Doubled Resonances and Unitarity*

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The constraints of unitarity on a multichannel partial-wave amplitude dominated by two nearby or coincident resonances are derived. The factorization conditions are discussed and used in the solution of these equations. The solutions permit variation in the relative heights of the two peaks in the cross section and variation in the depth of the dip. Single peaks appear in some cross sections, while doubled peaks appear in others. The single peak may be centered at the energy of the dip or displaced to either side, and the amplitude may be imaginary or real at the top of the single peak. The doubled-resonance amplitude is Reggeized and the narrow-width limit is investigated. The amplitude becomes a simple pole with an unfactorizable residue in this limit. Numerical examples of the solutions are also presented.

I. INTRODUCTION

THE recent discovery of the split-peak structure of the A_2 meson[1–5] has focused attention on the possible existence of double poles in the scattering amplitude.[6–12] Many physicists have argued that the narrow dip results from a coherent interference of two resonances having the same quantum numbers.[13] It is then an obvious question to ask for the nature of the constraints which unitarity imposes on an amplitude when the two poles are very close or even coincide. Although many papers have dealt with doubled resonances, we feel that the role of unitarity has not been completely investigated.

Consider a many-channel partial-wave amplitude which is dominated by two nearby resonances. If we assume the dominance of two-body channels, then this amplitude must satisfy many-channel two-body unitarity. We derive the set of equations which the resonance parameters must satisfy and then we solve the equations for any separation of the two resonance poles. The technique of solution makes essential use of the factorization property of the residues, when these two poles do not coincide.[14] The purpose of this paper is to solve the unitarity constraints and to discuss some of the features of their many-channel solution.

The single-channel solution is well known and has a number of similarities to the general multichannel solution. In the limit where the two resonances coincide, the amplitude is a sum of a simple pole and a dipole. The dipole is always accompanied by a single pole.[15] Moreover, the cross section varies smoothly as the separation of the two poles increases from zero to a finite value. However, many of the specific features of the single-channel partial-wave cross section no longer apply to the many-channel case.

In the single-channel case, the amplitude is restricted to lie on the "unitarity circle," so that the amplitude A is pure imaginary at the top of the peaks and $A=0$ at the bottom of the dip. This implies that the cross section must have two peaks of equal height, separated by a dip to zero.[7] But in the many-channel case, where the elastic amplitudes are only constrained to lie within the unitarity circle, the peaks may have any relative height, and the dip may be varied and may even vanish. In general, it is possible for peaks to be doubled in some cross sections and single in others. The single peak may appear at the energy of the dip or it may appear shifted, and the amplitude may be real or imaginary at the top of the single peak. Moreover, variation of the energy of the incident particle in

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[15] A "dipole" is a second-order pole in the sense of complex variable theory, i.e., its coefficient has no E dependence. The amplitude must depend on E on account of unitarity [see Eq. (2)].

production experiments can change the peak structure in a given decay mode. Many of these features appear to be present experimentally for the A_2, and all are consistent with unitarity and the hypothesis that both A_2's are 2^+ mesons. A detailed examination of these features is given in this paper.

The doubled-resonance amplitude may be studied in the "narrow-width" resonance limit. In almost all respects, the doubled resonances appear like simple resonance poles in this limit. Particularly interesting is the case in which the two resonances coincide. Then one finds in the narrow-width limit that:

(i) The dipole vanishes, leaving only a single pole. The coefficient of the dipole is an order of magnitude of the width smaller than the coefficient of the single pole.

(ii) The Regge behavior is $\beta(t)s^{\alpha(t)}$, just as an ordinary Regge pole. The terms behaving like $s^{\alpha(t)}\ln s$, introduced in the Regge formula by the dipole, again have coefficients that vanish in the limit.

(iii) The coefficient of the single pole does not factor. Thus, a doubled resonance becomes a single pole with an unfactorizable residue in the narrow-width limit.

In Sec. II we review in detail the single-channel solution. The algebra is simple and displays many of the features of the more complicated many-channel case.

The unitarity constraints on the doubled-resonance amplitude in the many-channel case are derived in Sec. III. The factorization properties of the amplitude are also discussed.

With the aid of the factorization properties, we solve the unitarity relations in Sec. IV, and examine the limit where the two resonances coincide. The interpretation of the arbitrary parameters is given. Possible experimental implications of the solution, especially related to the A_2 meson, are also discussed.

We "Reggeize" the dipole formula in Sec. V and then discuss the narrow-width limit.

In Sec. VI we present some numerical examples of the solutions for the two-channel case, which exhibit the features of the many-channel solution. Both transition probabilities and Argand diagrams are plotted.

II. SINGLE-CHANNEL CASE

In this section we outline the primary features of doubled resonances using as an example the algebraically simple single-channel situation. A unitary partial-wave S matrix representing a doubled resonance is[7]

$$S=\frac{(E-M_1-i\Gamma_1)(E-M_2-i\Gamma_2)}{(E-M_1+i\Gamma_1)(E-M_2+i\Gamma_2)}, \tag{1}$$

where we have neglected the background. (The Γ_i are half the usual Γ's for the widths.) The amplitude A $[=(S-1)/2i]$ is

$$A=-\frac{(\Gamma_1+\Gamma_2)E-M_1\Gamma_2-M_2\Gamma_1}{(E-M_1+i\Gamma_1)(E-M_2+i\Gamma_2)}. \tag{2}$$

Unitarity requires the zero in the numerator which gives the split peak. At the top of the peaks in the cross section, $A=i$; $A=0$ at the bottom of the dip.

