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UNITED WE VOTE

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Abstract

This paper studies the advantages that a coalition of agents in a larger electorate can obtain by forming a voting bloc to pool their votes and cast them all in one direction. We show under which conditions an agent will benefit from the formation of the voting bloc, whether being part of it or stepping out is most advantageous for an individual agent and what are the different optimal internal voting rules to aggregate preferences within the coalition.

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1 Introduction

Individuals often form groups to further their goals. Acting as a team enables them to obtain what perhaps they could not attain working alone. From the pack of wolves hunting together to the complexities of the International Space Station, pooling resources and effort is advantageous. Coalitions, teams, alliances, groups are constantly formed and we see them operate in all fields of human life.

However, despite the advantages of collaboration, alliances are often broken, groups are dissolved, coalitions split, or fail to be formed in the first place. Any union of heterogeneous agents may at times fail to act to the benefit of some of its members. Individual freedom of action is partially curtailed by joining a group and committing to follow its rules. This creates an incentive to abandon the group and proceed alone in a different course of action. There is a trade off between the furthered chances group-action enables, and the sacrifice of individual freedom involved in the process of forming a group.

In this paper we examine such trade-off in the context of political competition between agents who know each other and can communicate to form a coalition.

Several non-cooperative theories of coalition formation with economic applications are surveyed in Carraro [4]. Here we study coalition formation in a political setting in which agents wish to win an electoral vote: Agents coalesce because doing so increases the probability of getting the outcome they want in the election.

We draw inspiration from two examples: One is a legislature composed of independent politicians each with their own individual policy preferences. A group of these politicians can agree to form a political party. Members of a party vote on the platform the party will stand for and then party discipline will force all members to vote for the party's platform in the legislature. A politician joins a party with the hope that his preferences become the platform of the party and thus carry wider support. On the other hand, party discipline would force him to cast votes against his wishes if these don't become part of the party platform.

A second one is international politics: About 200 sovereign nations are engaged in constant interaction. Most countries may find that their influence in the outcome of the international decision making process is marginal or negligible and their chances of getting their desired outcome are not too high. For instance, how likely is it that [name a country out of the G-8 here] sees its favorite candidate becoming the next president of the IMF or the UN? Maybe forming a cohesive coalition to speak with a single voice in the world arena, a group of countries that were individually neglected can gain some significant leverage to get political outcomes closer to their wishes.

This is not only a theoretical possibility, but a very real one, at the heart of the Non-Aligned-Countries movement created in the Sixties, which grouped dozens of poor, former colonial countries trying to gain some influence aside of the American-led Western nations and the Communist camp. An arguably more successful venture in the same lines was the European Economic Community, which along other goals, shaped a common trade policy for its members.

We can imagine a coalition of countries facing a constitutional decision: Whether or not to unify their foreign policies and mold them into a common position, committing and binding themselves to support the coalition's, or Union's, future policies. Costs and benefits must be weighted beforehand, for we assume that the Union process is irreversible (the decision might involve transferring some crucial resources, or the control of the armed forces to a central or federal power).

A crucial problem with the project is the design of the common position: Each country knows what it wants and it also knows that a coalition of nations will have a better chance of being granted its wishes than a single country. Thus there is an incentive to form such coalition. But, if all its members actually have different goals, what is the coalition going to stand for?

The coalition requires to adopt an internal decision-making rule to aggregate the preferences of its members. Buchanan and Tullock [3] praise the virtues of unanimity, both as Constitutional rule (all members have to agree to join a Union) and as internal voting rule for the coalition or Union (the Union only acts collectively if all members agree on some course of action; otherwise each member can pursue its own policies). Barberá and Jackson [1] let agents choose among different rules, and they define "self-stable" voting rules as those that will not be beaten by any other rule if the given voting rule is used to choose among rules. We let unanimity be the Constitutional rule used to choose among rules, and we show that in most cases agents will prefer to choose some other rule as internal decision-making rule.

Maggi and Morelli's [11] study self-enforcing rules to determine whether collective action will be taken or not by a group of agents that requires each and every agent to participate in the collective effort. Our model can also be seen as a collective action problem where the collective action is to coordinate the disparate voting intentions into a unified voting behavior, to the advantage of the majority of the members of the voting coalition thus formed. A substantial distinction is that Maggi and Morelli consider only an homogeneous society in which agents are ex-ante identical, whereas we study the case with different ex-ante types. In heterogeneous coalitions, we find out that supermajority internal voting rules are optimal, whereas in Maggi and Morelli's setting no other internal rule but simple majority or unanimity is ever optimal.

A different approach to coalition formation comes from the voting power literature in the work of Gelman [8], who concentrates on the probability of casting a decisive vote in an election and the effect of coalitions over such probability. We focus on the probability of getting the desired outcome out of the election, not on the probability of casting a decisive vote, and we want to analyze the potential benefits of forming a voting bloc, coalescing with other agents to cast all votes in the same direction. Gelman works with a random voter model in which the probability of voting "yes" is one half for every agent. Our model develops one of the extensions mentioned in his paper by considering heterogeneity of types.

This responds to a different focus over a similar question: Gelman focuses on "a dynamic view of coalitions in which groups of voters choose of their own will to form and disband coalitions in a continuing struggle to maintain their voting power", we care about how the existence of a single coalition that forms a voting bloc will affect the degree of satisfaction of its members, how the heterogeneity of such members may affect their satisfaction resulting from forming the voting bloc and which internal voting rules in the coalition may make the voting bloc satisfactory for a broader range of parameters.

We believe these theoretical questions are particularly relevant to the ongoing debates about the need or desire for a common foreign policy in the EU, a purpose that was first vaguely stated in the Maastricht Treaty (1992), but that has been recently the subject of much deeper debates and controversy during the negotiations towards a Constitution (started in 2002) and will probably continue to be in the European political agenda for years to come. Therefore, we will frequently refer to the EU as a motivating example along our exposition.

After introducing the model and showing that there is a surplus to be gained by forming a voting bloc in Section 2, in Section 3 we ask whether the formation of the voting bloc will benefit every member of the coalition, and we find this to be more likely if the voting bloc uses a supermajority internal voting rule than if it uses simple majority. In Section 4 we study an "opt-out" rule that allows one agent to stay out of the voting bloc and we discuss under what conditions introducing such a rule will benefit all the members of the coalition. In Section 5 we conclude and propose a future agenda of research and an Appendix contains algebraic calculations and proofs.

2 The Model. Gains from forming a Voting Bloc.

We consider a society formed by $M + N + 1$ agents, where M and N are even. These agents (legislators, countries, etc.) face a binary decision: Either to keep the status quo, or to vote for an alternative a to replace it. All agents are called to vote either for a (*yes*), or against a (*no*). If the number of favorable votes is equal or higher than a threshold T, then a is implemented.

Each agent strictly prefers either the status quo or the alternative a, and we assume no intensity on preferences. Preferences over lotteries will simply be determined in favor of the lottery that assigns the higher weight to the preferred alternative.

M agents lack coordination powers and will thus vote individually. The remaining $N + 1$ agents can coordinate among themselves and may at wish form a voting bloc. We will call this set of agents the coalition C.

If each and everyone of its members agrees to do so, coalition C forms a voting bloc. In this case coalition C will hold an internal meeting to predetermine its voting behavior in the general vote. In the internal meeting, all members of the coalition will vote *yes* or *no* according to their preferences for or against a. Then:

1. If the majority in this internal vote has strictly more than $t(N+1)$ votes, where $t \in [\frac{1}{2}, \frac{N}{N+1}]$, then the majority prevails and all members of coalition C will vote as a bloc in the general election casting $N+1$ votes according to the preferences (either *yes* or *no*) of the majority of the coalition. The outcome of the coalitional internal meeting is binding and the $N+1$ agents cannot fail to act according to this collectively decided voting behavior.

2. If the majority gathers no more than $t(N+1)$ votes in the internal vote, then the coalition fails to act as a bloc in the general election and each member is free to vote according to individual preferences.

Note that threshold t defines the rule used by the coalition to decide whether or not it will act as a bloc rolling its internal minorities. A threshold $t \in [\frac{1}{2}, \frac{1}{2} + \frac{1}{N+1})$ corresponds to simple majority, $t \in [\frac{1}{2} + \frac{1}{N+1}, \frac{N}{N+1})$ to a supermajority and $t = \frac{N}{N+1}$ to unanimity.

Forming a voting bloc with unanimity as internal voting rule is in essence identical to not forming a voting bloc, for the coalition will only cast its votes as a bloc if all its members share the same preference, in which case votes will be cast as they would in the absence of a voting bloc.

If the coalition does not form a voting bloc, then all the member of the coalition will vote according to their individual preferences in the general election.

Coalition members decide to form or not a voting bloc with a $t - majority$ internal voting rule before the alternative a is specified, so agents do not know if they will prefer alternative a or the status quo. Agents have no power over the specification of alternative a, which comes exogenously.

Every agent i has a type p_i, which is the probability that agent i will prefer alternative a to the status quo, once alternative a is revealed. This type can be interpreted as a propensity for change, or as a displeasure with the status quo in general. Types are common knowledge, and so are true preferences once agents learn what the alternative a is.

Each realization of preferences is independent from the others, so once alternative a is revealed, each of possibly many agents with a type p_o has an independent probability p_o of supporting alternative a and typically several of them will end up supporting a, whereas some others will prefer the status quo.

If the coalition forms a voting bloc, in the internal vote voting will be sincere and there will be no abstention. With simple majority as internal decision making rule, if the number of *yes* votes surpasses the number of negative ones, then the whole coalition (now a voting bloc) will cast a total of $(N+1)$ *yes* votes in the general vote which includes all agents in the society. If the number of *no* votes surpasses the number of favorable ones, then the coalition accordingly votes as a bloc casting $(N+1)$ *no* votes in the general vote.

The voting bloc behavior we have described consists on rolling internal minorities to present a common front in the general vote, strengthening the position of the coalition's majority with the minority votes which are "converted" or "swayed" to the majoritarian camp, increasing the chances of eventually getting the outcome the majority wishes (of course, in doing so the probability of getting what the minority wishes decreases).

Proposition 1 *Let type $p_i \in (0,1)$ for each agent i in the society. Then, for any $N \geq 2$ (number of agents in the coalition), $M \geq 2$ (number of agents not in the coalition), and T (threshold to accept alternative a), a coalition of $N+1$ members strictly increases the aggregated expected utility of its members by forming a voting bloc with either simple majority or any supermajority as internal voting rule. Simple majority rule is the internal voting rule that maximizes the aggregated expected utility of the members of the voting bloc.*

The proof is straightforward. We offer a sketch here and details in the Appendix.

Forming a voting bloc only has an effect in utilities if the formation of a bloc and the subsequent rolling of minority votes within the coalition alters the outcome of the general vote. If so, every member of the coalition who is in the coalitional majority benefits from the voting bloc formation, at the cost of every voter in the minority. Since the majority is by definition bigger than the minority, there are more members benefiting than suffering from the bloc, and since the intensity of preferences is set to be equal for every member, in the aggregate forming a voting bloc generates a surplus of utility for the coalition. Any other rule that in some cases fails to roll a minority is giving away this net gain in utility and therefore underperforms in comparison to simple majority in terms of aggregated gains in utility.

It follows from Proposition 1 that if all the members of the coalition share a common type, then forming a voting bloc increases the utility of every member in the coalition and therefore an homogeneous coalition of agents who have a same type should always form a voting bloc with simple majority as internal voting rule to maximize their probability of winning the final vote in a larger electorate. Also from Proposition 1, we derive the following Corollary

Corollary 1 *If all but one of the members of the coalition share a common type, all the homogeneous members benefit from the formation of a voting bloc.*

Proof. Let $p_j = p_h \; \forall j, h \in C \backslash i$ and let $p_i \neq p_j$. Then if member i is in the rolled minority, more of the homogeneous members are in the majority benefiting from the rolling of votes the voting bloc imposes than in the hurt minority, thus in the aggregate the homogeneous members strictly benefit from the bloc.

