pointed out in a previous note[1] that the expression for the energy-momentum principle in a non-conservative mechanics could naturally be taken as given by the equation

$$(T^{\mu\nu})_\nu = B^\mu = b_0 \frac{dx^\mu}{ds}, \tag{1}$$

where $T^{\mu\nu}$ is the energy-momentum tensor, b_0 the proper rate of energy generation per unit volume, and dx^μ/ds the "velocity" of matter at the point of interest, all the quantities being regarded from a macroscopic point of view. It is immediately evident, nevertheless, that this expression would not agree with the usual relativistic relation between the energy-momentum tensor $T^{\mu\nu}$ and the contracted Riemann-Christoffel tensor $R^{\mu\nu}$

$$-8\pi T^{\mu\nu} = R^{\mu\nu} - \frac{1}{2} Rg^{\mu\nu}, \tag{2}$$

since the tensor divergence of the right-hand side of (2) vanishes identically.

To obtain a generalization of this equation suitable for a non-conservative mechanics we have the following considerations to guide us. In the first place the final postulate should be expressed in a covariant form, equally valid in all sets of coördinates, in order to avoid unsuspected assumptions that might be dependent on the use of some particular set of coördinates. Secondly, in accordance with the known successes of the theory of relativity, the postulate should be reducible in the absence of any energy creation to the older expression (2). Finally, if we are to assume a reality for the process of energy production, we should also wish to assume that any newly created energy would have the same kind of effect on gravitational field and metric as is produced by energy already present.

The first two of these considerations are, of course, very essential criteria for the acceptability of any final postulate. They have, however, practically no directive action which could lead us to an actual choice On the other hand, however, if we make use of the methods discovered by Einstein[2] for the approximate treatment of weak gravitational fields, it can be shown that the third consideration, requiring newly created energy to have the same kind of effect as other energy in producing a gravitational field, does have considerable force to suggest a definite method of attack.

2. *Weak Fields with Energy Conservation.*—Let us take a gravitational field weak enough so that deviations from the "flat" space-time of the special theory of relativity will be small. We can then use coördinates (x, y, z, t), which are approximately Galilean in character, with the components of the metrical tensor given by the expression

$$g_{\mu\nu} = \delta_{\mu\nu} + h_{\mu\nu}, \tag{3}$$

where the $\delta_{\mu\nu}$ are the Galilean values ± 1, 0, of the $g_{\mu\nu}$, and the $h_{\mu\nu}$ are small correction terms. Furthermore, denoting the Galilean values of the $g^{\mu\nu}$ by $\delta^{\mu\nu}$, we may also introduce the quantities

$$h^\nu_\mu = \delta^{\alpha\nu} h_{\mu\alpha} \text{ and } h = h^\alpha_\alpha = \delta^{\alpha\beta} h_{\alpha\beta}, \tag{4}$$

and may regard the $h_{\mu\nu}$ and h^ν_μ, together with their derivatives with respect to the coördinates, as terms of the first order, whose squares can be neglected.

Using such quasi-Galilean coördinates, it is then known[3] that the general expression for the contracted Riemann-Christoffel tensor can be written correct to the first order in the approximate but otherwise general form

$$R_{\mu\nu} = \frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\beta} - \frac{1}{2}\frac{\partial^2}{\partial x^\nu \partial x^\epsilon}\left(h^\epsilon_\mu - \frac{1}{2}\delta^\epsilon_\mu h\right) - \frac{1}{2}\frac{\partial^2}{\partial x^\mu \partial x^\epsilon}\left(h^\epsilon_\nu - \frac{1}{2}\delta^\epsilon_\mu h\right). \tag{5}$$

Furthermore, it is known[4] that an infinitesimal transformation of coördinates

$$x'^\mu = x^\mu + \phi^\mu, \text{ where } \delta^{\alpha\beta}\frac{\partial^2 \phi_\mu}{\partial x^\alpha \partial x^\beta} = \frac{\partial}{\partial x^\epsilon}\left(h^\epsilon_\mu - \frac{1}{2}\delta^\epsilon_\mu h\right), \tag{6}$$

can in general be made which, after dropping primes, will reduce the above expression to the simple form

$$R_{\mu\nu} = \frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\beta}, \tag{7}$$

holding in the specialized quasi-Galilean coördinates thus obtained.