Suppose for the moment that $M_1=M_2$ and $\Gamma_1=\Gamma_2$ are not simultaneously true. Then Eq. (2) may be written as a sum of two poles:

$$A=-\frac{\Gamma_1}{E-M_1+i\Gamma_1}\frac{M_1-M_2-i(\Gamma_1+\Gamma_2)}{M_1-M_2-i(\Gamma_1-\Gamma_2)}+\frac{\Gamma_2}{E-M_2+i\Gamma_2}\frac{M_2-M_1-i(\Gamma_1+\Gamma_2)}{M_1-M_2-i(\Gamma_1-\Gamma_2)}. \tag{3}$$

The residue of each pole is modified by the presence of the other pole, and the couplings become complex in a well-defined way. Moreover, in the limit $M_1\to M_2$ and $\Gamma_1\to\Gamma_2$, the pole residues themselves develop poles. The singular part of the residues is easily isolated by rewriting Eq. (2) in the form

$$A=-\frac{\Gamma_2}{E-M_1+i\Gamma_1}-\frac{\Gamma_1}{E-M_2+i\Gamma_2}-\frac{(M_1-M_2)(\Gamma_1-\Gamma_2)-i(\Gamma_1^2+\Gamma_2^2)}{(E-M_1+i\Gamma_1)(E-M_2+i\Gamma_2)}. \tag{4}$$

The limit $M_1\to M_2$ and $\Gamma_1\to\Gamma_2$ is smooth in Eq. (4) and gives an amplitude that is a pole plus a dipole:

$$A=-\frac{2\Gamma}{E-M+i\Gamma}+\frac{2i\Gamma^2}{(E-M+i\Gamma)^2}. \tag{5}$$

It is important to note that the amplitude is not given by an isolated dipole.[15] Unitarity requires that the dipole is necessarily accompanied by a simple pole. In an electrostatic analog, one would say that an isolated dipole charge distribution is impossible—that whenever a dipole occurs, it must be accompanied by a charge.[16]

We list here a number of features of the single-channel solution: (i) The two peaks in the cross section (or, more precisely, in $|A|^2$) have the same height, regardless of the separation or relative orientation of the two poles; (ii) the amplitude goes around the unitarity circle twice, i.e., the phase shift goes from below $\frac{1}{2}\pi$ to above $\frac{3}{2}\pi$ as E goes over the doubled-resonance region; (iii) the amplitude has a zero between the two peaks; and (iv) the cross section always has a double peak. None of these features need be true of the many-channel amplitude.

[16] We are pleased to thank Professor H. P. Dürr for a discussion of this point.

The doubled-resonance amplitude may be studied in the narrow-width resonance limit. The idea of the narrow-width limit is old,[17] and today a beautiful example of a narrow-width amplitude possessing many nice features is given by the Veneziano formula.[18] It is interesting to study the implications that doubled resonances have for this limit. There has been some concern on how to accommodate the split A_2 in such a scheme. However, several important features of Eq. (4) suggest that the doubled resonances can be incorporated into the narrow-width models without difficulty.

The limit consists of dividing the amplitude by the scale of the width, then letting the two poles approach the real axis.

(i) The limit does not contain the dipole:

$$A/\epsilon \underset{\epsilon\to 0}{\longrightarrow} -\frac{\Gamma_2/\epsilon}{E-M_1}-\frac{\Gamma_1/\epsilon}{E-M_2}. \tag{6}$$

If we also take the limit $M_1\to M_2$, then Eq. (2) reduces to a single pole, not a dipole. The residue of the single pole ($M_1=M_2$) is the sum of the residues of the simple resonance poles.

(ii) When Eq. (2) is unitarized in the complex J plane and the asymptotic formulas for large $\cos\theta$ are derived by performing the Watson-Sommerfeld transformation, then we find "Regge" behavior of the form

$$A(s,t)=b(t)(s/s_0)^{\alpha(t)}+\tilde{b}(t)(s/s_0)^{\alpha(t)}\ln(s/s_0), \tag{7}$$

where $b(t)=O(\mathrm{Im}\alpha)$ and $\tilde{b}(t)=O[(\mathrm{Im}\alpha)^2]$. The dipole contribution is of order $(\mathrm{Im}\alpha)^2$ and should be dropped in the narrow-width limit. For phenomenology, $\mathrm{Im}\alpha$ is small, so that $\tilde{b}(t)$ is expected to be an order of magnitude smaller than $b(t)$.

III. UNITARITY CONSTRAINTS ON MULTICHANNEL AMPLITUDES

The constraints that unitarity and factorization impose on the doubled-resonance amplitude in the multichannel case are discussed in this section.