If member i is in the majority, there are at least the same number of homogeneous members in the majority as in the minority, thus in the aggregate the bloc is at worst neutral to the homogeneous members. Since both cases are possible, overall there is a surplus for the homogeneous members (maybe not so for the heterogeneous one). ■

In Section 3 and 4 we will show under which conditions will every member of an heterogeneous coalition benefit from the formation of a voting bloc. We now ask whether the formation of a voting bloc benefits or harms the interests of the agents who are not part of the coalition. The answer will depend on the voting rule in the general election:

If the rule in the general election is unanimity, then each agent has a veto power over changes to the status quo. If a coalition forms a voting bloc, it removes the veto power to its members, but not to non-members, who therefore benefit from the formation of a voting bloc by the coalition. If the rule in the general election is simple majority, a voting bloc will only change the outcome to make a minority the winner. That is contrary to the interests of a majority of non-members of the coalition.

Proposition 2 *Let type $p_i \in (0,1)$ for each agent i in the society and let coalition C form a voting bloc with any internal rule other than unanimity. Then, if the voting rule in the general election is unanimity, every agent in the society strictly benefits from the formation of the voting bloc. If the general voting rule is simple majority, there is an aggregated loss in expected utility for the M agents not in the coalition.*

Proof. Let j be an agent not in the coalition. If $T = M + N + 1$, then if j or any other agent not in the coalition does not support alternative a, it has the power to veto it by voting "*no*". So the formation of a bloc has no impact whenever j or any other agent not in the coalition opposes alternative a.

If all M agents, including j, support alternative a, then without a bloc it would be necessary for each and every member of the coalition to support it as well in order for a to pass the vote. Should a bloc be formed, it is enough that a sufficient majority within the coalition supports a, for then the minority of negative votes will be rolled and a would be implemented. With types $p_i \in (0,1)$ and any rule other than unanimity as internal voting rule for the coalition, there is a positive probability of rolling a minority of negative votes in the voting bloc. This probability increases the utility of the agents not in the coalition.

If $T = \frac{M+N}{2} + 1$, then the aggregated utility for the whole society is maximized without a voting bloc, because the maximal social welfare is achieved by always implementing the wishes of the majority. Since a voting bloc generates extra utility for the members of the coalition whilst it reduces total social welfare by making minorities win, it must then be the case that non-members are harmed by the voting bloc. ■

Even if the formation of a voting bloc is in the aggregate hurting non-members of the coalition, this effect will in general not be uniform: Some agents not in the coalition will win, some lose expected utility if the coalition forms a voting bloc. For instance, suppose that the members of the coalition have types such that almost always the *yes* wins in the coalitional internal vote and a small but significant *no* minority is rolled. Then the bloc behavior by the coalition tilts the general vote in favor of alternative a. Agents with a high type, who are likely to prefer alternative a, will be then happy to see the coalition form a voting bloc. Of course, the behavior of the voting bloc hurts in our example those with a lower type.

Recapitulating what we have learnt in this section: Forming a voting bloc is inconsequential if the coalition uses unanimity as internal voting rule, but with any other rule, forming a bloc gives a surplus in utility to the coalition and simple majority is the internal voting rule that maximizes such surplus. Every agent in the rest of the society benefits from the formation of a bloc by the coalition if the general voting rule is unanimity, but if the general voting rule is simple majority these agents suffer an aggregate loss in utility, though some of them may still benefit from the formation of a voting bloc by the coalition.

In the next section we investigate under what conditions the coalition can reach unanimous agreement among its members to proceed with the formation of a voting bloc and appropriate the surplus in utility that comes with the voting bloc.

3 Achieving consensus to form a Voting Bloc:

We wish to find an internal rule for the coalition to aggregate preferences in such a way that maximizes the aggregated utility of its members, whilst not hurting individually any of them relative to a default in which the coalition uses unanimity as internal voting rule and all members always vote according to their true preferences in the general election.

Throughout this section we assume that the general election rule is simple majority.

We assume that each of the members of the coalition can block the formation of a voting bloc, so we need to find a rule to aggregate internal preferences such that every member prefers this rule better than unanimity.

Given an internal voting rule v, we say that it **Pareto-dominates** unanimity **for C** if every member in C is weakly better off using v rather than unanimity as internal voting rule and some member in C is strictly better off.

The purpose of this section is to find out which majority rules Pareto-dominate unanimity for C. If there are several majority rules that Pareto-dominate unanimity for C, we focus on whichever one maximizes the overall surplus for the coalition. We recall from Proposition 1 that the internal voting rule that maximizes the aggregated utility for the coalition is simple majority. However, in an heterogeneous coalition, some members may not benefit from pooling votes in a voting bloc with simple majority.

In this section, we compare the default rule of unanimity with $t-majority$ rules.

We label unanimity rule as $\varnothing$ as a reminder that using unanimity as internal voting rule is identical to not forming a voting bloc, or no member joining the voting bloc. It is the default rule in the absence of agreement, thus it is the benchmark we use as comparison with other rules.

We label as t a $t-majority$ rule in which every member of the coalition participates in the voting bloc, and minorities of size strictly less than $t(N+1)$ are rolled to join the position of the majority of the coalition in the general vote. Simple majority, denoted Sm, refers to the special case in which $t=\frac{1}{2}$.

Let A and B be two internal voting rules for coalition C. For $i \in C$, let p_{-i} be a vector with the types of all agents in the society except for i. Let us then define two functions, which depend on the rules A and B, and the vector of types p_{-i}:

$\alpha_i(B, A, p_{-i})$ = Probability that, given that member i prefers *yes,* the outcome in the general election is *yes* if the coalition uses rule B and *no* if it uses rule A.

$\beta_i(B, A, p_{-i})$ = Probability that, given that member i prefers *no,* the outcome in the general election is *no* if the coalition uses rule B and *yes* if it uses rule A.

Given two rules A and B, a profile of types for all the agents in the society and a member i, we can partition the set of realizations of preferences for or against alternative a in just three events:

E_o contains all the realizations of preferences for which the outcome in the general election is the same regardless of whether the coalition uses A or B as internal voting rule.

E_1 contains all the realizations of preferences for which the outcome coincides with the preference of member i if the coalition uses rule B but not if it uses rule A.

E_2 contains all the realizations of preferences for which the outcome coincides with the preference of member i if the coalition uses rule A but not if if uses rule B.

Then member i prefers to use rule B if and only if the event E_1 occurs with higher probability than the event E_2. The event E_1 occurs with probability $p_i * \alpha_i(B, A, p_{-i}) + (1 - p_i)\beta_i(B, A, p_{-i})$, the event E_2 occurs with probability $p_i * \alpha_i(A, B, p_{-i}) + (1 - p_i)\beta_i(A, B, p_{-i})$.

The following lemma completes this intuition, relating preferences over rules to individual types:

Lemma 1 *Given two internal voting rules B and A for coalition C, a member i of the coalition C is indifferent between them if:*
$p_i = \frac{\beta_i(A,B,p_{-i})-\beta_i(B,A,p_{-i})}{\alpha_i(B,A,p_{-i})-\alpha_i(A,B,p_{-i})+\beta_i(A,B,p_{-i})-\beta_i(B,A,p_{-i})}$.

Proof. Let $EU_i(A)$ denote the utility for agent i of forming a voting bloc with internal rule A when simple majority is the rule used in the general election and let $EU_i(B)$ be analogously defined. Then $EU_i(B) - EU_i(A) = p_i[\alpha_i(B, A, p_{-i}) - \alpha_i(A, B, p_{-i})] + (1 - p_i)[\beta_i(B, A, p_{-i}) - \beta_i(A, B, p_{-i})]$.

Equating to zero and solving for p_i we get

$$p_i = \frac{\beta_i(A, B, p_{-i}) - \beta_i(B, A, p_{-i})}{\alpha_i(B, A, p_{-i}) - \alpha_i(A, B, p_{-i}) + \beta_i(A, B, p_{-i}) - \beta_i(B, A, p_{-i})}.$$

■

Before presenting our results, we need to make some assumptions on the types of the agents:

Assumption 1 *The number of favorable votes cast by the M agents not in coalition C follows a symmetric distribution around $\frac{M}{2}$ with some positive probability of casting a quantity of favorable votes different than $\frac{M}{2}$.*

A sufficient condition for this assumption to be true is that all M agents can be paired in such way that for each pair (j, j'), $p_j + p_{j'} = 1$, and at least two agents have a type strictly between zero and one. We let $f(x)$ denote the probability that the M members cast exactly x favorable votes for alternative a, and we let $F(x) = \sum_{k=0}^{x} f(x)$ be the distribution function of the number of favorable votes cast by the M agents not in the coalition.

We make a milder assumption on the types of the members of coalition C, namely, we assume that coalition C "leans towards" accepting alternative a. Let $g_{i,j}(x)$ denote the probability that x members of $C \backslash \{i, j\}$, of the coalition without i or j, prefer alternative a. Then we require the following:

Assumption 2 *For all* $k \in [0, \frac{N}{2} - 1]$ *and for all* $i, j \in C$, $g_{ij}(\frac{N}{2} + k) > g_{ij}(\frac{N}{2} - k - 1)$.

Note that $g_{lh}(k) = \sum\limits_{\substack{A \subseteq C \backslash \{l,h\} \\ |A| = k}} [\prod\limits_{\substack{i \in A \\ j \in C \backslash (A \cup \{l,h\})}} p_i(1 - p_j)]$.

Assumption 2 states that given any $N - 1$ members of the coalition and given any particular majority-minority split of votes in this subset of the coalition, it is more probable that this majority in the subset is for the *yes* side. A sufficient condition for our assumption to hold is that excluding any three members, we can pair the rest in such a way that for each pair (i, i'), $p_i + p_{i'} \geq 1$ and at least one pair is different from $(0.5, 0.5)$.

Let $g_i(x)$ denote the probability that exactly x members of $C \backslash i$, of the coalition without i, prefer alternative a. Formally $g_l(k) = \sum\limits_{\substack{A \subseteq C \backslash l \\ |A| = k}} [\prod\limits_{\substack{i \in A \\ j \in C \backslash (A \cup l)}} p_i(1 - p_j)]$. From Assumption 2 it follows that for all $k \in [1, \frac{N}{2}]$ and for all $i \in C$, $g_i(\frac{N}{2} + k) > g_i(\frac{N}{2} - k)$. We show this in the Appendix

With these two assumptions on the types of the agents and simple majority as voting rule in the general election, we find that a member of the coalition will like to form a voting bloc with a $t - majority$ rule as internal voting rule if her type is "high enough": If a given member would benefit from forming a voting bloc with a $t - majority$, then every other member with a higher type would benefit even further:

Lemma 2 *Let* $l, h \in C$ *such that* $p_h \geq p_l$. *Then* $EU_h[t] - EU_h[\varnothing] \geq EU_l[t] - EU_l[\varnothing]$.

The proof is in the Appendix.

We can then focus only on the member with the lowest type to see if she benefits from the formation of a voting bloc with a $t - majority$. If she does, then every member in the coalition benefits from forming a voting bloc with a $t - majority$ rule:

Proposition 3 *Let* $l \in C$ *be the member with the lowest type. Then a* $t - majority$ *rule Pareto-dominates unanimity as internal voting rule for coalition* C *if and only if* $p_l > p_l^{t,\varnothing}(p_{-l})$.

