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ON FORMATION OF CLOSE BINARIES BY TWO-BODY TIDAL CAPTURE*

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ABSTRACT

We calculate in detail the two-body tidal capture mechanism of Fabian, Pringle, and Rees: when two unbound stars have a close encounter, they may become bound by the energy that each deposits into nonradial oscillations of the other. After dimensional scalings are removed, the process depends only on a single dimensionless parameter, and on the dimensionless envelope structure of the stars. General formulae are derived; for definiteness, we apply them to the specific case of stars with an $n = 3$ polytropic structure. Capture cross sections as a function of velocity and capture rates for an isothermal distribution are given for the case of equal-mass stars; other cases can easily be computed from the formulae given.

Subject headings: stars: binaries — stars: stellar dynamics

I. INTRODUCTION

Fabian, Pringle, and Rees (1975, hereafter FPR) have drawn attention to a novel mechanism for forming close binary stars: if two unbound stars should happen to have a close encounter, then their mutual tidal forces, which are time dependent near periastron, will excite nonradial oscillations in the stars. The energy deposited into oscillatory modes comes from the orbital motion. If enough energy is thus absorbed, the stars will be left as a bound system. The mode excitation process will repeat on subsequent periastron passages, gradually circularizing the orbit. As FPR note, the details of the dissipation within the stars are unimportant as long as the damping time scale for the oscillations is long compared with the duration of periastron passage, and short compared with the (long) time between periastron passages. Even if the latter condition fails, the mechanism will proceed as long as the phase of oscillations is random with respect to the times of periastron—and it should take many periastron passages for a tidal-locking resonance to destroy this randomness (if it can at all).

The novelty of the FPR mechanism is that it forms binaries by *capture* (not during star formation *ab initio*), so that the components of a binary system need not have shared a consistent evolutionary history; and that the capture requires a close approach of only two (not three) bodies, so that capture rates will scale as the square (not cube) of stellar population densities.

FPR proposed the tidal capture mechanism as a means for creating close binaries (and thus X-ray sources) in globular clusters. If the new X-ray "bursters" (Grindlay *et al.* 1976) are in fact associated with globular clusters (cf. Bahcall and Ostriker 1976), then models which produce close binary X-ray sources on the model of normal galactic binary sources become less compelling. For this reason we are not active proponents of the FPR model. However, the tidal capture *mechanism* seems interesting enough in its own right, and it may be applicable to other astronomical situations. For example, Rees (private communication) has noted that the mechanism might form close binaries from dense associations of new, early-type stars. Another possible application is in the evolving core of relaxed globular clusters: Sanders (1970) has traced this evolution to the point of stellar collisions. Tidal capture, as we shall see, becomes important before this point; and the influence of binary stars, once formed, on subsequent core relaxation is not at all clear (Heggie 1974, 1976; Aarseth and Lecar 1975; Hills 1975), and may be important. The tidal capture of two galaxies has been discussed by Alladin (1965), Sastry and Alladin (1970), and Toomre and Toomre (1972).

The purpose of this paper is just to calculate, in some quantitative detail, the amount of orbital energy which is deposited into oscillatory modes during a close periastron passage. We will see that the "tidal capture susceptibility" of a star can be boiled down to one or two dimensionless functions of a suitable dimensionless variable. These functions depend on the envelope structure of the star (through its normal modes), but do not depend explicitly on

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the mass or radius of the star. Therefore we can compute the functions numerically for a polytropic stellar model with $n = 3$, and have some confidence that the results are applicable (with fair approximation) to a wide range of early-type stars. Of course, the calculation of this paper could also be repeated using an actual accurate stellar model, if there were any compelling reason for doing so.

The result of this paper is to make the following question instantly answerable for any particular parameters: Given the masses of two stars, their radii, and their periastron distance of close approach, how much orbital energy is absorbed in the encounter? (Equivalently, what is the cross section for capture between two stars of given relative velocity at infinity?)