The most general form for an amplitude dominated by a doubled resonance and in which the background is neglected is[19]

$$A=-\frac{\alpha E+\beta}{(E-M_1+i\Gamma_1)(E-M_2+i\Gamma_2)}, \tag{8}$$

where A, α, and β are $N\times N$ matrices in channel space. An E^2 term in the numerator of Eq. (8) would correspond to a background. We require that Eq. (8) satisfy partial-wave unitarity for many two-body channels:

$$A-A\dagger=2iA\rho A\dagger=2iA\dagger\rho A. \tag{9}$$

The phase-space factor ρ is a diagonal matrix with zero elements for the channels whose thresholds are greater than E. The second form of unitarity in Eq. (9) gives one additional constraint on Eq. (8) [Eq. (10e) below] and should not be ignored. After substituting Eq. (8) into Eq. (9) and equating coefficients of E, we find this set of equations:

$$\alpha=\alpha\dagger, \tag{10a}$$

$$\sigma=\alpha\rho\alpha-(\Gamma_1+\Gamma_2)\alpha, \tag{10b}$$

$$-(M_1+M_2)\sigma=(M_1\Gamma_2+M_2\Gamma_1)\alpha-(\Gamma_1+\Gamma_2)\lambda+\alpha\rho\lambda+\lambda\rho\alpha, \tag{10c}$$

$$(M_1M_2-\Gamma_1\Gamma_2)\sigma=(M_1\Gamma_2+M_2\Gamma_1)\lambda+\lambda\rho\lambda+\sigma\rho\sigma, \tag{10d}$$

$$\sigma\rho\lambda=\lambda\rho\sigma, \tag{10e}$$

where

$$\beta=\lambda-i\sigma,\quad \beta\dagger=\lambda+i\sigma. \tag{10f}$$

We have separated β into Hermitian and anti-Hermitian parts, so that α, λ, and σ are all Hermitian matrices. A more convenient form of Eqs. (10) is obtained by taking various linear combinations:

$$\alpha\rho\tau+\tau\rho\alpha=2\mu\gamma\alpha+2\Gamma\tau, \tag{11a}$$

$$(-\Gamma^2-\mu^2+\gamma^2)\sigma=\tau\rho\tau+\sigma\rho\sigma-2\mu\gamma\tau, \tag{11b}$$

$$\sigma=\alpha\rho\alpha-2\Gamma\alpha, \tag{11c}$$

$$\sigma\rho\tau=\tau\rho\sigma, \tag{11d}$$

$$\alpha\dagger=\alpha,\quad \tau\dagger=\tau,\quad \sigma\dagger=\sigma, \tag{11e}$$

where

$$\begin{aligned}\tau&=\lambda+M\alpha,\\ M&=\tfrac{1}{2}(M_1+M_2),\\ \Gamma&=\tfrac{1}{2}(\Gamma_1+\Gamma_2),\\ \mu&=\tfrac{1}{2}(M_1-M_2),\\ \gamma&=\tfrac{1}{2}(\Gamma_1-\Gamma_2).\end{aligned} \tag{11f}$$

Time-reversal invariance gives the further condition that α, τ, and σ are real symmetric matrices.

It seems that it should be very difficult to solve Eqs. (11), since they are a set of coupled nonlinear matrix equations. However, we can greatly simplify the calculation by imposing causality, which requires that the residues of simple poles factor. This factorization theorem, which is a consequence of unitarity and analyticity, remains valid as long as the pole positions do not coincide.[14] We first discuss the case where $M_1=M_2$ and $\Gamma_1=\Gamma_2$ are not both true. Then we prove a new factorization theorem for the case when the two poles coincide.

When the two poles do not coincide, the amplitude may be written as

$$A_{ij}=-\frac{g_ig_j}{E-M_1+i\Gamma_1}-\frac{g_i'g_j'}{E-M_2+i\Gamma_2}. \tag{12}$$

[17] R. F. Dashen and M. Gell-Mann, Phys. Letters **17**, 142 (1965); S. Mandelstam, Phys. Rev. **166**, 1539 (1968).
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[19] Again, we define Γ to be half the usual Γ for convenience.

The couplings g_i and g_i' are related to α_{ij} and β_{ij} by

$$\alpha_{ij}=g_ig_j+g_i'g_j', \tag{13a}$$

$$\beta_{ij}=-g_ig_j(M_2-i\Gamma_2)-g_i'g_j'(M_1-i\Gamma_1). \tag{13b}$$

The inverse relations are

$$g_ig_j=\frac{\alpha_{ij}(\mu-i\Gamma_1)+\tau_{ij}-i\sigma_{ij}}{2(\mu-i\gamma)}, \tag{14a}$$

$$g_i'g_j'=\frac{\alpha_{ij}(\mu+i\Gamma_2)-\tau_{ij}+i\sigma_{ij}}{2(\mu-i\gamma)}. \tag{14b}$$

Equations (14) show that, in general, the g_i and g_i' are complex, and in the limit $\mu\to 0$, $\gamma\to 0$, these couplings may develop poles. (In general, α and β are nonsingular and nonzero in this limit.) Just as in the single-channel case, it is possible to rewrite the amplitude so that the singular term (as $\mu,\gamma\to 0$) becomes a nonsingular coefficient of a dipole:

$$A_{ij}=-\frac{\frac{1}{2}\alpha_{ij}}{E-M_1+i\Gamma_1}-\frac{\frac{1}{2}\alpha_{ij}}{E-M_2+i\Gamma_2}+\frac{i\Gamma\alpha_{ij}-\tau_{ij}+i\sigma_{ij}}{(E-M_1+i\Gamma_1)(E-M_2+i\Gamma_2)}. \tag{15}$$

The limit $M_1=M_2=M$ and $\Gamma_1=\Gamma_2=\Gamma$ is a smooth limit in Eq. (15), but the factorization conditions, Eqs. (14), are still singular. However, it is easy to prove a new factorization theorem using the same techniques by which one proves the factorization of a simple-pole residue.[14] Since the most singular part of the amplitude is $(E-M+i\Gamma)^{-2}$, it turns out that the coefficients of this dipole must factor. The amplitude may be written as

$$A_{ij}=-\frac{\alpha_{ij}}{E-M+i\Gamma}-\frac{f_if_j}{(E-M+i\Gamma)^2}. \tag{16}$$

From Eq. (15), it is apparent that

$$f_if_j=\tau_{ij}-i(\Gamma\alpha_{ij}+\sigma_{ij}). \tag{17}$$

Again we emphasize that the residue of the simple pole need not factor in this limit, but the coefficient of the dipole must factor. Actually, the factorization of the dipole coefficient follows from the factorization of the two separated poles if the limiting procedure is done carefully.