In the Appendix we find the exact expression of $p_l^{t,\varnothing}(p_{-l})$ as a function of the types of the agents and the threshold t and we proof the proposition.

Since the coalition "leans" towards accepting alternative a, the majority within the coalition will more often than not be in favor of alternative a, with the result that the negative votes will be rolled more often than the favorable ones, making it more likely that alternative a wins the general election. Member l only likes such voting behavior if her type is high "enough", where the exact meaning of "enough" is given by the threshold in the Proposition.

The threshold $p_l^{t,\varnothing}(p_{-l})$ converges to 1 as every type p_i, $i \in C\backslash l$ converges to 1. On the other hand as $g_l(\frac{N}{2}+k) - g_l(\frac{N}{2}-k)$ converges to zero (as the distribution of votes by the other members of the coalition converges to a symmetric one), the threshold $p_l^{t,\varnothing}(p_{-l})$ converges to $-\infty$. A threshold below zero indicates that member l supports the creation of a voting bloc regardless of her own type.

We illustrate Proposition 3 with the aid of Figure 1, for the specific case of simple majority as internal voting rule.

To be able to plot the threshold $p_l^{Sm,\varnothing}(p_{-l})$ with respect to only one variable, we assume that the distribution of votes by the agents not in coalition C follows a binomial $Bi(M, \frac{1}{2})$ and that all the members of coalition C except l share a common type r. A single parameter r is sufficient (captures all the relevant information about the types of the rest of members of C) to determine if member l would benefit from the formation of a voting bloc: For any heterogeneous coalition in which all other members but l did not share a common type, that coalition is mapped to one particular value of r such that l evaluates coalition C as if all the other members had a common type r. Therefore, Figure 1 indirectly captures all possible coalitions of size N. In Figure 1 we set $M = 176$ and $N = 24$ to approximate our European Union example. We will use these values in most of our figures.

Our model corresponds to the right half the graph: If the common type r of the N members other than l is bigger than one half, member l will support the formation of a voting bloc with simple majority if the type p_l is above the depicted threshold. The left half of the picture is a symmetric case in which the coalition leans towards rejecting a. Then member l will only support the formation of a bloc if her type is below the threshold.

The following proposition tells us that some coalitions that can't form a voting bloc with simple majority can form a voting bloc with some supermajority rule in such a way that every member's utility increases.

The threshold function $p_l^{t,\varnothing}(p_{-l})$ is decreasing in t. The higher the $t-majority$ rule used as internal voting rule, the more type profiles for which the $t-majority$ rule Pareto-dominates unanimity for coalition C:

Proposition 4 *Let $t' = t + \frac{1}{N+1}$. Then for any $t \in [\frac{1}{2}, \frac{N-1}{N+1})$, the subset of type profiles for which a $t-majority$ rule Pareto-dominates unanimity for C is strictly contained in the subset of types for which a $t'-majority$ rule Pareto-dominates unanimity for C.*

The proof is in the Appendix.

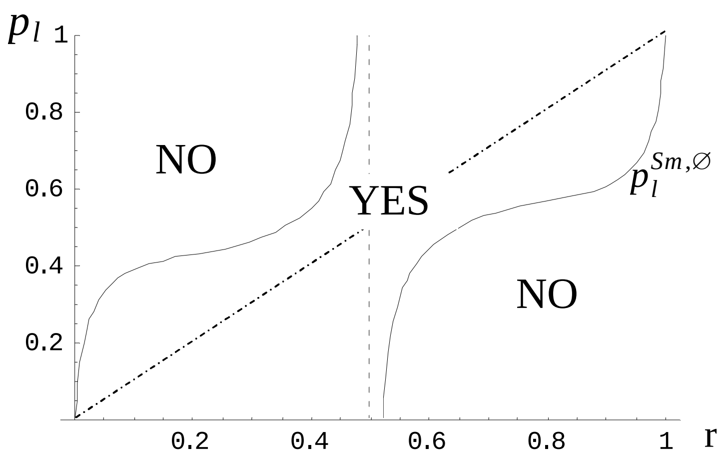

Figure 1: Consensus to form a voting bloc with simple majority

The more stringent the supermajority rule the coalition uses, the lower that the type of member l can be and yet allow l to benefit from the formation of a voting bloc with the $t-majority$ internal voting rule.

We depict this result in Figure 2 for $N = 24$, $M = 176$, F is a binomial $Bi(M, \frac{1}{2})$ and every member of C except for l has a type $r > \frac{1}{2}$.

Figure 2 presents four different possible rules a coalition the size of the EU: Simple majority, two-thirds majority, four-fifths majority, and nine-tenths majority.

We see how the range of parameters for which a voting bloc would benefit every member increases as the supermajority rule becomes more stringent. However, since simple majority maximizes the overall surplus for the coalition, setting higher thresholds for approval of a common position diminishes the value of the voting bloc, though it may help to bring an outlier on board.

Aiming to maximize the utility of the coalition subject to not hurting any member, the idea would be to find the lowest possible supermajority threshold that would benefit (or at least leave indifferent) the member with the lowest type. A consequence of Proposition 4 is that for any $t' \in [\frac{1}{2}, \frac{N-1}{N+1})$ there exists a profile of types such that t' maximizes the surplus for the coalition among the class of $t-majority$ rules which Pareto-dominate unanimity.

In the remainder of this section we investigate how changes in the size of the coalition or the heterogeneity of types of its members affect which rules the coalition will be able to use to the benefit of all its members.

We find that if the size of the coalition is too large, then no coalition in which all members but l share a common type $r > p_l$ can form a voting bloc with simple majority.

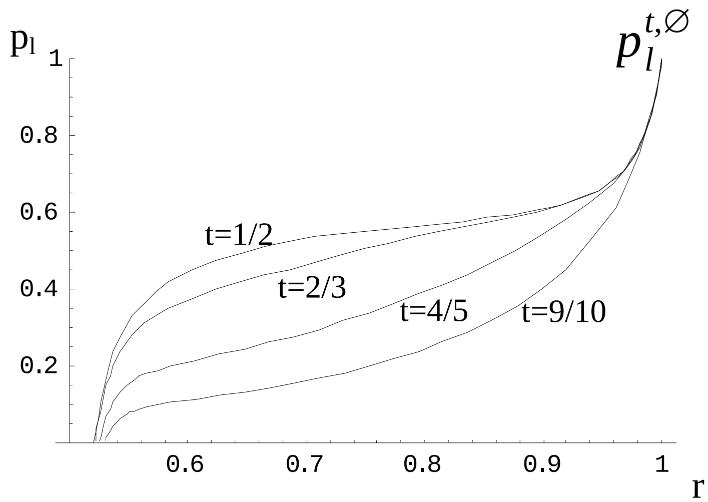

Figure 2: Supermajority rules

Proposition 5 *Let M be fixed. Let $p_l < r$ for $l \in C$ and $p_i = r$ for all $i \in C \backslash \{l\}$. There exists some $\overline{N}$ such that if $N > \overline{N}$, simple majority does not Pareto dominate unanimity for C.*

We proof this in the Appendix.

As the coalition becomes very large relative to M, the internal majority coincides with the external majority unless the coalition is almost evenly split. The coalition is more likely to vote for a than against a. Since member l is the member with the lowest type, conditional on the coalition being evenly split, member l is more likely to be against a, thus in the losing side. Therefore, if the coalition becomes so large that rolling its votes only affects the outcome when the coalition is almost evenly split, the member with the lowest type rejects the formation of a bloc with simple majority and in the limit only a fully homogeneous coalition where every member has the same type could form a voting bloc with simple majority.

This finding runs contrary to the perceived experience in the EU, where successive enlargements have put pressure to lower the supermajority thresholds. The European Union traditionally used very stringent supermajority requirements, often unanimity: In a larger Union, the inadequacy of such rules becomes obvious and the EU now faces a pressure to lower the degree of consensus necessary to take a common decision. Had the Union been using simple majority, it should have experienced a pressure to rise, not lower the supermajority requirement as the Union got larger.

Figure 3 shows the convergence of the threshold $p_l^{Sm,\varnothing}(r, N)$ to r as N gets large, given that F is a binomial $Bi(M, \frac{1}{2})$, $M = 60$ and $p_i = r\ \forall i \in C \backslash l$. The

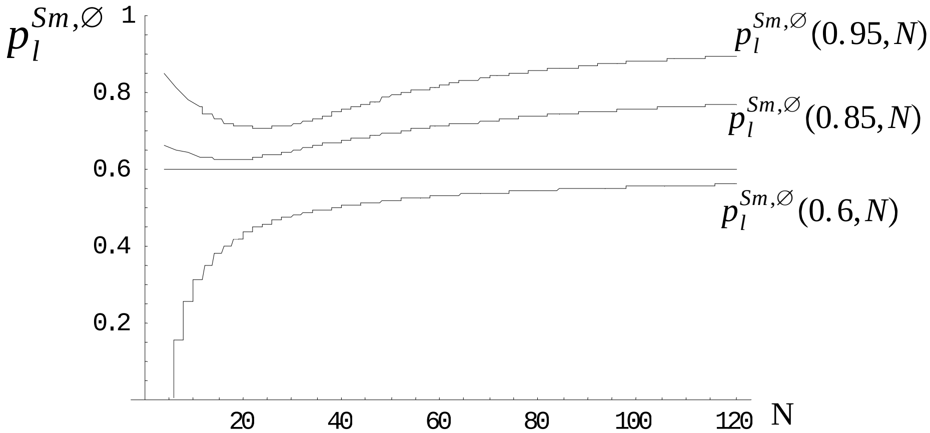

Figure 3: Convergence of $p_l^{Sm,\varnothing}(r, N)$ to r in a dichotomous coalition.

three plots (from bottom to top) correspond to a common value $r = 0.6$ (we show the convergence asymptote as well), a common value 0.85 and a common value 0.95 for the N members not l in the coalition.

Beyond size, we ask how does heterogeneity affect the chances of a coalition to form a voting bloc. We know that an homogeneous coalition will always find simple majority to Pareto dominate unanimity for the coalition, whereas heterogeneous coalitions may run into obstacles. Nevertheless, we show by means of an example that the possibility of forming a voting bloc with simple majority is not monotonic with heterogeneity:

We compare three coalitions with the same mean type and the same lowest type. We measure heterogeneity by the standard deviation of types. We find that the most homogeneous and the least homogeneous of the three cannot form a voting bloc with simple majority, whereas the intermediate one can.

Example 1 *Let all agents not in C have a type $p_m = 0.5$ and let there be 10 of them.*

Let C_1 be a coalition of agents with types {0.445, 0.75, 0.75, 0.75, 0.75}. The mean type is 0.689. The standard deviation 0.1368. If $C = C_1$, then $p_l^{Sm,\varnothing}(p_{-l}) = 0.4943$ and l rejects the formation of a voting bloc with simple majority as internal voting rule.

Let C_2 be another coalition of agents with types {0.445, 0.5, 0.5, 1, 1}, mean type 0.689, standard deviation 0.2847. If $C = C_2$, then $p_l^{Sm,\varnothing}(p_{-l}) = 0.441$ so forming a voting bloc with simple majority benefits every member of the coalition.

Let C_3 be yet another coalition of agents with types {0.445, 0.45, 0.55, 1, 1}, mean type 0.689, standard deviation 0.2869. If $C = C_3$, then $p_l^{Sm,\varnothing}(p_{-l})$ $= 0.446$ so once again member l vetoes the formation of a voting bloc with simple *majority.*

For coalitions C_1 and C_3 in Example 1, using a two-thirds majority or a three-quarters majority (or any other value of t that requires a majority of 4 to 1 to roll the minority) every member benefits from forming a voting bloc: In C_1, using $t = 3/4$, $p_l^{3/4,\varnothing}(p_{-l}) = 0.39$ so member l favors the formation of a voting bloc that rolls only minorities of size one. Similar results hold for coalition C_3.