In the ordinary theory of relativity, in which energy conservation is preserved, (7) and (2) may then be combined to give as a result, valid in the above specialized coördinates,

$$-8\pi\left(T_{\mu\nu} - \frac{1}{2}T\delta_{\mu\nu}\right) = \frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 h_{\mu\nu}}{\partial x^\alpha \partial x^\beta}, \tag{8}$$

which, since $\delta^{\alpha\beta}\partial^2/\partial x^\alpha \partial x^\beta$ is the d'Alembertian operator, has the well-known solution

$$h_{\mu\nu} = -4\int \frac{\left[T_{\mu\nu} - \frac{1}{2}T\delta_{\mu\nu}\right]_{t-r}}{r}\,dv, \tag{9}$$

familiar from the theory of retarded potentials, where r is the distance from the element of volume dv to the point of interest.

3. *Weak Fields without Energy Conservation.*—If now we desire to change to a theory in which the creation of energy is to be permitted, it is evident that equation (5) must still be taken as valid in weak fields, since the Riemann-Christoffel tensor is in any case a perfectly definite defined function of the metrical tensor and its derivatives. Furthermore, it is evident that we can maintain our hypothesis that energy whether newly created or not exerts the same effect on the gravitational field described by the $h_{\mu\nu}$ if we also adopt equation (9) as valid in our new theory, with the understanding that we shall insert therein the actual

values of $T_{\mu\nu}$ and T at the location of dv at the time $t - r$, without any distinction between energy which has just been created inside the element dv and energy which has been there for some time or has entered from the outside. Nevertheless, if we desire to base our new theory on equations (5) and (9), it is evident that we can now no longer take the specialized quasi-Galilean coördinates in which (9) holds as being also a special set in which (5) reduces to the simple form (7), since this would at once lead to the usual relation between distribution and metric which requires the conservation laws.

These considerations are sufficient to give a relation between the $R_{\mu\nu}$ and the $T_{\mu\nu}$, since by combining the general equation (5) with the consequences of (9), we can readily obtain the result

$$R_{\mu\nu} = -8\pi(T_{\mu\nu} - \frac{1}{2}T\delta_{\mu\nu}) + 2\,\frac{\partial^2}{\partial x^\nu \partial x^\epsilon}\int \frac{[T^\epsilon_\mu]_{t-r}}{r}\,dv + 2\,\frac{\partial^2}{\partial x^\mu \partial x^\epsilon}\int \frac{[T^\epsilon_\nu]_{t-r}}{r}\,dv. \quad (10)$$

Furthermore, since a displacement in the point of interest denoted by $\partial/\partial x^\epsilon$ can be allowed for by differentiating back of the integral sign to give terms of the form $[\partial T^\epsilon_\mu/\partial x^\epsilon]_{t-r}$ with r left unchanged, it is evident that this expression will reduce to the usual relation between metric and distribution in case the conservation laws do hold, and in any case will approach thereto as we go to regions further and further from those where the conservation laws fail.

By applying the d'Alembertian operator to (10), we can re-express this equation in a form suitable for covariant generalization

$$\delta^{\alpha\beta}\,\frac{\partial^2 R_{\mu\nu}}{\partial x^\alpha \partial x^\beta} = -8\pi\delta^{\alpha\beta}\,\frac{\partial^2}{\partial x^\alpha \partial x^\beta}\,(T_{\mu\nu} - \frac{1}{2}T\delta_{\mu\nu}) + 8\pi\,\frac{\partial^2 T^\epsilon_\mu}{\partial x^\nu \partial x^\epsilon} + 8\pi\,\frac{\partial^2 T^\epsilon_\nu}{\partial x^\mu \partial x^\epsilon}. \quad (11)$$

4. *Covariant Generalization.*—In accordance with the spirit of the principles of covariance and equivalence, we may now generalize (11) by replacing the d'Alembertian operator by the corresponding covariant operator, the Galilean values of the metrical tensor by their covariant values, the ordinary derivative by the covariant derivative and the ordinary divergence by the tensor divergence. Doing so, and introducing the expression for the divergence of the energy-momentum tensor given by (11), we obtain the covariant relation

$$g^{\alpha\beta}(R_{\mu\nu})_{\alpha\beta} = -8\pi g^{\alpha\beta}(T_{\mu\nu} - \frac{1}{2}Tg_{\mu\nu})_{\alpha\beta} + 8\pi(B_\mu)_\nu + 8\pi(B_\nu)_\mu. \quad (12)$$