IV. MANY-CHANNEL SOLUTION

The complete solution of the unitarity condition in the many-channel case is given in this section. At first, we assume that the two poles of the amplitude do not coincide. The simplicity of the solution depends crucially on the fact that both residues must factor. We find that only two possible forms are consistent with factorization and the reality properties of the amplitude. These forms are then substituted into Eq. (11) and the final constraints of unitarity are derived. Having satisfied unitarity, we then cast the result into a form in which $M_1\to M_2$ and $\Gamma_1\to\Gamma_2$ is a smooth limit. The dipole factorization theorem follows directly. Finally, we discuss the parameterization of the solution.

The factorized amplitude, Eq. (12), provides a simple starting point. Let

$$g_i=x_i+iy_i,\qquad g_i'=u_i+iv_i. \tag{18}$$

Although the couplings may be complex, α is always a real symmetric matrix. It follows from Eq. (13a) that

$$x_iy_j+y_ix_j+u_iv_j+v_iu_j=0. \tag{19}$$

A careful analysis reveals that all the solutions of Eq. (19) are included in the following two solutions:

$$y_i=cu_i,\quad v_i=-cx_i, \tag{20a}$$

$$x_i=cu_i,\quad v_i=-cy_i, \tag{20b}$$

where c is a proportionality constant independent of channel. Thus, Eqs. (20a) and (20b) represent a complete solution to the factorization and reality constraints on the amplitude. Since unitarity at a point in the E plane is used to derive the factorization theorem, we may expect the remaining restrictions of unitarity to be simple in nature. These restrictions are obtained by substituting Eqs. (20a) or (20b) into the definitions of α, τ, and σ [Eq. (13)] and then substituting these matrices into Eq. (11). Applying this procedure to Eq. (20b), one finds that either Γ_1 or Γ_2 must be negative, which is unphysical. Thus, we confine our attention to Eq. (20a).

Equations (11) are satisfied if

$$\mathbf{x}\cdot\mathbf{x}=\frac{\Gamma_1-c^2\Gamma_2}{(1-c^2)^2}, \tag{21a}$$

$$\mathbf{u}\cdot\mathbf{u}=\frac{\Gamma_2-c^2\Gamma_1}{(1-c^2)^2}, \tag{21b}$$

$$\mathbf{x}\cdot\mathbf{u}=-\frac{2\mu c}{(1-c^2)^2}. \tag{21c}$$

The scalar product $\mathbf{x}\cdot\mathbf{u}$ is defined by

$$\mathbf{x}\cdot\mathbf{u}=\sum_i x_i\rho_iu_i, \tag{22}$$

where the phase-space factor ρ_i is zero for closed channels and positive for open channels. The squares of the vectors x and u must be non-negative, and must satisfy the inequality $(\mathbf{x}\cdot\mathbf{u})^2\leq\mathbf{x}^2\mathbf{u}^2$. Equations (20a) and (21) completely solve the unitarity constraints as long as the two poles do not coincide. It is an important feature of this solution that $c^2\neq 1$, unless the poles

coincide. Indeed, from the properties of the scalar product, one finds that c^2 is bounded by

$$1-c^2\geq\frac{2(\mu^2+\gamma^2)^{1/2}}{\Gamma^2-\gamma^2}[(\Gamma^2+\mu^2)^{1/2}-(\mu^2+\gamma^2)^{1/2}], \quad (23)$$

where μ and γ are defined in Eqs. (11).

We now examine the limit $\mu=\gamma=0$. We saw in Eqs. (14) that g_ig_j could develop a pole in this limit, which would imply that $c^2=1$, since $\mathbf{x}\cdot\mathbf{x}$ would also be singular. Thus, it is convenient to rescale x_i and u_i so that the limit of small separation is smooth. We define some new quantities:

$$1-c^2=2s/\xi\kappa,$$
$$x_i=(\xi\kappa\Gamma/2s)^{1/2}X_i, \quad (24)$$
$$u_i=(\xi\kappa\Gamma/2s)^{1/2}U_i,$$

where

$$\kappa=(1+s^2\cos^2\theta)^{1/2}+s,$$
$$\mu=s\Gamma\cos\theta, \quad (25)$$
$$\gamma=s\Gamma\sin\theta.$$