We quantify the impact that the formation of a voting bloc by coalition C_1 in Example 1 would have over the outcome in the general election. We show the results in Table 1. The numbers indicate the probability that the event indicated in each row occurs, given the internal rule the coalition uses. In the first column, the coalition uses unanimity or forms no bloc, in the second column it forms a bloc with simple majority and in the third column it uses a 3/4 majority.

TABLE 1	No bloc	1/2 maj	3/4 maj
a approved	69.52%	79.58%	73.45%
a approved given l likes a	79.82%	90.00%	84.72%
a approved given l dislikes a	61.26%	71.22%	64.42%
l satisfied with outcome	57.02%	56.02%	57.44%
$j \in C \backslash l$ satisfied with outcome	67.26%	74.70%	70.57%
$m \notin C$ satisfied with outcome	59.44%	53.51%	57.61%

Since the coalition leans towards a, forming a voting bloc makes approval of a more likely. All the members except for l benefit and all non-members are hurt forming a voting bloc. Member l is hurt if simple majority is used, so in order to benefit all its members, the coalition has to select a supermajority that makes l better off, but attenuates the advantage for all the other members. We see that forming a voting bloc with simple majority would have a substantial impact: The probability of approving a increases ten percentage points, the probability of getting the desired outcome out of the election would increase seven percentage points for all members of coalition C_1 but l. Using a 3/4 majority reduces this benefit of a voting bloc to roughly a half, but it makes all members of C_1 more likely to see their preference prevail in the general election.

In this section we have described the necessary and sufficient condition for a coalition to be able to form a voting bloc with a majority rule. We show that although simple majority maximizes the aggregate surplus, there are type profiles for which simple majority does not Pareto dominate unanimity but some supermajority rules do and the coalition can choose one of them to gain some of the surplus of a voting bloc benefiting all its members.

4 An Opt-Out rule:

In this section we explore a more nuanced rule, which consists on forming a voting bloc with all but one of the members of the coalition: The excluded member does not participate in the internal vote of the voting bloc, but votes directly and according to her true preferences in the general election.

This scheme differs from expelling one member from the coalition in a crucial detail: The exclusion is voluntary, the member who does not participate in the voting bloc agrees to the formation of the voting bloc without her, hence it opts to be out, or "opts-out". The member who opts out has to benefit from the formation of the voting bloc by the other members, otherwise she would rather veto the whole project and keep unanimity in place as the voting rule to aggregate votes in the coalition.

Famous opt-outs in the European Union include the UK and Denmark with regards to the European Monetary Union: Their approval to the Maastricht Treaty was necessary for the monetary union to bring about the euro, and they supported the implementation of the treaty, whilst staying out of the project. If they had deemed it harmful to their interests, they could have refused to sign it.

We denote by *Out* the "Opt-Out for l" rule in which the member l with the lowest type does not participate in the voting bloc, which is formed by every other member of the coalition and simple majority is chosen as internal voting rule.

We could extend the results in this section to consider opting-out rules in which l stayed out and the participating members chose a supermajority as internal voting rule, but for simplicity we focus on the rule that will maximize the surplus for the members who participate in the bloc given that l will not join them.

Throughout the section we assume that the general election rule is simple majority.

Proposition 6 *The formation of a voting bloc with simple majority rule by every member of coalition C except l, benefits member l if and only if $p_l > p_l^{Out,\varnothing}(p_{-l})$.*

We provide the expression of $p_l^{Out,\varnothing}(p_{-l})$ and a proof in the Appendix.

The threshold function $p_l^{Out,\varnothing}(p_{-l})$ is always positive given our assumptions on types, it is not increasing with respect to the type of all other members of the coalition and it does not always converge to one as the types of the members of the coalition do. This last feature guarantees that in some cases in which member l rejects forming a voting bloc with simple majority she benefits from the formation of a bloc with an "Opt-Out for l" rule. If all the other members also benefit from the "Opt-Out for l" rule, then this rule Pareto-dominates unanimity and it offers a solution for a coalition which couldn't form a bloc with $t - majority$ rules. The next Proposition states this result:

Proposition 7 *If $4 \leq N \leq M$, there exist type profiles for which an "Opt-Out for l" internal voting rule Pareto-dominates unanimity for C and no $t-majority$ rule does. If $N = 2$ or $N > M$, there exists no type profile for which "Opt-Out for l" Pareto-dominates unanimity for C and simple majority doesn't.*

The Appendix contains a proof of Proposition 7. Here we sketch the intuition: For the negative statement, if $N = 2$, allowing one member to step-out

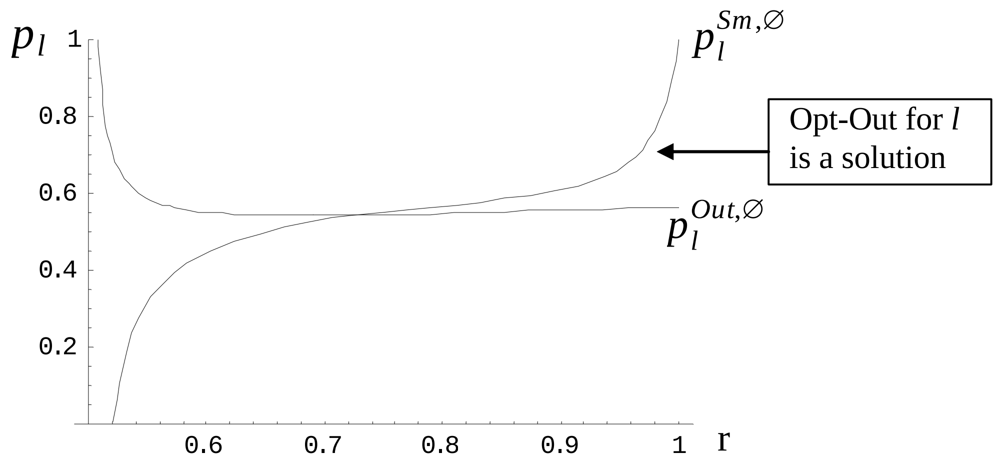

Figure 4: An Opt-Out rule as a solution

reduces the bloc to size two, which is identical to not forming a bloc at all, or forming it with unanimity. If $N > M$, then the coalition acts as a dictator even if one member opts-out, thus the member who opts-out cannot be better off out of the voting bloc than in the voting bloc.

We use Figure 4 to gain some insight about the threshold $p_l^{Out,\varnothing}(p_{-l})$ and its comparison with $p_l^{Sm,\varnothing}(p_{-l})$. As in all figures, we assume that the distribution of votes by the agents not in coalition C follows a binomial $Bi(M, \frac{1}{2})$ and that all the members of coalition C except l share a common type r. We set $M = 176$ and $N = 24$.

Looking at Figure 4, we notice that if types are in the area below the threshold $p_l^{Sm,\varnothing}(p_{-l})$ and above $p_l^{Out,\varnothing}(p_{-l})$, l would veto forming a voting bloc with simple majority if it included all the members, but allowing l to stay out, the coalition can form a voting bloc with simple majority with every other member and in expectation raise the utility of every member including l.

This result casts a favorable light over "opt-out" rules. On the other hand, "opt-out" rules have two setbacks:

If the coalition is heterogeneous (and not just dichotomous with all members but l sharing a common type) and the member with the lowest type opts out, it is possible that the member with the second lowest type opposes the formation of the reduced bloc in which he is now the member with the lowest type. Allowing the member with the second lowest type to opt-out as well may lead the member with the third lowest type to oppose the formation of a bloc, and so on until the voting bloc fully unravels and every member but two opt-out, which negates the purpose of a voting bloc.

Even if this unravelling does not take place, there is a second latent complication to opt-out rules: If the coalition allows for the member with the lowest type to opt-out, then other members may also request to opt-out, even if they benefit from the formation of a voting bloc, simply because they would benefit even further by opting-out. If the coalition lets every member join in or stay out of the voting bloc, then it faces a "free-rider" problem, where some members who would benefit from joining the voting bloc, may prefer to opt-out and passively take advantage of the pooling of votes by other coalition partners. We exclude from our concept of free-riding situations in which a member opts-out of a voting bloc and benefits from the pooling of votes by the other members if the member who opts-out would be hurt by a voting bloc that included him:

Definition 1 *Member l "free-rides" if she would have benefitted from forming and participating in a voting bloc, but benefits even more as a result of opting-out.*

With no opt-out rules, there is no chance to free-ride, since the coalition faces an "all-or-none" binary decision: Either every member joins the voting bloc, or the bloc is not formed. If instead members can individually choose whether to join in or to stay out, some may not choose to join in. In the following Proposition we explore whether a member would prefer to participate in or to stay out of a voting bloc formed by every other member of the coalition.

Proposition 8 *Member l prefers to participate in a voting bloc formed by the coalition with simple majority as internal decision rule better than to opt-out and not participate in the pooling of votes by the rest of the members of the coalition if and only if $p_l > p_l^{Sm,Out}(p_{-l})$.*

We provide the exact expression of $p_l^{Sm,Out}(p_{-l})$ and a proof in the Appendix. We could extend this result to consider opt-out rules where the members that form the voting bloc use a supermajority instead of simple majority.

From Proposition 3 we obtain the condition for l to benefit from forming and participating in a voting bloc with simple majority. Proposition 8 now states when will member l prefer to opt-out from such a bloc. Combining Propositions 3 and 8 we obtain Proposition 9. It outlines the downside of Opt-Out rules: The creation of a free-riding problem when member l would benefit from participating in the bloc but prefers to opt-out.

Proposition 9 *An "Opt-Out for l" rule creates a free-rider problem if and only if $p_l^{Sm,\varnothing}(p_{-l}) < p_l < p_l^{Sm,Out}(p_{-l})$. If $4 \leq N \leq M$, there exist type profiles for which this condition is met. If $N = 2$ or $N > M$, this condition cannot hold and free-riding cannot occur.*

A proof is in the Appendix.

If $p_l < p_l^{Sm,Out}(p_{-l})$, member l prefers not to participate in the voting bloc and she free-rides if participating would be better for l than not forming a voting bloc.

When member l compares the utility of being in the bloc, or out of the bloc, she compares the probability of affecting the general outcome to make it

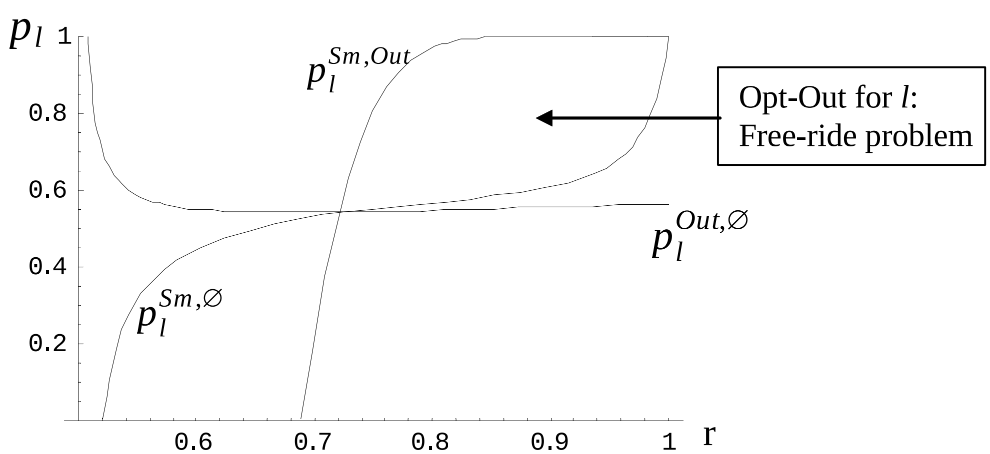

Figure 5: Opt-out and free-ride

coincide with her wishes when she votes directly in the general election, to the same probability voting through the voting bloc.