In § II we derive a general expression for the energy deposited into oscillatory modes by the perturbing force. In § III we resolve the tidal perturbation into spherical harmonics in the standard way. The dimensionless functions mentioned above are defined in § IV. The normal modes of the star enter in § V. The Fourier transform of the perturbing force is evaluated in § VI. Results are given in § VII and discussed in § VIII.

II. THE ENERGY DEPOSITION FORMULA

To high accuracy, each star perceives the other as a point mass only (tide-tide coupling being of second order in the small parameter which measures the amplitude of the tidal perturbations). Therefore we focus attention on one star, of mass M_* and radius R_*, and imagine it to orbit a point object of mass M.

The rate at which energy is deposited in the star M_* is

$$\frac{dE}{dt} = \int d^3x \rho \boldsymbol{v} \cdot \nabla U . \tag{1}$$

Here ρ is the density of the star, $\boldsymbol{v}$ is the velocity of a fluid element in the star, and U is the gravitational potential of the point mass:

$$U(\boldsymbol{r}, t) = \frac{GM}{|\boldsymbol{r} - \boldsymbol{R}(t)|} , \tag{2}$$

where $\boldsymbol{R}(t)$ is the relative orbit of the point mass. For notational convenience, define the scalar product of any two vector functions $\boldsymbol{C}(\boldsymbol{r})$ and $\boldsymbol{D}(\boldsymbol{r})$ by

$$\langle \boldsymbol{C} | \boldsymbol{D} \rangle \equiv \int d^3x \rho \boldsymbol{C}(\boldsymbol{r}) \cdot \boldsymbol{D}(\boldsymbol{r}) . \tag{3}$$

Then equation (1) can be rewritten

$$\frac{dE}{dt} = \langle \boldsymbol{v} | \nabla U \rangle . \tag{4}$$

We shall assume that the effect of ∇U on the equilibrium static star can adequately be described by a linearized perturbation analysis. Then ρ in equation (1) can be taken to be the unperturbed stellar density, while the quantity $\boldsymbol{v}$ can be expressed in terms of the Lagrangian displacement $\boldsymbol{\xi}$ of a fluid element from its unperturbed position by (e.g., Chandrasekhar 1969)

$$\boldsymbol{v} = \frac{\partial \boldsymbol{\xi}}{\partial t} . \tag{5}$$

Define the Fourier transforms of U and $\boldsymbol{\xi}$ by

$$U(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \tilde{U}(\omega) d\omega , \tag{6}$$

$$\boldsymbol{\xi}(t) = \int_{-\infty}^{\infty} e^{-i\omega t} \tilde{\boldsymbol{\xi}}(\omega) d\omega . \tag{7}$$

Since U and $\boldsymbol{\xi}$ are real,

$$U(\omega) = U^*(-\omega) , \qquad \boldsymbol{\xi}(\omega) = \boldsymbol{\xi}^*(-\omega) , \tag{8}$$

where an asterisk denotes complex conjugation. Substituting equations (5), (6), and (7) in equation (4) gives

$$\frac{dE}{dt} = \int_{-\infty}^{\infty} d\omega \int_{-\infty}^{\infty} d\omega' (-i\omega) e^{-i(\omega + \omega')t} \langle \tilde{\boldsymbol{\xi}}(\omega) | \nabla \tilde{U}(\omega') \rangle . \tag{9}$$

Thus the total energy deposited is

$$\Delta E = \int_{-\infty}^{\infty} dt \frac{dE}{dt} = 2\pi \int_{-\infty}^{\infty} d\omega(-i\omega)\langle\tilde{\xi}(\omega)|\nabla\tilde{U}^*(\omega)\rangle . \tag{10}$$