The separation s between the poles is given in units of Γ, and the angle θ gives the direction of the dipole; it is the angle between the real axis and the direction from the pole at M_2, Γ_2 to the pole at M_1, Γ_1. We have defined κ so that $\kappa(s=0)=1$. Then, by Eq. (23), ξ is restricted to the range

$$\xi_{\min}\leq\xi\leq 1, \quad (26a)$$

where

$$\xi_{\min}=2s/\kappa. \quad (26b)$$

With these definitions, the general solution to the unitarity constraints is

$$\alpha_{ij}=\Gamma(X_iX_j+U_iU_j), \quad (27a)$$

$$\tau_{ij}=\Gamma^2[(\xi\kappa-s)\cos\theta\,(X_iX_j-U_iU_j)+\xi\kappa_1\sin\theta\,(X_iU_j+U_iX_j)], \quad (27b)$$

$$\sigma_{ij}=\Gamma^2[(\xi\kappa-s)\sin\theta\,(X_iX_j-U_iU_j)-\xi\kappa_1\cos\theta\,(X_iU_j+U_iX_j)]-\Gamma\alpha_{ij}, \quad (27c)$$

where

$$\mathbf{X}\cdot\mathbf{X}=1+(\xi\kappa-s)\sin\theta, \quad (27d)$$
$$\mathbf{U}\cdot\mathbf{U}=1-(\xi\kappa-s)\sin\theta, \quad (27e)$$
$$(\mathbf{X}\cdot\mathbf{U})=-\xi\kappa_1\cos\theta, \quad (27f)$$
$$\kappa_1=\kappa(1-2s/\xi\kappa)^{1/2}. \quad (27g)$$

Although the phase of κ_1 is arbitrary, the expressions for α, τ, and σ do not depend on this phase.

Equation (27) is the most general solution to the unitarity constraints. For N channels, there are $2N+2$ arbitrary parameters, including M, Γ, s, and θ. It is straightforward to check this solution by inserting it into Eqs. (11).

We now examine the limit in which the poles coincide and recover the factorization of the dipole, Eq. (16). In the limit $s=0$, κ and κ_1 become $\kappa=\kappa_1=1$ for all allowed values of ξ. If we substitute Eq. (27) into the expression for the amplitude, then we recover Eq. (16) in the explicit form[20]

$$A_{ij}=-\frac{\Gamma(X_iX_j+U_iU_j)}{E-M+i\Gamma}-\xi\Gamma^2e^{-i\theta}\frac{(X_i+iU_i)(X_j+iU_j)}{(E-M+i\Gamma)^2}. \quad (28)$$

Thus, the factorization of the dipole follows from the factorization of the two separated poles.

The shapes of the cross sections are very insensitive to the separation s of the poles. The matrices may change linearly as a function of s, but the coefficients of the linear term are small by suitably scaling ξ. An example comparing a double peak for $s=0$ with one for $s=0.5$ is given in Sec. VI, where we see that the shapes are indistinguishable. Thus, the question "Do the poles coincide?" cannot be easily resolved from an analysis of the experimental data.

The parameter ξ measures the strength of the dipole. When $\xi=0$, the dipole in Eq. (28) vanishes. In general, $\xi=\xi_{\min}$ corresponds to a simple resonance solution in which the two resonances are coupled to orthogonal sets of channels. This is a possible solution since the resonances might have different internal quantum numbers. All cross sections have single peaks for $\xi=\xi_{\min}$.

The maximum value of ξ, $\xi=1$, corresponds to the single-channel solution of Sec. II. The vectors $\mathbf{X}$ and $\mathbf{U}$ are parallel, and $\sigma_{ij}=0$. This means that doubled peaks of equal height appear in all cross sections and that the amplitude is factorizable. This solution is obtained by multiplying the single-channel solution by a factorizable matrix to obtain a many-channel solution.

The cross sections exhibit interesting new features when ξ is between $\xi_{\min}$ and 1. The residue of the single pole ($\mu=\gamma=0$) is never factorizable when ξ is not at an extreme value. Variation of ξ controls the depth of the dip, and it is no longer necessary that the two peaks in the cross section be of equal height. The direction of the dipole θ controls the relative height of the peaks, even when $s=0$.

In the general case, it is also possible to have doubled peaks in some cross sections and single peaks in others. For example, it is very easy to arrange ξ and θ in the two-channel problem so that a pronounced double peak appears in one elastic cross section and single peaks appear in the production cross section and the other elastic cross section at the energy of the elastic dip. In a similar manner, it is possible to obtain both doubled and single peaks in a given channel by changing the production mechanism of the doubled resonance. An example of this has apparently been observed experimentally. The $K\bar{K}$ decay mode of the A_2 produced in

[20] To some extent, one may think of Eq. (28) as a generalization of the Breit-Wigner formula to doubled resonances.

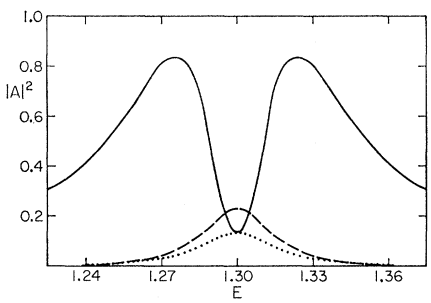

FIG. 1. Transition probabilities. The solid line is $|A_{11}|^2$, the dashed line is $|A_{12}|^2$, and the dotted line is $|A_{22}|^2$, plotted as functions of energy. The parameters of Eqs. (35) are $\xi=0.8$, $\theta=\frac{1}{2}\pi$, $s=0$, and $\psi=0$. The energy scale, which is determined by the choice $M=1.30$ GeV and $\Gamma=25$ MeV, is chosen merely to guide the eye and is not intended to fit the split A_2 data. This choice of M and Γ is unchanged in the following figures.