Treating this expression by familiar methods of tensor manipulation, the result can also be expressed in the form

$$-8\pi T_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + D_{\mu\nu}, \tag{13}$$

where $D_{\mu\nu}$, the *discrepancy* from the usual expression for the dependence of the energy-momentum tensor on the metric, would be subject to the equation

$$g^{\alpha\beta}(D_{\mu\nu})_{\alpha\beta} = -8\pi(B_\mu)_\nu - 8\pi(B_\nu)_\mu + 8\pi(B^\alpha)_\alpha g_{\mu\nu}. \tag{14}$$

5. *Conclusion.*—The final result certainly does exhibit some of the properties for a satisfactory relation between metric and distribution in a non-conservative mechanics. It is covariantly expressed and hence would be true in all sets of coördinates if true in one. In fields of any strength, it can be taken as reducing to the usual relation between metric and distribution in the complete absence of energy creation or destruction, and as approaching thereto as we recede from regions where the conservation laws fail. Furthermore, in fields of any strength, the discrepancy from the usual relation would be propagated, through regions where the conservation laws do hold, in accordance with a type of wave equation which is characteristic of the metric. Lastly, in fields weak enough so that approximate methods can be used, we can obtain the solution (9), which makes newly created energy have the same effect in producing a gravitational field as that which has survived destruction.

On the other hand, the result has two quite unsatisfactory features. In the first place, it has been obtained by a fairly complicated generalization that starts from an equation (9) which at best can be true only in very special sets of quasi-Galilean coördinates. In the second place, in the process of generalization we have changed from equations of the second order to equations of the fourth order in the derivatives of the metrical tensor, without obtaining any specific rules for choosing those integrals of the latter which have physical significance, beyond general notions as to the behavior of the equations in weak fields or under certain limiting conditions.[5] Both of these points have also been specially emphasized by my colleague Professor J. R. Oppenheimer whose criticism has been very helpful.

Finally, it may be well to stress again as in an earlier note,[6] that the failure or validity of our older ideas as to the conservation of energy is still a matter for empirical decision. Furthermore, in the case of apparent failure, it would still be necessary to decide whether the addition of new terms to the expressions for energy and momentum in order to retain conservation might not be a convenient rather than a purely formal procedure.

[1] Tolman, these PROCEEDINGS, **20,** 437 (1934).

[2] Einstein, *Berl. Ber.*, **1916**, p. 688.

[3] See, for example, Eddington *The Mathematical Theory of Relativity*, § 57, Cambridge, 1923; or Tolman, *Relativity, Thermodynamics and Cosmology*, § 93, Oxford, 1934.

[4] Hilbert, *Goettingen Nachrichten*, **1917**, p. 53.

[5] In the case of cosmological problems the limiting relation when the conservation laws do hold might have to be taken in the form $-8\pi T_{\mu\nu} = R_{\mu\nu} - {}^1/_2 R g_{\mu\nu} + \Lambda g_{\mu\nu}$ with Λ a constant.

[6] Tolman, these PROCEEDINGS, **20,** 379 (1934).

THE TEMPERATURE-STRUCTURE-COMPOSITION BEHAVIOR OF CERTAIN CRYSTALS

BY M. J. BUERGER

DEPARTMENT OF GEOLOGY, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated May 24, 1934

Structure and Notation of Polyatomic Crystals.—Substances which are polyatomic in any sense, have the possibility of forming crystals of two distinct types. These two types and the consequences of their existence may be illustrated by considering crystals formed of elements A and B.

Suppose elements A and B form an unbroken series of solid solution crystals in which the A atoms and B atoms proxy for one another. Crystals of composition half A and half B, then, may have either of two structures:

1. A possible configuration of the atoms is one in which either A or B occupies a given structural position in the solid solution structure without regard to its neighbors. There is thus a random distribution of A and B which has the same identity periods as the general solid solution crystal structure. In this solid solution series, crystals of pure A may be designated by the formula A, crystals of pure B by the formula B, and in conformity with this, crystals of half A and half B in random distribution may be designated by the formula:

$$\begin{bmatrix} A_{1/2} \\ B_{1/2} \end{bmatrix}. \tag{1}$$

2. The structure of the crystal may consist of some sort of regular alternation of A and B within the general scheme of the $A-B$ solid solution crystal structure. As a necessary consequence of the alternation of A and B atoms, one or more of the primitive identity periods of the crystal become small multiples of the corresponding identity periods of the general