This analysis of the utility of being in or being out of a voting bloc bears some resemblance, but is not equivalent, to the comparison of probabilities of being decisive (of being pivotal in the final outcome) that occupy the voting power literature. In a nutshell, in the voting power literature the agents seek to maximize their probability of being able to alter the outcome, whereas in our model they only care to alter the outcome towards their preference:

If an agent can change the outcome against his preference by joining a voting bloc and casting a vote against his preference, whereas if he stays out of the bloc his vote is irrelevant, then the agent in our model is indifferent between being out of the bloc and being irrelevant but getting the desired outcome, or being in the bloc and being crucial to obtain the desired outcome. For a rigorous study of the relation and differences between voting power and probability of success or satisfaction (the approach we take), we recommend Laruelle and Valenciano [10].

Let us visualize when will a member prefer to opt-out and free-ride on her coalition partners with the aid of Figure 5, where again $N = 24$, $M = 176$, the 24 members other than l in coalition C share a common type r and the number of favorable votes by agents not in the coalition follows a binomial $Bi(M, \frac{1}{2})$.

If type p_l is above $p_l^{Sm,\varnothing}(p_{-l})$ but below $p_l^{Sm,Out}(p_{-l})$, member l would have supported a voting bloc with simple majority and no opt-outs better than nothing, but she prefers to opt-out if she can. If she opts-out, the overall utility for the coalition is reduced.

If not only l but any member can opt-out, then coalitions of size less than M face an even worse problem: For some configurations of types all members would prefer to opt-out. If $N > M$, then the coalition forming a voting bloc is a dictator and thus votes of agents not in the voting bloc don't count at all and no member can gain anything opting out.

As a summary, allowing a member to opt-out can be a good solution in a coalition with great homogeneity of types and one outlier, but in many other instances it can generate free-rider problems, aggravated if the possibility to opt-out is extended to every member.

5 Conclusions and Extensions:

A coalition of agents who are part of a larger electorate facing a vote may choose to form a voting bloc, thus deciding according to some internal voting rule which way the whole coalition will vote in the general election, after rolling the internal minorities.

We've shown that forming a voting bloc generates a surplus in the aggregate utility of the members of the coalition and we've checked that simple majority is the internal rule for the voting bloc that maximizes such surplus.

However, if there is heterogeneity among the members of the coalition, the surplus will not be evenly shared, and in the absence of transfers, the formation of a voting bloc may be detrimental to some members of the coalition. Ordering members from "least likely" to "most likely" to support changes to the status quo, we find a single cutting point or threshold separating those members of the coalition who support the formation of a voting bloc, and those who reject it. This implies that either every agent in the coalition supports the voting bloc, or at most agents at one tail of the distribution of types reject it, but it will never be the case that extremists from both tails reject the formation of a voting bloc.

We find it interesting to compare this result with a different model of federalism, by Crémer and Palfrey [5] and [6]. Crémer and Palfrey argue that moderate voters, with preferences closer to the median of the Union, will advocate federalism and unified policies. Complementing their work, our paper provides a rationale for at least one set of extreme voters to wish for a common foreign policy: If the binary choice is between the status quo and "change" and forming a voting bloc the aggregate vote for the whole coalition will most likely be "change", then voters within each country who want "change" see a better chance of getting it through a unified federal government.

Under the motivation that each member of the coalition may have a veto over the formation of any kind of voting bloc, we analyze possible solutions to achieve unanimous support by all members of the coalition to proceed with some form of voting bloc when some member of the coalition opposes the solution which maximizes the total surplus of utility: A voting bloc formed by the whole coalition with simple majority as internal decision rule.

We find that, for some range of parameters, allowing an extreme agent who is opposed to the formation of a voting bloc to opt-out and not participate in the bloc is sufficient to achieve unanimous support (including support by the member who chooses to opt-out) to the formation of a voting bloc by the rest of the coalition. Though it may look strange to require an agent that does not participate in the bloc to acquiesce to its formation, we think the relevant example of the EU (where each country has a veto power over changes on fundamental treaties and thus can stop an initiative regardless of whether it includes or excludes the vetoing country) provides enough motivation to find this result important.

Another solution that also reduces the overall surplus in utility but helps to achieve unanimous support for the formation of a voting bloc is the use of qualified majority rules (supermajorities) as internal voting rules. The higher the threshold of the qualified majority rule, the less likely that any agent would be hurt by the formation of a bloc. This result contrasts with the findings of Maggi and Morelli's paper [11] in which only simple majority or unanimity are ever found to be optimal.

The implications for the European Union are straight forward: Assuming that the preferences of Union members, albeit different, are not squarely opposed to each other, unanimity is not the best rule that the Union can use to aggregate the votes of its members: Every one of the 25 countries in the EU would be more likely to see its preference prevail at a UN Assembly vote if the Union first pre-determined how it will cast all its 25 votes according to an internal voting rule that rolled minorities within the EU. The wider range of foreign policy goes beyond the scope of this paper; we just focus on pooling votes in an scenario in which all members are symmetric: In terms of winning votes in international forums that grant one vote per country, the EU (or any other collection of countries with some similarity in their policy preferences) would do better by forging a common foreign policy that was not based on unanimity.

This model could be extended to incorporate decision costs of forming a voting bloc and of course that would make it harder to achieve unanimous support for the voting bloc. Alternatively, a more favorable setting for voting blocs could be envisioned by considering economies of scale, where the joint expression of will by a united coalition has more power in the general electorate than the individual sum of the votes of the coalition members. That is difficult to justify in terms of votes, but is much more reasonable if lobbying, exerting political pressure or otherwise influencing others are parts of the actions that come with voting in one or the other direction.

Another extension within the framework of the model would be to considerate correlation in the realization of the preferences of the members of the coalition, possibly through a correlation matrix. A way to partially incorporate correlation without adding too substantial complications would be to define a "leader of the coalition" and then let the type of every other agent be a two-dimensional vector, stating the probability of supporting the alternative if the leader does and if the leader doesn't. Then we could use the model we have presented here for each of the two cases taking them separately and aggregate them to obtain expected utilities.

A more ambitious extension that we think deserves future research consists on allowing several coalitions, and not just one to form voting blocs. Ideally, any subset of agents would be allowed to form a voting bloc and we would look for stable partitions of the space of agents into voting blocs, stability implying that no agent wishes to abandon the voting bloc that it belongs to. We find strong justification for our assumption of a single coalition considering the formation of a voting bloc in the case of international relations, where the 25 countries in the EU participate in a project that is, to a large extent, unique. However, another very natural scenario to look for voting blocs, in fact an even more appropriate one, is any legislature in which political parties may be formed. If we want to explain party formation, we need to allow for different parties to exist. Starting with a set of individual legislators, "parties" would be each one of the voting blocs that are formed.

Some models studying the incentives to party formation are grounded on a distributive politics setting, where parties help agents to "get a share of the pie", as in Baron [2]. Jackson and Moselle [9] attempt to model coalition and party formation with both distributive and ideological dimensions. Our approach would try to explain party formation solely on the grounds of enhanced probabilities of getting the desired ideological outcome.

We leave these and other developments for future research.

6 Appendix:

6.1 Proposition 1:

Proof. Let s denote the size of the minority in the coalitional internal vote, S the size of the majority in such vote, thus $s+S=N+1$, let $EU_i^T(t)$ denote the expected utility for agent $i \in C$ given that the whole coalition C forms a voting bloc with t as the internal proportion threshold above which the coalition will roll its minorities in the general election vote and given that T is the number of votes needed in the general election for alternative a to be implemented. Similarly let $EU_i^T(\varnothing)$ denote the expected utility for agent $i \in C$ if the coalition does not form a voting bloc and T is the threshold in the general election. Then:

$\sum\limits_{i \in C} EU_i(t) - \sum\limits_{i \in C} EU_i^T(\varnothing) = \sum\limits_{k=1}^{N/2} (N+1-2k) * \Pr[s=k] * \Pr[\textit{minority is rolled}] * \Pr[\textit{rolling k votes alters outcome}]$.

With either simple majority or supermajority rules short of unanimity, $\Pr[\textit{minority is rolled}]$ is equal to one if $s=1$. All the other terms in the expression are strictly positive for types $p_i \in (0,1)$, thus the aggregated expected surplus in utility for the coalition generated by the voting bloc is strictly positive.

Proposition 1 also notes that no other rule yields a higher surplus in utility (in probability of getting the desired option after the final vote) than simple majority. The optimality of simple majority rule as internal aggregation rule for a set of agents (in our case coalition C) for any common type p is mentioned in Rae [12] and proved in Taylor [13]. We check that we can extend this result to fit our model with an individual type for each agent:

Let Sm denote simple majority as internal voting rule for coalition C, and let vr denote any other internal voting rule for coalition C. Under Sm, all the votes of the coalition are always cast in favor of the position chosen by the majority of the coalition, thus no other rule can give more votes to the position favored by a majority of the coalition. Therefore, if for some voting behavior the outcome in the general election depends on whether the coalition uses vr or Sm, it must be that the outcome under Sm is the one favored by the majority, and under vr the one favored by the minority. Let $v \in \{0,1\}^{M+N+1}$ be a realization of the preferences of all members of the society, where 0 represents a *no* preference and 1 a *yes* preference. Let $\Pr(v)$ be the probability that the realization v occurs, according to the vector of types of all agents. Let $q(vr|v)$ denote the probability that given preferences v, the outcome in the general election depends on whether C uses rule vr or Sm as internal voting rule. Then:

$$\sum_{i\in C} EU_i(Sm) - \sum_{i\in C} EU_i^T(vr) = \sum_{v\in\{0,1\}^{M+N+1}} (S-s)\Pr(v)q(vr|v).$$

This term is always non-negative and it is strictly positive for any rule vr that with positive probability will bring about a different outcome than Sm. ■

6.2 Claim from Assumption 2:

Assumption 2: For all $k \in [0, \frac{N}{2}-1]$ and for all $i,j \in C$, $g_{ij}(\frac{N}{2}+k) > g_{ij}(\frac{N}{2}-k-1)$.

We want to show: For all $k \in [1, \frac{N}{2}]$ and for all $i \in C$, $g_i(\frac{N}{2}+k) > g_i(\frac{N}{2}-k)$.
Proof. $g_{ij}(x)$ is the distribution of a sum of $N-1$ independent Bernoulli trials, each trial taking the type of a member of $C\backslash\{i,j\}$ as probability of success. The sum of independent Bernoulli trials is a unimodal distribution, as shown by Darroch [7]. Therefore, $g_{ij}(\frac{N}{2}+k-1) > g_{ij}(\frac{N}{2}-k)$ implies $g_{ij}(\frac{N}{2}+k-1) > g_{ij}(\frac{N}{2}-k-1)$.