π^-p reactions has a single peak,[2] but the $K\bar{K}$ decay mode of the A_2 produced in $\bar{p}p$ annihilations has a doubled peak.[4] The single peak need not appear at the energy of the dip, as is shown in Sec. VI. We emphasize that the occurrence of a double peak or the location of a single peak in a given decay mode depends not only on the decay amplitude, but also on the production amplitude. The data of Refs. 2–4 are consistent with the hypothesis that both A_2 are 2^+ mesons.

Some numerical examples of these features are given in Sec. VI.

V. REGGEIZATION AND NARROW-WIDTH LIMIT

All of the considerations of the preceding sections may be repeated, almost unaltered, in the complex angular momentum plane. The Froissart-Gribov continuation allows one to define a unique amplitude for complex J. If we assume that the partial-wave amplitudes are dominated by two nearby or coincident Regge poles, then the amplitude may be written

$$A_{ij}(J,t)=-\frac{a_{ij}(t)J+b_{ij}(t)}{[J-\alpha_1(t)][J-\alpha_2(t)]}, \tag{29}$$

where the background has again been neglected, and $a(t)$ and $b(t)$ are $N\times N$ matrices in channel space.

Two-body unitarity may be continued to complex J. The multichannel generalization is

$$\begin{aligned}A_{ij}(J,t)-A_{ij}^*(J^*,t)&=2i\sum_k A_{ik}(J,t)\rho_k A_{kj}^*(J^*,t)\\&=2i\sum_k A_{ik}^*(J^*,t)\rho_k A_{kj}(J,t),\end{aligned} \tag{30}$$

where t is above threshold.

Inserting Eq. (29) into the unitarity equation (30), we obtain a set of constraints that are formally identical to Eqs. (11), provided that the J-plane quantities are identified with the appropriate E-plane quantities. It is

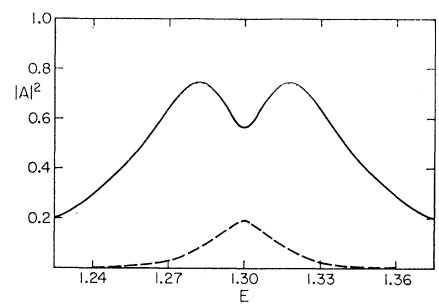

FIG. 2. $|A_{11}|^2$ (solid line) and $|A_{12}|^2$ (dashed line) as functions of energy; $\xi=0.5$, $\theta=\frac{1}{2}\pi$, $s=0$, and $\psi=0$.

clear that the solution obtained in Sec. IV may be immediately extended to the quantities $a_{ij}(t)$ and $b_{ij}(t)$ of Eq. (29).

There is little point in discussing the features of the solution in the J plane, since they are similar to those found in Sec. IV. However, we should call attention to two interesting points: The first concerns phenomenological applications of the Reggeized doubled resonance; the second deals with the narrow-width limit.

For simplicity, let $\alpha_1(t)=\alpha_2(t)=\alpha(t)$. As in Sec. IV, we find

$$A_{ij}(J,t)=-\frac{a_{ij}(t)}{J-\alpha(t)}-\frac{f_i(t)f_j(t)}{[J-\alpha(t)]^2}. \tag{31}$$

Once again, unitarity requires the dipole to be accompanied by a single pole in the complex J plane.

The implication of Eq. (31) for high-energy phenomenology is that one should never expect a "pure" dipole high-energy Regge behavior at presently available energies, but one should expect the "mixed" behavior:

$$\mathrm{Im}A(s,t)\underset{s\to\infty}{\longrightarrow}\left(a_{ij}(t)+f_i(t)f_j(t)\frac{d}{d\alpha(t)}\right)(s/s_0)^{\alpha(t)}. \tag{32}$$

Moreover, at least for t above the first threshold, $f_i(t)f_j(t)$ is expected to be of one order of magnitude of Imα smaller than $a_{ij}(t)$, according to Eq. (28). In the region where most resonances are observed, Imα appears to be about 0.1. This value should give an estimate of

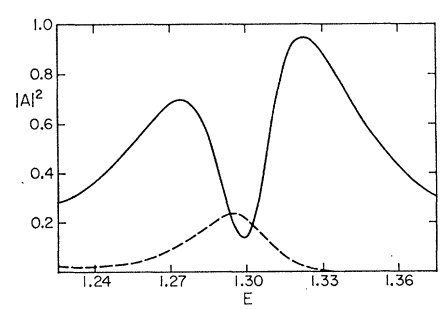

FIG. 3. $|A_{11}|^2$ (solid line) and $|A_{12}|^2$ (dashed line) for $\xi=0.8$, $\theta=\frac{1}{4}\pi$, $s=0$, and $\psi=0$.

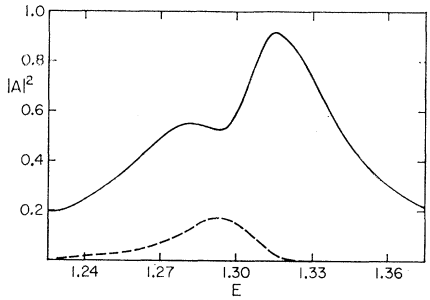

FIG. 4. $|A_{11}|^2$ (solid line) and $|A_{12}|^2$ (dashed line) for $\xi=0.5$, $\theta=\frac{1}{4}\pi$, $s=0$, and $\psi=0$.