We now analyze $\tilde{\xi}$ into normal modes. The normal modes satisfy a linear, self-adjoint eigenvalue equation of the form

$$(\mathscr{L} - \rho\omega_n^2)\xi_n = 0 , \tag{11}$$

(Chandrasekhar 1964), where the detailed form of the operator $\mathscr{L}$ is not particularly important. The eigenvectors are orthogonal with weight ρ

$$\langle\xi_n|\xi_k\rangle = \delta_{nk} , \tag{12}$$

and without loss of generality they can be taken to be real. The quantity $\tilde{\xi}$ satisfies an equation of the same form as equation (11), with $\nabla\tilde{U}$ providing a driving term on the right-hand side:

$$(\mathscr{L} - \rho\omega^2)\tilde{\xi} = \rho\nabla\tilde{U} . \tag{13}$$

Expand $\nabla\tilde{U}$ as a superposition of normal modes, i.e.,

$$\nabla\tilde{U} = \sum_n A_n(\omega)\xi_n , \tag{14}$$

where the (complex) quantities A_n are given by

$$A_n(\omega) = \langle\xi_n|\nabla\tilde{U}\rangle . \tag{15}$$

Then

$$\tilde{\xi} = \sum_n B_n(\omega)\xi_n , \tag{16}$$

where by equations (11), (13), and (14)

$$B_n = \frac{A_n}{\omega_n^2 - \omega^2} . \tag{17}$$

In order that the response $\xi(t)$ to the perturbing force be retarded and not advanced, we adopt the usual prescription of replacing the denominator $\omega_n^2 - \omega^2$ by $\omega_n^2 - \omega^2 - i\omega\epsilon$ and taking the limit as $\epsilon \to 0$.

Substituting equations (14), (16), and (17) in equation (10) gives

$$\Delta E = 2\pi \sum_n \int_{-\infty}^{\infty} d\omega \frac{-i\omega|A_n(\omega)|^2}{\omega_n^2 - \omega^2 - i\omega\epsilon} = 2\pi^2 \sum_n |A_n(\omega_n)|^2 , \tag{18}$$

where we have used the fact that $|A_n|^2$ is even in ω by equation (8).

The above analysis is of general applicability to any linear physical system and leads to the simple prescription (18) for the energy deposited, where A_n is determined by the "overlap integral" (15).

III. ORBITAL MOTION AND TIDAL FORCES

For situations of interest, the relative velocity of the stars at periastron will be hundreds of km s^{-1} or more, while their velocities at infinity will be tens of km s^{-1} or less. Therefore, the eccentricity of the orbit will be within 10^{-2} of unity, and we can well approximate the orbit as parabolic (especially near periastron). Let the periastron distance between the bodies be $R_{\min}$. The relative orbit of the point mass is given parametrically by the equations

$$R = R_{\min}(1 + x^2) , \tag{19}$$

$$t = \left[\frac{2R_{\min}^3}{G(M_* + M)}\right]^{1/2}(x + x^3/3) , \tag{20}$$

$$x = \tan(\Phi/2) , \tag{21}$$

where Φ is the true anomaly.

Now to compute the tidal perturbation on the star M_*, we go to a comoving coordinate system $\boldsymbol{r}$ (or r, θ, ϕ) whose origin is at M_*'s center with the orbital plane at $\theta = \pi/2$. The tidal potential due to M is then given by expanding equation (2) in spherical harmonics (cf. Jackson 1962, eq. [3.70]):

$$U = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \frac{4\pi GM}{2l+1} \frac{r^l}{R^{l+1}} Y_{lm}\left(\frac{\pi}{2}, \Phi\right) Y_{lm}^*(\theta, \phi) . \tag{22}$$

Here we have deleted from the sum the constant $l = 0$ and vanishing $l = 1$ contributions. The normal modes of the star will be resolved into spherical harmonics in § V, so we can focus attention on the coefficient of a single Y_{lm}^* in equation (22) and rewrite it, using spherical harmonic identities, as

$$U_{lm} = W_{lm} \frac{GMr^l}{R(t)^{l+1}} e^{im\Phi(t)}, \tag{23}$$

where W_{lm} is the numerical coefficient

$$W_{lm} = (-)^{(l+m)/2}\left[\frac{4\pi}{2l+1}(l-m)!(l+m)!\right]^{1/2} \Big/ \left[2^l\left(\frac{l-m}{2}\right)!\left(\frac{l+m}{2}\right)!\right]. \tag{24}$$

Here the symbol $(-)^k$ is to be interpreted as zero when k is not an integer.