For any i,j, $g_i(\frac{N}{2}+k) - g_i(\frac{N}{2}-k)$ is equal to:

$$\begin{aligned}
& p_j g_{ij}(\frac{N}{2}+k-1) + (1-p_j)g_{ij}(\frac{N}{2}+k) - p_j g_{ij}(\frac{N}{2}-k-1) - (1-p_j)g_{ij}(\frac{N}{2}-k) \\
> \;& (2p_j-1)g_{ij}(\frac{N}{2}+k-1) - (2p_j-1)g_{ij}(\frac{N}{2}-k-1) \\
> \;& (2p_j-1)[g_{ij}(\frac{N}{2}+k-1) - g_{ij}(\frac{N}{2}-k-1)].
\end{aligned}$$

Since this is true for any $i,j,$ and since by Assumption 2 at least two members have a type over a half, then for any i this last expression is true for at least one j with $p_j > \frac{1}{2}$. Then $g_i(\frac{N}{2}+k) - g_i(\frac{N}{2}-k) > 0$. ■

6.3 Lemma 2:

We want to show that $EU_h[t] - EU_h[\varnothing] - (EU_l[t] - EU_l[\varnothing]) \geq 0$.
Proof. $EU_l[t] - EU_l[\varnothing] =$

$$p_l * \alpha_l(t,\varnothing,p_{-l}) + (1-p_l)*\beta_l(t,\varnothing,p_{-l}) - p_l*\alpha_l(\varnothing,t,p_{-l}) - (1-p_l)*\beta_l(\varnothing,t,p_{-l})$$

and similarly $EU_h[t] - EU_h[\varnothing] =$

$p_h{*}\alpha_h(t,\varnothing,p_{-h}){+}(1{-}p_h){*}\beta_h(t,\varnothing,p_{-h}){-}p_h{*}\alpha_h(\varnothing,t,p_{-h}){-}(1{-}p_h){*}\beta_h(\varnothing,t,p_{-h}).$

In Step 1 we show that

$p_h{*}\alpha_h(t,\varnothing,p_{-h}){+}(1{-}p_h){*}\beta_h(t,\varnothing,p_{-h}){-}p_l{*}\alpha_l(t,\varnothing,p_{-l}){-}(1{-}p_l){*}\beta_l(t,\varnothing,p_{-l})$

is positive. In Step 2, we show that

${-}p_h{*}\alpha_h(\varnothing,t,p_{-h}){-}(1{-}p_h){*}\beta_h(\varnothing,t,p_{-h}){+}p_l{*}\alpha_l(\varnothing,t,p_{-l}){+}(1{-}p_l){*}\beta_l(\varnothing,t,p_{-l})$

is also positive, thus adding all the terms, $EU_h[t] - EU_h[\varnothing] - (EU_l[t] - EU_l[\varnothing])$ is also positive.

Step 1:

Let $\lceil x \rceil$ denote the smallest integer equal or larger than x and similarly let $\lfloor x \rfloor$ denote the largest integer smaller or equal to x. With this conventin, for $i = \{l, h\}$,

$$\alpha_i(t,\varnothing,p_{-i}) = \sum_{k=\lceil tN \rceil}^{N-1} g_i(k)[F(\frac{M+N}{2} - k - 1) - F(\frac{M-N}{2} - 1)].$$

Then, writing $g_l(k)$ as $g_l(k) = p_h g_{lh}(k-1) + (1-p_h)g_{lh}(k)$, we obtain:

$$p_l{*}\alpha_l(t,\varnothing,p_{-l}) = \sum_{k=\lceil tN \rceil}^{N-1} p_l[p_h{*}g_{lh}(k{-}1){+}(1{-}p_h)g_{lh}(k)][F(\frac{M+N}{2}{-}k{-}1){-}F(\frac{M-N}{2}{-}1)]$$

and

$$p_h{*}\alpha_h(t,\varnothing,p_{-h}) = \sum_{k=\lceil tN \rceil}^{N-1} p_h[p_l{*}g_{lh}(k{-}1){+}(1{-}p_l)g_{lh}(k)][F(\frac{M+N}{2}{-}k{-}1){-}F(\frac{M-N}{2}{-}1)],$$

so:

$$p_h{*}\alpha_h(t,\varnothing,p_{-h}){-}p_l{*}\alpha_l(t,\varnothing,p_{-l}) = \sum_{k=\lceil tN \rceil}^{N-1} [(p_h{-}p_l)g_{lh}(k)][F(\frac{M+N}{2}{-}k{-}1){-}F(\frac{M-N}{2}{-}1)].$$

which relabeling the counter in the summation becomes:

$$\sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2}-1} [(p_h - p_l)g_{lh}(\frac{N}{2} + k)][F(\frac{M}{2} - k - 1) - F(\frac{M-N}{2} - 1)].$$

Now, noting that for $i = \{l, h\}$,

$$\beta_i(t,\varnothing,p_{-i}) = \sum_{k=1}^{\lfloor (1-t)N \rfloor} g_i(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k)].$$

and omitting a very similar step we directly obtain that

$$(1-p_h)\beta_h(t,\varnothing,p_{-h}) - (1-p_l)\beta_l(t,\varnothing,p_{-l}) =$$
$$= -\sum_{k=1}^{\lfloor (1-t)N \rfloor} [(p_h - p_l)g_{lh}(k-1)][F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k)]$$

which, since $\lceil tN \rceil + \lfloor (1-t)N \rfloor = N$, relabeling the counter in the summation becomes:

$$(1-p_h)\beta_h(t,\varnothing,p_{-h}) - (1-p_l)\beta_l(t,\varnothing,p_{-l}) =$$
$$= -\sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2}-1} [(p_h - p_l)g_{lh}(\frac{N}{2} - k - 1)][F(\frac{M+N}{2}) - F(\frac{M}{2} + k)].$$

By Assumption 1,

$$F(\frac{M+N}{2}) - F(\frac{M}{2} + k) = F(\frac{M}{2} - k - 1) - F(\frac{M-N}{2} - 1),$$

it follows

$$(1-p_h)\beta_h(t,\varnothing,p_{-h}) + p_h * \alpha_h(t,\varnothing,p_{-h}) - (1-p_l)\beta_l(t,\varnothing,p_{-l}) - p_l * \alpha_l(t,\varnothing,p_{-l}) =$$
$$= \sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2}-1} [(p_h - p_l)[F(\frac{M+N}{2}) - F(\frac{M}{2} + k)][g_{lh}(\frac{N}{2} + k) - g_{lh}(\frac{N}{2} - k - 1)]$$

which is positive by Assumption 2.

Step 2:

Noting that for $i = \{l, h\}$,

$$\alpha_i(\varnothing,t,p_{-i}) = \sum_{k=0}^{\lfloor (1-t)N \rfloor - 1} g_i(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k - 1)] \text{ and}$$

$$\beta_i(\varnothing,t,p_{-i}) = \sum_{k=\lceil tN \rceil + 1}^{N} g_i(k)[F(\frac{M+N}{2} - k) - F(\frac{M-N}{2} - 1)],$$

and repeating the same steps as in Step 1, we get:

$$p_h * \alpha_h(\varnothing,t,p_{-h}) - p_l * \alpha_l(\varnothing,t,p_{-l}) = \sum_{k=0}^{\lfloor (1-t)N \rfloor - 1} [(p_h - p_l)g_{lh}(k)][F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k - 1)]$$

which relabeling the counter in the summation becomes:

$$p_h * \alpha_h(\varnothing,t,p_{-h}) - p_l * \alpha_l(\varnothing,t,p_{-l}) = \sum_{k=\lceil tN \rceil - \frac{N}{2}}^{\frac{N}{2}-1} [(p_h - p_l)g_{lh}(\frac{N}{2} - k - 1)][F(\frac{M+N}{2}) - F(\frac{M}{2} + k)]$$

and

$$(1-p_h)\beta_h(\varnothing,t,p_{-h})-(1-p_l)\beta_l(\varnothing,t,p_{-l})=$$
$$=-\sum_{k=\lceil tN\rceil+1}^{N}[(p_h-p_l)g_{lh}(k-1)][F(\frac{M+N}{2}-k)-F(\frac{M-N}{2}-1)]$$

which, relabeling once again, becomes

$$(1-p_h)\beta_h(\varnothing,t,p_{-h})-(1-p_l)\beta_l(\varnothing,t,p_{-l})=$$
$$=-\sum_{k=\lceil tN\rceil-\frac{N}{2}}^{\frac{N}{2}-1}[(p_h-p_l)g_{lh}(\frac{N}{2}+k)][F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)].$$

Therefore,

$$-(1-p_h)\beta_h(\varnothing,t,p_{-h})-p_h*a_h(\varnothing,t,p_{-h})+(1-p_l)\beta_l(\varnothing,t,p_{-l})+p_l*\alpha_l(\varnothing,t,p_{-l})=$$
$$\sum_{k=\lceil tN\rceil-\frac{N}{2}}^{\frac{N}{2}-1}[(p_h-p_l)[F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)][g_{lh}(\frac{N}{2}+k)-g_{lh}(\frac{N}{2}-k-1)]$$

which is also positive by Assumption 2.

It follows that $EU_h[t]-EU_h[\varnothing]-(EU_l[t]-EU_l[\varnothing])\geq 0$. ■

6.4 Proposition 3:

Proof. By Lemma 1, member l is indifferent between a voting bloc with supermajority t or a voting bloc with unanimity (identical to no voting bloc) if:
$p_l^{t,\varnothing}(p_{-l})=\frac{\beta_l(\varnothing,t,p_{-l})-\beta_l(t,\varnothing,p_{-l})}{\alpha_l(t,\varnothing,p_{-l})-a_l(\varnothing,t,p_{-l})+\beta_l(\varnothing,t,p_{-l})-\beta_l(t,\varnothing,p_{-l})}$,
where:

$$\begin{aligned}
\alpha_l(t,\varnothing,p_{-l}) &= \sum_{k=\lceil tN\rceil}^{N-1} g_l(k)[F(\frac{M+N}{2}-k-1)-F(\frac{M-N}{2}-1)];\\
\beta_l(t,\varnothing,p_{-l}) &= \sum_{k=1}^{\lfloor(1-t)N\rfloor} g_l(k)[F(\frac{M+N}{2})-F(\frac{M+N}{2}-k)];\\
a_l(\varnothing,t,p_{-l}) &= \sum_{k=0}^{\lfloor(1-t)N\rfloor-1} g_l(k)[F(\frac{M+N}{2})-F(\frac{M+N}{2}-k-1)]\\
\text{and } \beta_l(\varnothing,t,p_{-l}) &= \sum_{k=\lceil tN\rceil+1}^{N} g_l(k)[F(\frac{M+N}{2}-k)-F(\frac{M-N}{2}-1)].
\end{aligned}$$

The derivative with respect to p_l of the surplus for member l generated by the voting bloc with internal voting rule t is equal to the denominator of $p_l^{t,\varnothing}(p_{-l})$, which relabeling the counter in the four summations in the denominator, is equal to:

$$\sum_{k=\lceil tN\rceil-\frac{N}{2}}^{\frac{N}{2}-1} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]$$
$$-\sum_{k=\lceil tN\rceil-\frac{N}{2}}^{N/2} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k)]$$
$$+\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k)-F(\frac{M-N}{2}-1)]$$
$$-\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k-1)]$$

$$= \sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k)-F(\frac{M-N}{2}-1)+F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]$$
$$+g_l(\lceil tN\rceil)[F(\frac{M+N}{2}-1-\lceil tN\rceil)-F(\frac{M-N}{2}-1)$$
$$-\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k-1)+F(\frac{M+N}{2})-F(\frac{M}{2}+k)]$$
$$-g_l(N-\lceil tN\rceil)[F(\frac{M+N}{2})-F(\frac{M-N}{2}+\lceil tN\rceil)].$$

then note that, by assumption,

$$F(\frac{M}{2}-k-1)=1-F(\frac{M}{2}+k); F(\frac{M}{2}-k)=1-F(\frac{M}{2}+k-1)$$

and $2F(\frac{M-N}{2}-1)=2-2F(\frac{M+N}{2})$.