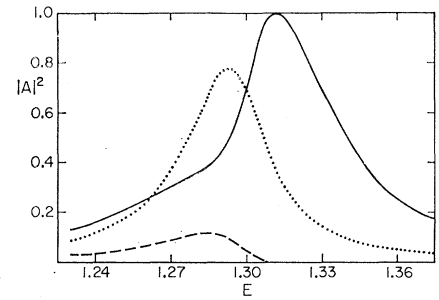

FIG. 6. $|A_{11}|^2$ (solid line), $|A_{12}|^2$ (dashed line), and $|A_{22}|^2$ (dotted line) for $\xi=0.5$, $\theta=0$, $s=0$, and $\psi=0$.

the relative magnitude of $f_i(t)f_j(t)$ compared to $a_{ij}(t)$. If the same ratio persists below the threshold to $t<0$, then the dipole contribution to Eq. (32) will not dominate until extremely high energies are reached.

The discussion of the narrow-width limit closely parallels the discussion of Sec. II, except for some new multichannel features. If we scale the widths Γ_1 and Γ_2 and the amplitude by ϵ, and then let $\epsilon\to 0$, we find that

$$A_{ij}=-\frac{X_iX_j}{E-M_1}-\frac{U_iU_j}{E-M_2}. \tag{33}$$

The proof of Eq. (33) follows immediately from Eq. (23), where one finds that $c=0$ for $\epsilon=0$. In the case where $M_1=M_2$, the narrow-width amplitude contains a single pole with an unfactorizable residue.

The dipole term of Eq. (31) is one order higher in ϵ than the single-pole term, so that the high-energy Regge behavior in the narrow-width limit is

$$\mathrm{Im}A_{ij}(s,t)\underset{s\to\infty}{\longrightarrow} a_{ij}(t)(s/s_0)^{\alpha(t)}, \tag{34}$$

where $a_{ij}(t)$ is not factorizable, but is a matrix of rank 2.

Thus, in the narrow-width limit, double resonances appear like normal Regge poles in all respects, except that the residue functions do not factorize.

VI. NUMERICAL EXAMPLES

In this section we present examples of the two-channel solution. The two-channel problem has all the freedom and arbitrariness of the arbitrary case, so we give explicit examples of the features described in Sec. IV. It is apparent that the "anomalies" observed in the A_2 meson are consistent with the doubled-resonance hypothesis and unitarity.[1–5]

There are six arbitrary parameters in the two-channel problem: M, Γ, s, ξ, θ, and a new "angle" ψ. M and Γ merely locate the doubled resonance and set the energy scale. We recall that s is the separation of the two poles, θ is the angle between the real axis and the line connecting the two poles, and ξ measures the strength of the dipole term. We let $M=1.3$ BeV and $\Gamma=25$ MeV. This choice is merely to guide the eye and does not represent a fit to the doubled A_2 data. Equation (27) is used to construct α, τ, and σ. The X_i and U_i are defined by

$$X_1=x\cos\psi,\quad X_2=x\sin\psi, \tag{35a}$$

$$U_1=u\cos\phi,\quad U_2=u\sin\phi, \tag{35b}$$

where

$$x=[1+(\xi\kappa-s)\sin\theta]^{1/2}, \tag{35c}$$

$$u=[1-(\xi\kappa-s)\sin\theta]^{1/2}, \tag{35d}$$

and ϕ is computed from Eq. (27f):

$$\phi=\psi\pm\left[\pi-\cos^{-1}\left(\frac{\xi\kappa_1\cos\theta}{xu}\right)\right]. \tag{35e}$$

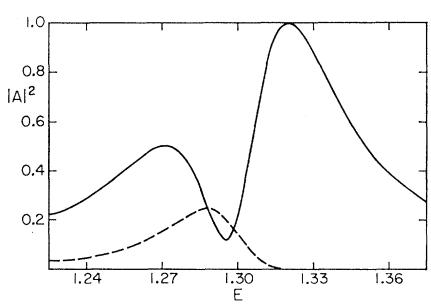

FIG. 5. $|A_{11}|^2$ (solid line) and $|A_{12}|^2$ (dashed line) for $\xi=0.8$, $\theta=0$, $s=0$, and $\psi=0$.

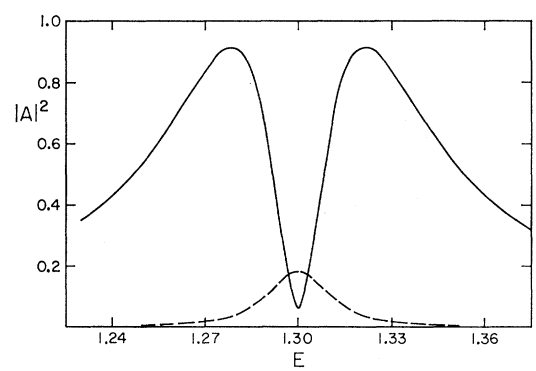

FIG. 7. Variation with s. $|A_{11}|^2$ (solid line) and $|A_{12}|^2$ (dashed line) as functions of E for $s=0.5$, $\xi=0.93$, $\theta=\frac{1}{2}\pi$, and $\psi=0$ (ξ has been rescaled with respect to $\xi_{\min}$). This figure should be compared with Fig. 1.