IV. REDUCTION TO A NONDIMENSIONAL PROBLEM: THE FUNCTIONS $T_l(\eta)$

Natural units of mass, length, time, and energy in this problem are respectively M_*, R_*, $(R_*^3/GM_*)^{1/2}$, and GM_*^2/R_*. Consider one value of l, say $l = 2$ (the quadrupole tide). Then in equation (23) the perturbing mass M and distance $R(t)$ enter into the strength of the perturbing potential only as multiplicative powers. (A mass eight times as great at twice the distance would give the same potential.) However, the time dependence of the perturbation would be different, since M and $R_{\min}$ enter into equation (20). In natural units, the time dependence is controlled by a single parameter, which we call η. From equation (20),

$$\eta \equiv \left(\frac{M_*}{M_* + M}\right)^{1/2}\left(\frac{R_{\min}}{R_*}\right)^{3/2}. \tag{25}$$

The quantity η measures the duration of periastron passage, relative to the hydrodynamic time of the star: large η means slow passage.

For a fixed value of η, the amplitude of the tidally induced oscillations scales with M and $R_{\min}$ just like the perturbing potential U_{lm}, because the time dependence is the same, and because the response of the star's normal modes is (assumed!) linear. The energy of the oscillations is quadratic in the amplitude. Therefore the energy deposited into oscillations of spherical harmonic index l must be

$$\Delta E_l = \left(\frac{GM_*^2}{R_*}\right)\frac{(M/M_*)^2}{(R_{\min}/R_*)^{2l+2}} T_l(\eta), \tag{26}$$

where $T_l(\eta)$ is a dimensionless function of η alone. Energy is independently deposited into the modes of each value of l. Therefore, the general formula for the total energy deposition is the sum,

$$\Delta E = \left(\frac{GM_*^2}{R_*}\right)\left(\frac{M}{M_*}\right)^2 \sum_{l=2,3,\ldots}\left(\frac{R_*}{R_{\min}}\right)^{2l+2} T_l(\eta). \tag{27}$$

In the next section we will give an explicit expression for computing $T_l(\eta)$. However, let us first ask what the relevant ranges of l and η are.

We will see later that all of the T_l are about the same order, so successive terms in the sum (27) are smaller by the ratio $(R_*/R_{\min})^2$. If this number is larger than $\sim 1/10$, the stars are so close to grazing incidence that one ought to doubt the whole validity of a linear-mode calculation anyway. Therefore we might as well assume that $(R_*/R_{\min})^2 \lesssim 1/10$, in which case the neglect of $l = 4, 5, \ldots$ should contribute a $\sim 1\%$ error. Thus we concentrate attention on the $l = 2$ (quadrupole) and $l = 3$ (octopole) tides (although all our equations are valid for higher l as well).

What is the interesting range of η? An upper limit is set by the fact that the energy deposited becomes very small as η increases. When it is too small, there is no interesting possibility of tidal capture, which corresponds to such slow, distant encounters that the tides respond to the potential quasi-statically. So the upper limit on η will be set *a posteriori*. There is a more fundamental lower limit on η: our calculation is certainly not valid if the star comes within the Roche limit of the other mass. Within the Roche limit, one would expect actual hydrodynamic disruption of the star. This might well be inelastic enough to leave the stars bound (or even merged), but the normal mode calculation of this paper has nothing to say about this violent case (nor about the even more violent case of an actual collision!).

The Roche limit for a parabolic orbit is not accurately known, but numerical evidence (Nduka 1971; see also Chandrasekhar 1969, p. 12) indicates that disruption is avoided when

$$(R_*^3/M_*)(M/R_{\min}^3) < 0.0724, \tag{28}$$

i.e. (cf. eq. [25]), when

$$\eta > 3.72\left(\frac{M}{M_* + M}\right)^{1/2}. \tag{29}$$

On the other hand, since $R_{\min}/R_* > 1$,

$$\eta > \left(\frac{M_*}{M_* + M}\right)^{1/2}. \tag{30}$$

Inequalities (29) and (30) imply

$$\eta > 0.966 \approx 1.0; \tag{31}$$

We need consider no smaller values.