Substitute accordingly to get:

$$\sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} [g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][2F(\frac{M+N}{2})-F(\frac{M}{2}+k)-F(\frac{M}{2}+k-1)]$$
$$+g_l(\lceil tN\rceil)[\sum_{k=0}^{N-1-\lceil tN\rceil} g_l(\frac{M-N}{2}+k)]-g_l(\lfloor(1-t)N\rfloor)\sum_{k=\lceil tN\rceil+1}^{N} f(\frac{M-N}{2}+k)=$$

$$= \sum_{k=\lceil tN\rceil-\frac{N}{2}+1}^{N/2} [g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][2F(\frac{M+N}{2})-F(\frac{M}{2}+k)-F(\frac{M}{2}+k-1)]$$

$$+[g_l(\lceil tN\rceil)-g_l(\lfloor(1-t)N\rfloor)] \sum_{k=0}^{N-1-\lceil tN\rceil} f(\frac{M-N}{2}+k)$$

Since $\lceil tN\rceil + \lfloor(1-t)N\rfloor = N$ and $t \geq \frac{1}{2}$, if follows $g_l(\lceil tN\rceil) > g_l(\lfloor(1-t)N\rfloor)$ and thus the denominator is positive.

Therefore,

$$EU_l[t] - EU_l[\varnothing] > 0 \Longleftrightarrow p_l > p_l^{t,\varnothing}.$$

Then, by Lemma 2, $EU_l[t] - EU_l[\varnothing] > 0 \Longrightarrow EU_i[t] - EU_i[\varnothing] > 0$ for all $i \in C$.

As a corollary note, if the internal voting rule is simple majority, then $\lceil tN\rceil = (1-t)N = \frac{N}{2}$, $[g_l(\lceil tN\rceil) - g_l(\lfloor(1-t)N\rfloor)] = 0$ and the threshold $p_l^{t,\varnothing}(p_{-l})$ simplifies to:

$$p_l^{Sm,\varnothing}(p_{-l}) = \frac{\sum_{k=\frac{N}{2}+1}^{N} g_l(k)[F(\frac{M+N}{2}-k)-F(\frac{M-N}{2}-1)] - \sum_{k=1}^{N/2} g_l(k)[F(\frac{M+N}{2})-F(\frac{M+N}{2}-k)]}{\sum_{k=1}^{N/2}[g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)] * [2F(\frac{M+N}{2})-F(\frac{M}{2}+k)-F(\frac{M}{2}+k-1)]}.$$

■

6.5 Proposition 4:

Let simple majority be the general election rule. Then the formation of a voting bloc by the coalition with t as the internal decision rule benefits member $l \in C$ if and only if $p_l > p_l^{t,\varnothing}(p_{-l})$.

Let $V^t \subset V$ be the subset of type profiles such that a $t-majority$ rule Pareto-dominates unanimity and let $t' = t + \frac{1}{N+1}$.

WTS: For any $t \in [\frac{1}{2}, \frac{N-1}{N+1}), V^t$ is strictly contained in $V^{t'}$.

Proof. $V^t = \{\overrightarrow{p} \in \Re^{N+M} : A1, A2 \text{ hold and } p_l > p_l^{t,\varnothing}(p_{-l})\}$.

$V^{t'} = \{\overrightarrow{p} \in \Re^{N+M} : A1, A2 \text{ hold and } p_l > p_l^{t+\frac{1}{N+1},\varnothing}(p_{-l})\}$.

It suffices to show that $p_l^{t,\varnothing}(p_{-l}) > p_l^{t+\frac{1}{N+1},\varnothing}(p_{-l})$ for any p_{-l}.

Suppose $p_l = p_l^{t,\varnothing}(p_{-l})$. Then $EU_l[t] - EU_l[\varnothing] = 0$

Let $S_{l|k}$ denote the probability that member l is in the majority of the coalition, given that the minority is of size k and let $s_{l|k} = 1 - S_{l|k}$ denote the probability that member l is in the minority of the coalition, given that the minority is of size k. Then $EU_l[t] - EU_l[\varnothing] =$

$$\sum_{k=1}^{\lfloor(1-t)N\rfloor} (S_{l|k}-s_{l|k}) prob[Minority\ size = k] prob[rolling\ k\ votes\ alters\ outcome].$$

Note that $S_{l|k} - s_{l|k}$ is decreasing in k. The bigger the minority, the more likely l is in it. Then, $EU_l[t] - EU_l[\varnothing] = 0$ implies that for $k = \lfloor (1-t)N \rfloor > 1$, $(S_{l|k} - s_{l|k}) < 0$. Then:

$$\sum_{k=1}^{\lfloor (1-t)N \rfloor -1} (S_{l|k} - s_{l|k}) prob[Minority\ size = k] prob[rolling\ k\ votes\ alters\ outcome] > 0.$$

But

$$\sum_{k=1}^{\lfloor (1-t)N \rfloor -1} (S_{l|k} - s_{l|k}) prob[Minority\ size = k] prob[rolling\ k\ votes\ alters\ outcome]$$

is equal to $EU_l[t'] - EU_l[\varnothing]$, so $p_l > p_l^{t',\varnothing}(p_{-l})$. ■

6.6 Proposition 5:

Proof. Let M be fixed. Let all the N members in $C \backslash \{l\}$ have a common type r. Then:

$$EU_l[Sm] - EU[\varnothing] = p \sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}+k) F(\frac{M}{2}-k-1) + (1-p) \sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}-k)[1 - F(\frac{M}{2}+k)]$$
$$-p \sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}-k-1)[1-F(\frac{M}{2}+k)] - (1-p) \sum_{k=0}^{\frac{M}{2}-1} g_l(\frac{N}{2}+k+1)[F(\frac{M}{2}-k-1)]$$

Since $F(\frac{M}{2}-k-1) = [1 - F(\frac{M}{2}+k)]$, this is equal to:

$$\sum_{k=0}^{\frac{M}{2}-1} \frac{N!}{(\frac{N}{2}-k)!(\frac{N}{2}+k)!} [p * r^{\frac{N}{2}+k}(1-r)^{\frac{N}{2}-k} + (1-p) r^{\frac{N}{2}-k}(1-r)^{\frac{N}{2}+k}] F(\frac{M}{2}-k-1)$$
$$-\sum_{k=0}^{\frac{M}{2}-1} \frac{N!}{(\frac{N}{2}-k-1)!(\frac{N}{2}+k+1)!} [p * r^{\frac{N}{2}-k-1}(1-r)^{\frac{N}{2}+k+1}$$
$$+(1-p) r^{\frac{N}{2}+k+1}(1-r)^{\frac{N}{2}-k-1} [F(\frac{M}{2}-k-1)].$$

Equating to zero and simplifying:

$$\sum_{k=0}^{\frac{M}{2}-1} p\{\frac{1}{N-2k}[r^{2k+1}(1-r) - r(1-r)^{2k+1}] - \frac{1}{N+2k+2}[(1-r)^{2k+2} - r^{2k+2}]\}$$
$$= \sum_{k=0}^{\frac{M}{2}-1} (\frac{r^{2k+2}}{N+2k+2} - \frac{(1-r)^{2k+1}}{N-2k})$$

Now we break this equation into $\frac{M}{2}$ different equations, imposing that for each $k \in \{0, \frac{M}{2} - 1\}$,

$$\begin{aligned} & p\{\frac{1}{N-2k}[r^{2k+1}(1-r) - r(1-r)^{2k+1}] - \frac{1}{N+2k+2}[(1-r)^{2k+2} - r^{2k+2}]\} \\ = \ & (\frac{r^{2k+2}}{N+2k+2} - \frac{(1-r)^{2k+1}}{N-2k}) \end{aligned}$$

A solution to this system of equations (with just one unknown) also solves the original equation. For each individual equation:

$$p = \frac{(N-2k)r^{2k+2} - (N+2k+2)r(1-r)^{2k+1}}{r(1-r)[r^{2k} - (1-r)^{2k}](N+2k+2) - (N-2k)[(1-r)^{2k+2} - r^{2k+2}]}$$

which, as $N \longrightarrow \infty$, converges to

$$\frac{r^{2k+2} - r(1-r)^{2k+1}}{r^{2k+1}(1-r) - r(1-r)^{2k+1} - (1-r)^{2k+2} + r^{2k+2}} = r$$

So $\lim_{N-->\infty} p_l^{Sm,\varnothing}(p_{-l}) = r$. Since by assumption $p_l < r$, this implies that for N large enough, $p_l < p_l^{Sm,\varnothing}(p_{-l})$ and then member l would be hurt if C forms a voting bloc with simple majority. ■

6.7 Proposition 6:

Proof. Let Out denote the internal voting rule for the coalition under which N members form a voting bloc with simple majority and member l stays out of the bloc and does not pool her vote with the rest of the coalition. By Lemma 1 member l is indifferent between rules Out and $\varnothing$ if:
$p_l^{Out,\varnothing}(p_{-l}) = \frac{\beta_l(\varnothing,Out,p_{-l}) - \beta_l(Out,\varnothing,p_{-l})}{\alpha_l(Out,\varnothing,p_{-l}) - \alpha_l(\varnothing,Out,p_{-l}) + \beta_l(\varnothing,Out,p_{-l}) - \beta_l(Out,\varnothing,p_{-l})}$, where:

$$\begin{aligned} \alpha_l(Out,\varnothing,p_{-l}) &= \sum_{k=\frac{N}{2}+1}^{N-1} g_l(k)[F(\frac{M+N}{2} - k - 1) - F(\frac{M-N}{2} - 1)]; \\ \beta_l(Out,\varnothing,p_{-l}) &= \sum_{k=1}^{\frac{N}{2}-1} g_l(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2} - k)]; \\ \alpha_l(\varnothing,Out,p_{-l}) &= \sum_{k=1}^{\frac{N}{2}-1} g_l(k)[F(\frac{M+N}{2} - 1) - F(\frac{M+N}{2} - k - 1)] \\ \text{and } \beta_l(\varnothing,Out,p_{-l}) &= \sum_{k=\frac{N}{2}+1}^{N-1} g_l(k)[F(\frac{M+N}{2} - k) - F(\frac{M-N}{2})]. \end{aligned}$$

Relabeling the counters in the summations, we can write the denominator

$\alpha_l(Out, \varnothing, p_{-l}) - \alpha_l(\varnothing, Out, p_{-l}) + \beta_l(\varnothing, Out, p_{-l}) - \beta_l(Out, \varnothing, p_{-l})$ as:

$$\begin{aligned}
&\sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k-1) - F(\frac{M-N}{2}-1)] \\
&- \sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2}-1) - F(\frac{M}{2}+k-1)] \\
&+ \sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k) - F(\frac{M-N}{2})] - \sum_{k=1}^{\frac{N}{2}-1} g_l(\frac{N}{2}-k)[F(\frac{M+N}{2}) - F(\frac{M}{2}+k)].
\end{aligned}$$

Since

$$\begin{aligned}
&[F(\frac{M}{2}-k) - F(\frac{M-N}{2}) + F(\frac{M}{2}-k-1) - F(\frac{M-N}{2}-1)] = \\
&[F(\frac{M+N}{2}) - F(\frac{M}{2}+k) + F(\frac{M+N}{2}-1) - F(\frac{M}{2}+k-1)],
\end{aligned}$$

the denominator simplifies to

$$\sum_{k=1}^{\frac{N}{2}-1}[g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][F(\frac{M}{2}-k)-F(\frac{M-N}{2})+F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]$$

and $p_l^{Out,\varnothing}(p_{-l}) = \frac{\sum_{k=1}^{\frac{N}{2}-1}\{g_l(\frac{N}{2}+k)[F(\frac{M}{2}-k)-F(\frac{M-N}{2})]-g_l(\frac{N}{2}-k)[F(\frac{M+N}{2})-F(\frac{M}{2}+k)]\}}{\sum_{k=1}^{\frac{N}{2}-1}[g_l(\frac{N}{2}+k)-g_l(\frac{N}{2}-k)][F(\frac{M}{2}-k)-F(\frac{M-N}{2})+F(\frac{M}{2}-k-1)-F(\frac{M-N}{2}-1)]}$.