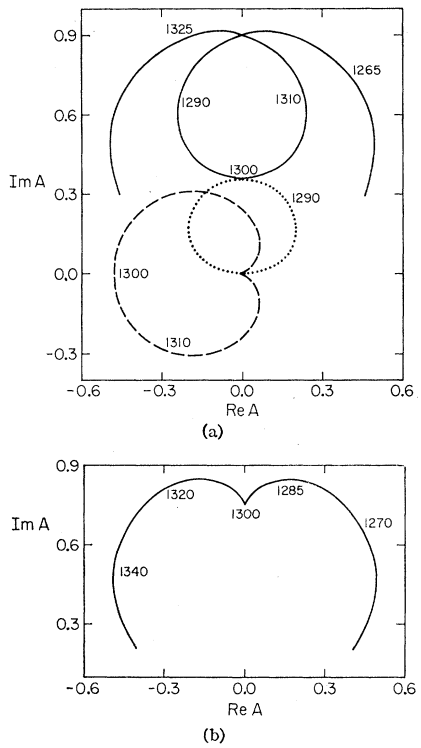

FIG. 8. (a) Argand diagrams for A_{11} (solid line), A_{12} (dashed line), and A_{22} (dotted line), for $\xi=0.8$, $\theta=\frac{1}{2}\pi$, $s=0$, and $\psi=0$, as in Fig. 1. The numbers refer to the energy. (b) Argand diagrams for A_{11} for $\xi=0.5$, $\theta=\frac{1}{2}\pi$, $s=0$, and $\psi=0$ as in Fig. 2.

In the plots we have always chosen the + sign in Eq. (35e). The expressions for α, τ, and σ, constructed from these couplings, are substituted into the formulas for $|A_{ij}|^2$, $\mathrm{Re}A_{ij}$, and $\mathrm{Im}A_{ij}$, and then plotted as functions of E in Figs. 1–10.

The figures are self-explanatory. We call attention to the following features:

(i) The depth of the dip in the elastic cross section is a function of ξ, for fixed θ, s, and ψ.

(ii) The relative height of the two peaks in the elastic cross section is a function of θ for fixed ξ, s, and ψ.

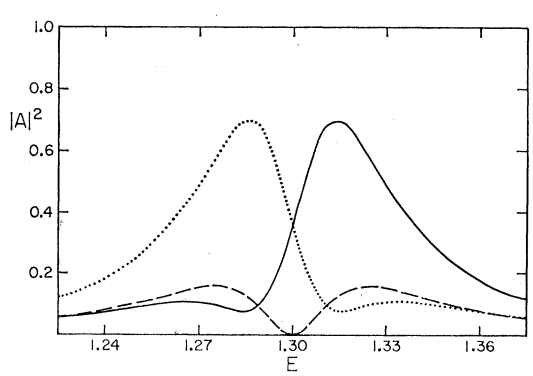

FIG. 9. Variation with ψ. $|A_{11}|^2$ (solid line), $|A_{12}|^2$ (dashed line), and $|A_{22}|^2$ (dotted line) for $\psi=\frac{1}{4}\pi$, $\xi=0.8$, $\theta=\frac{1}{2}\pi$, and $s=0$.

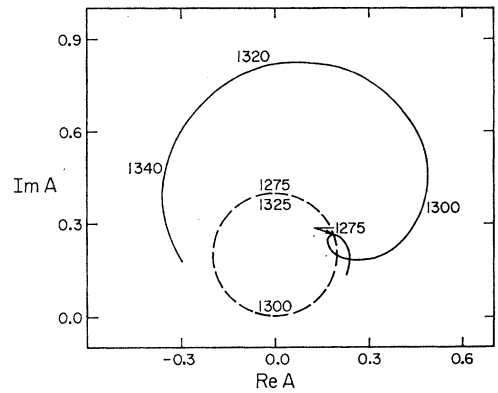

FIG. 10. Argand diagrams for A_{11} (solid line) and A_{12} (dashed line) for $\xi=0.8$, $\theta=\frac{1}{2}\pi$, $s=0$, and $\psi=\frac{1}{4}\pi$ as in Fig. 9. The single peak in $|A_{11}|^2$ occurs when A_{11} is pure imaginary. The little loop in A_{11} is the remains of the other peak. The amplitude A_{12} goes twice around a "small" unitarity circle.

(iii) The shapes of the cross section do not depend on s, provided ξ is appropriately scaled. In Fig. 7, $\xi=0.93$ for $s=0.5$ is equivalent to $\xi=0.8$ for $s=0$ (Fig. 1).

(iv) A single peak in the production cross section may occur when A_{12} is real [see Fig. 8(a)].

(v) The phase of the elastic amplitude need not go around the unitarity twice in order to obtain a doubled peak, as can be seen in Fig. 8(b).

(vi) Doubled peaks in some elastic cross sections are consistent with single peaks in other elastic cross sections (see Fig. 1).

(vii) The variable ψ "reshuffles" the peaks among the amplitudes. In Fig. 9, single peaks appear in the elastic cross sections $|A_{11}|^2$ and $|A_{22}|^2$, but a double peak appears in $|A_{12}|^2$. Here the single peaks are shifted away from the dip and the amplitudes are imaginary at the tops of the peaks (see Fig. 10).

(viii) The shape of the peak in a given decay mode is not *only* a property of the decay amplitude. The shape also depends on the production mechanism.

(ix) A calculation with an arbitrary number of channels would show similar features, except there would be more variables like ψ that permit reshuffling the peaks among the different cross sections.

We close with a comment on the solution $\xi=\xi_{\min}$, where two resonances coupled to orthogonal sets of channels. If there exists a background which connects the two sets of channels, then unitarity destroys the orthogonality of **X** and **U**. This is the same as increasing ξ above $\xi_{\min}$. The two resonances must then interfere with each other.

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