V. NORMAL MODES AND OVERLAP INTEGRALS

The normal modes of a spherical star can be labeled by spherical harmonic indices l and m, and by a "radial quantum number" n. We adopt the conventions of Chandrasekhar (1961) for vector spherical harmonics, except that our Y_{lm}'s are normalized as in Jackson (1962). Thus a particular normal mode can be written as the sum of radial and poloidal pieces:

$$\boldsymbol{\xi}_n(\boldsymbol{r}) \equiv \boldsymbol{\xi}_{nlm}(\boldsymbol{r}) = [\xi_{nl}{}^R(r)\boldsymbol{e}_r + \xi_{nl}{}^S(r)r\boldsymbol{\nabla}]Y_{lm}(\theta, \phi). \tag{32}$$

[Toroidal modes proportional to $(\boldsymbol{r} \times \boldsymbol{\nabla})Y_{lm}$ are not excited by a conservative perturbing force.] Strictly speaking, we should use real linear combinations of Y_{lm} and $Y_{l,-m}$ to agree with our conventions in § II; however, the representation (32) gives the same result.

The overlap integral (15) can be written, using the Fourier inverse of equation (6), in the form

$$A_{nlm}(\omega_n) = \int d^3x\rho\boldsymbol{\xi}_{nlm}(\boldsymbol{r})\cdot\boldsymbol{\nabla}\frac{1}{2\pi}\int_{-\infty}^{\infty} dtU(\boldsymbol{r}, t)\exp(i\omega_n t). \tag{33}$$

The angular integral in equation (33) can be done by using the expansion (22):

$$\boldsymbol{\nabla}U(\boldsymbol{r}, t) = \boldsymbol{\nabla}\sum_{lm} U_{lm}(r, t)Y_{lm}{}^*(\theta, \phi) = GM\sum_{lm} W_{lm}\frac{r^{l-1}}{R(t)^{l+1}}e^{im\Phi(t)}(l\boldsymbol{e}_r + r\boldsymbol{\nabla})Y_{lm}{}^*(\theta, \phi). \tag{34}$$

Equations (34) and (32) and the orthogonality relation

$$\int d\Omega[r\boldsymbol{\nabla}Y_{lm}(\Omega)]\cdot[r\boldsymbol{\nabla}Y_{l'm'}{}^*(\Omega)] = l(l+1)\delta_{ll'}\delta_{mm'} \tag{35}$$

enable one to reduce equation (33) to the form

$$A_{nlm}(\omega_n) = GM\int_0^{R_*} r^2 dr\rho l r^{l-1}[\xi^R + (l+1)\xi^S]\int_{-\infty}^{\infty} dt\,\frac{W_{lm}\exp\{i[\omega_n t + m\Phi(t)]\}}{2\pi R(t)^{l+1}}. \tag{36}$$

Now nondimensionalize the integrals in equation (36) by expressing each quantity in the integrands in the natural units of § IV. Thus

$$A_{nlm}(\omega_n) = \left(\frac{GM^2}{R_*}\right)^{1/2}\left(\frac{R_*}{R_{\min}}\right)^{l+1}Q_{nl}K_{nlm}, \tag{37}$$

where

$$Q_{nl} = \int_0^1 r^2 dr\rho l r^{l-1}[\xi_{nl}{}^R + (l+1)\xi_{nl}{}^S] \tag{38}$$

and

$$K_{nlm} = \frac{W_{lm}}{2\pi}\int_{-\infty}^{\infty} dt\left(\frac{R_{\min}}{R(t)}\right)^{l+1}\exp\{i[\omega_n t + m\Phi(t)]\}. \tag{39}$$

Note from the normalization (12) that we are measuring ξ in units of $M_*{}^{-1/2}$.