The difference in utility for agent l between the formation of a bloc without l and no bloc at all is

$$\begin{gathered}
EU_l[Out] - EU_l[\varnothing] = \\
= p_l[\alpha_l(Out, \varnothing, p_{-l}) - \alpha_l(\varnothing, Out, p_{-l})] + (1-p_l)[\beta_l(Out, \varnothing, p_{-l}) - \beta_l(\varnothing, Out, p_{-l})].
\end{gathered}$$

The derivative of $EU_l[Out] - EU_l[\varnothing]$ with respect to p_l coincides with the denominator of the threshold $p_l^{Out,\varnothing}(p_{-l})$, which is positive. Therefore, for p_l above the threshold $p_l^{Out,\varnothing}(p_{-l})$, member l prefers the formation of a voting bloc in which l does not participate better than not forming any bloc at all; whereas for p_l below $p_l^{Out,\varnothing}(p_{-l})$, member l prefers to form no bloc than to form a bloc in which l does not participate. ∎

6.8 Proposition 7:

Let V be the set of type profiles $(p_1, p_2, ..., p_{N+M+1})$ satisfying Assumptions 1 and 2. Let $V^t \subset V$ be the subset of type profiles such that a $t-majority$ rule Pareto-dominates unanimity for C, let $(V^t)^C$ be its complement such that $V^t \cup (V^t)^C \equiv V$ and let $V^{Out} \subset V$ be the subset of type profiles such that an "Opt-Out for l" rule Pareto dominates unanimity for C. Then, for any M and for any $4 \leq N \leq M$, $(V^t)^C \cap V^{Out}$ is not empty.

Proof. Suppose $4 \leq N \leq M$, $p_l = (1-\delta)$ and $p_j = (1-\varepsilon)$ $\forall j \in C\backslash\{l\}$. Then:

$$\begin{aligned} EU_l[Out] - EU_l[\varnothing] &= (1-\delta)\sum_{k=\lceil t(N-1)\rceil+1}^{N-1} g_l(k)[F(\frac{M+N}{2}-k-1) - F(\frac{M-N}{2}-1)] \\ &\quad +\delta \sum_{k=1}^{\lfloor (1-t)(N-1)\rfloor} g_l(k)[F(\frac{M+N}{2}) - F(\frac{M+N}{2}-k)] \\ &\quad -(1-\delta)\sum_{k=1}^{\lfloor (1-t)(N-1)\rfloor} g_l(k)[F(\frac{M+N}{2}-1) - F(\frac{M+N}{2}-k-1)] \\ &\quad -\delta \sum_{k=\lceil t(N-1)\rceil+1}^{N-1} g_l(k)[F(\frac{M+N}{2}-k) - F(\frac{M-N}{2})]. \end{aligned}$$

As ε converges to zero, $\frac{g_l(k)}{g_l(N-1)}$ converges to zero for any $k < N-1$ and $EU_l[Out] - EU_l[\varnothing]$ converges to:

$(1-\delta)g_l(N-1)[F(\frac{M-N}{2})-F(\frac{M-N}{2}-1)]-\delta g_l(N-1)[F(\frac{M-N}{2}+1)-F(\frac{M-N}{2})]$ which is positive for a sufficiently low δ, provided that $f(\frac{M-N}{2}) > 0$.

Then, there exist a $\delta > 0$ and $\overline{\varepsilon} > 0$ such that for all $\varepsilon < \overline{\varepsilon}$, $EU_l[Out] - EU_l[\varnothing] > 0$.

Therefore, if $4 \leq N \leq M$, and the types of all the members but l converge to 1, member l with type $p_l = (1-\delta)$ will benefit from a voting bloc with an "Opt-Out for l" rule. Given that all the other members of C share a common type, they would all benefit from forming a voting bloc without l. Since $p_l^{t,\varnothing}(p_{-l})$ converges to 1, member l would not benefit from a $t-majority$ internal voting rule in a voting bloc that includes every member. It follows that if $\varepsilon < \overline{\varepsilon}$, any profile of types in which $p_l = (1-\delta)$ and $p_j = (1-\varepsilon)$ $\forall j \in C\backslash\{l\}$ is in V^{Out} but not in V^t, thus $V^{Out} \nsubseteq V^t$.

If $N > M$, then $EU_l[Sm] - EU_l[Out] = 2g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})] > 0$, thus thus $V^{Out} \subset V^{Sm}$. If $N = 2$, then Out coincides with $\varnothing$ and V^{Out} is empty. ■

6.9 Proposition 8:

Proof. By Lemma 1, member l will be indifferent between participating in the voting bloc or opting out if

$p_l^{Sm,Out}(p_{-l}) = \frac{\beta_l(Out,Sm,p_{-l})-\beta_l(Sm,Out,p_{-l})}{\alpha_l(Sm,Out,p_{-l})-\alpha_l(Out,Sm,p_{-l})+\beta_l(Out,Sm,p_{-l})-\beta_l(Sm,Out,p_{-l})}$, where:

$\alpha_l(Sm,Out,p_{-l}) = g_l(\frac{N}{2})[F(\frac{M}{2}-1) - F(\frac{M-N}{2}-1)]$; $\beta_l(Sm,Out,p_{-l}) = g_l(\frac{N}{2})[F(\frac{M+N}{2})-F(\frac{M}{2})]$; $\alpha_l(Out,Sm,p_{-l}) = \sum_{k=0}^{\frac{N}{2}-1} g_l(k)f(\frac{M+N}{2})$; $\beta_l(Out,Sm,p_{-l}) = \sum_{k=\frac{N}{2}+1}^{N} g(k)f(\frac{M-N}{2})$.

Since $F(\frac{M}{2}-1)-F(\frac{M-N}{2}-1) = F(\frac{M+N}{2})-F(\frac{M}{2})$ and $f(\frac{M+N}{2}) = f(\frac{M-N}{2})$ it follows that $\alpha_l(Sm,Out,p_{-l}) = \beta_l(Sm,Out,p_{-l})$ and we can simplify the denominator to $\alpha_l(Out, Sm, p_{-l}) - \beta_l(Out, Sm, p_{-l}) =$

$$[\sum_{k=\frac{N}{2}+1}^{N} g(k) - \sum_{k=0}^{\frac{N}{2}-1} g_l(k)] f(\frac{M+N}{2}) = \sum_{k=1}^{N/2} [g_l(\frac{N}{2}+k) - g_l(\frac{N}{2}-k)] f(\frac{M+N}{2})$$

and $p_l^{Sm,Out}(p_{-l}) = \frac{\sum_{k=\frac{N}{2}+1}^{N} g(k) f(\frac{M-N}{2}) - g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})]}{\sum_{k=1}^{N/2} [g_l(\frac{N}{2}+k) - g_l(\frac{N}{2}-k)] f(\frac{M+N}{2})}$.

The advantage for member l of staying in, $EU_l[Sm] - EU_l[Out] =$

$$= p_l[\alpha_l(Sm, Out, p_{-l}) - a_l(Out, Sm, p_{-l})] + (1-p_l)[\beta_l(sm, Out, p_{-l}) - \beta_l(Out, Sm, p_{-l})].$$

Its derivative with respect to p_l coincides with the denominator of $p_l^{Sm,Out}(p_{-l})$. Since $g_l(\frac{N}{2}+k) > g_l(\frac{N}{2}-k)$ for all $k \in [1, \frac{N}{2}]$, the denominator and thus the derivative are positive. Therefore, member l prefers to stay in if type p_l is above the threshold $p_l^{Sm,Out}(p_{-l})$ and member l prefers to opt-out than to stay in if $p_l < p_l^{Sm,Out}(p_{-l})$. ∎

6.10 Proposition 9:

Proof. The first statement comes straightforward from Propositions 3 and 8. For the second one, suppose $p_i = p \ \forall i \in C$. Then all members benefit from the formation of a voting bloc with simple majority: $EU_i[Sm] > EU_i[\varnothing] \ \forall i \in C$. The extra gains of stepping out for member l when simple majority is the general voting rule are:

$$EU_l[Out] - EU_l[Sm] = p \sum_{k=0}^{\frac{N}{2}-1} g_l(k) f(\frac{M+N}{2}) + (1-p) \sum_{k=\frac{N}{2}+1}^{N} g_l(k) f(\frac{M-N}{2})$$
$$-pg_l(\frac{N}{2})[F(\frac{M}{2}-1) - F(\frac{M-N}{2}-1)] - (1-p)g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})]$$

For $p \in (1/2, 1)$ and any N, M :

$$\begin{aligned} EU_l[Out] - EU_l[Sm] &> (1-p) \sum_{k=\frac{N}{2}+1}^{N} g_l(k) f(\frac{M-N}{2}) - g_l(\frac{N}{2})[F(\frac{M+N}{2}) - F(\frac{M}{2})] \\ &> (1-p) f(\frac{M-N}{2}) g_l(N) - \frac{1}{2} g_l(\frac{N}{2}), \end{aligned}$$

which letting $\alpha = f(\frac{M-N}{2})$ and $\beta = \frac{N!}{\frac{N}{2}!\frac{N}{2}!}$ is equal to:

$\alpha(1-p)p^N - \frac{1}{2}\beta p^{N/2}(1-p)^{N/2} = (1-p)p^{N/2}[\alpha p^{N/2} - \frac{1}{2}\beta(1-p)^{\frac{N-2}{2}}]$, which is positive if and only if

$$2\alpha p^{N/2} \geq \beta(1-p)^{\frac{N-2}{2}} \iff \frac{p^{N/2}}{(1-p)^{\frac{N-2}{2}}} \geq \frac{\beta}{2\alpha} \iff \frac{p}{(1-p)^{\frac{N-2}{N}}} \geq \left(\frac{\beta}{2\alpha}\right)^{2/N}.$$

Letting $\gamma > 0$ be any number such that $p \geq \gamma$, this last inequality will be satisfied if

$$\frac{\gamma}{(1-p)^{\frac{N-2}{N}}} \geq \left(\frac{\beta}{2\alpha}\right)^{2/N} \Longleftrightarrow (1-p)^{\frac{N-2}{N}} \leq \gamma\left(\frac{2\alpha}{\beta}\right)^{2/N} \Longleftrightarrow (1-p) \leq \gamma\left(\frac{2\alpha}{\beta}\right)^{\frac{2}{N-2}}$$
$$\Longleftrightarrow \quad p \geq 1-\gamma\left(\frac{2\alpha}{\beta}\right)^{\frac{2}{N-2}},$$

which is less than one for $N \in [4, M]$, provided that $f(\frac{M-N}{2}) > 0$.

Note that if $N = 2$, then $\left(\frac{2\alpha}{\beta}\right)^{\frac{2}{N-2}} = 0$; whereas if $N > M$, $\alpha = 0$.

If $N = 2$, then $EU_l[Out] = EU_l[\varnothing]$, thus $EU_l[Sm] > EU_l[\varnothing]$ implies $EU_l[Sm] > EU_l[Out]$.

If $N > M$, then $f(\frac{M+N}{2}) = f(\frac{M-N}{2}) = 0$, thus

$$EU_l[Out] - EU_l[Sm] = -pg_l(\frac{N}{2})F(\frac{M}{2}-1) - (1-p)g_l(\frac{N}{2})[1-F(\frac{M}{2})] < 0.$$

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