Now from equation (18) we get

$$\Delta E = 2\pi^2\left(\frac{GM_*^2}{R_*}\right)\left(\frac{M}{M_*}\right)^2 \sum_{nlm}\left(\frac{R_*}{R_{\min}}\right)^{2l+2}|Q_{nl}|^2|K_{nlm}|^2 . \tag{40}$$

Comparison with equation (27) gives

$$T_l(\eta) = 2\pi^2 \sum_n |Q_{nl}|^2 \sum_{m=-l}^{l} |K_{nlm}|^2 . \tag{41}$$

The η-dependence in equation (41) can be made explicit by rewriting the integral (39) in terms of the parameter x of equations (19)–(21). This gives

$$K_{nlm} = \frac{W_{lm}}{2\pi} 2^{3/2}\eta I_{lm}(\eta\omega_n) , \tag{42}$$

where

$$I_{lm}(y) = \int_0^\infty dx(1 + x^2)^{-l} \cos\left[2^{1/2}y(x + x^3/3) + 2m \tan^{-1} x\right] . \tag{43}$$

The calculation of $T_l(\eta)$ is now seen to consist of three separate parts: (i) Determine the normal modes $\xi_{nl}(r)$ of the star (which are independent of m) and hence the Q_{nl}'s from equation (38). (ii) Without reference to any normal modes, compute and tabulate the functions $I_{lm}(y)$ of equation (43). (iii) Evaluating the functions from (ii) at the mode frequencies of (i), determine $T_l(\eta)$ by equation (41).

VI. THE FUNCTIONS $I_{lm}(y)$

The computation of the functions $I_{lm}(y)$ is facilitated by the following recurrence relations which are proved in the Appendix:

$$I_{l,m\pm 1} = \left(2 \mp \frac{2m}{l}\right)I_{l+1,m} - I_{lm} \mp \frac{2^{1/2}y}{l} I_{l-1,m} , \tag{44}$$

$$I_{l0} = \frac{2l-3}{2l-2} I_{l-1,0} + \frac{y^2}{(2l-2)(l-3)} I_{l-4,0} . \tag{45}$$

Equation (44) enables one to obtain all the I_{lm}'s from the I_{l0}'s, while equation (45) allows the I_{l0}'s for $l \geq 4$ to be computed from I_{00}, I_{10}, I_{20}, and I_{30}.

The first one, I_{00}, is a modified Bessel function or Airy function,

$$I_{00}(y) = 3^{-1/2}K_{1/3}(2^{3/2}y/3) = \pi(2^{1/2}y)^{-1/3} \text{ Ai}\left[(2^{1/2}y)^{2/3}\right] . \tag{46}$$

This is easily evaluated numerically from Abramowitz and Stegun (1965), equation (10.4.2) for small arguments and Table 10.11 for large arguments. For the remaining three functions, we have adopted the strategy of evaluating them numerically (by fast Fourier transforms), and fitting them to rational function approximations. The results are

$$I_{10}(y) = \frac{1.5288 + 0.79192y^{1/2} - 0.86606y + 0.14593y^{3/2}}{1 + 1.6449y^{1/2} - 1.2345y + 0.19392y^{3/2}} \exp\left(-\frac{2^{3/2}}{3}y\right) \quad \text{for } y \leq 4 , \tag{47}$$

$$I_{10}(y) = \frac{1.4119 + 18.158y^{1/2} + 22.152y}{1 + 12.249y^{1/2} + 28.593y} \exp\left(-\frac{2^{3/2}}{3}y\right) \quad \text{for } y \geq 4 , \tag{48}$$

$$I_{20}(y) = \frac{0.78374 + 1.5039y^{1/2} + 1.0073y + 0.71115y^{3/2}}{1 + 1.9128y^{1/2} + 1.0384y + 1.2883y^{3/2}} \left(1 + \frac{2^{3/2}}{3}y\right)^{1/2} \exp\left(-\frac{2^{3/2}}{3}y\right) , \tag{49}$$

$$I_{30}(y) = \frac{0.58894 + 0.32381y^{1/2} + 0.45605y + 0.15220y^{3/2}}{1 + 0.54766y^{1/2} + 0.76130y + 0.53016y^{3/2}} \left(1 + \frac{2^{3/2}}{3}y\right) \exp\left(-\frac{2^{3/2}}{3}y\right) . \tag{50}$$

The fractional accuracy of these approximations is around $\lesssim 0.1\%$, except for very small and very large values of y (which are not of interest anyway; there the error rises to a few percent). Our source of error is primarily in the

TABLE 1

EIGENFREQUENCIES AND OVERLAP INTEGRALS FOR AN $n = 3$ POLYTROPE

Mode	$l = 2$		$l = 3$	
	$\frac{4}{3}\omega_n^2$	$\lvert Q_{nl}\rvert$	$\frac{4}{3}\omega_n^2$	$\lvert Q_{nl}\rvert$
p_5	107.4	...	121.2	0.05834
p_4	79.25	0.06790	89.95	0.07896
p_3	55.30	0.09422	63.55	0.1087
p_2	35.63	0.1378	41.81	0.1521
p_1	20.35	0.2199	24.61	0.2137
f	10.90	0.3043	12.55	0.2184
g_1	6.553	0.2258	9.023	0.2062
g_2	3.771	0.1064	5.648	0.1297
g_3	2.430	0.07217	3.849	0.09794
g_4	1.694	0.04345	2.790	0.07182
g_5	1.248	0.03147	2.116	0.05460
g_6	0.9585	0.02024	1.660	0.04065
g_7	0.7593	0.01509	1.337	0.03079
g_8	0.6165	0.009930	1.100	0.02295
g_9	0.5106	0.007556	0.9218	0.01731
g_{10}	0.4300	0.004990	0.7834	0.01287
g_{11}	0.3671	0.003869	0.6740	0.009668
g_{12}	0.3171	0.002528	0.5861	0.007162
g_{13}	0.2768	0.002008	0.5144	0.005356
g_{14}	0.2437	0.001280	0.4551	0.003950

discrete Fourier transform itself, not in the rational fits, so these approximations could easily be improved were there any reason to do so.

The equations of this section render the functions $I_{lm}(y)$ "known," so we will not dwell on them any further.

VII. RESULTS FOR $n = 3$ POLYTROPES

We have computed eigenmodes for $n = 3$ polytropes using the method described by Robe (1968), essentially "parallel shooting" from surface and center to an intermediate point (cf. Keller 1968, § 2.4). Along with the eigenfunctions, we simultaneously integrate the overlap integral Q_{nl}. Results, for $l = 2$ and $l = 3$, are shown in Table 1. To facilitate comparison with Robe's work, the tabulated eigenfrequencies have been multiplied by the factor 4/3. Our eigenfrequencies agree with Robe's to about four significant figures for $l = 2$ (he does not list other l's).

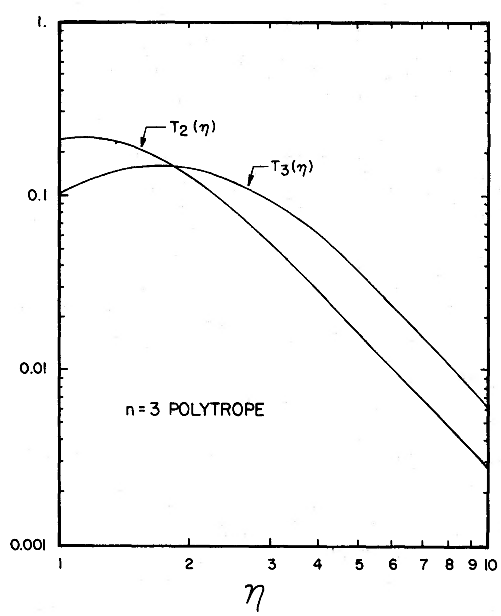

FIG. 1.—The dimensionless functions $T_2(\eta)$ and $T_3(\eta)$ which determine the energy deposition by the quadrupole and octopole tides, respectively.

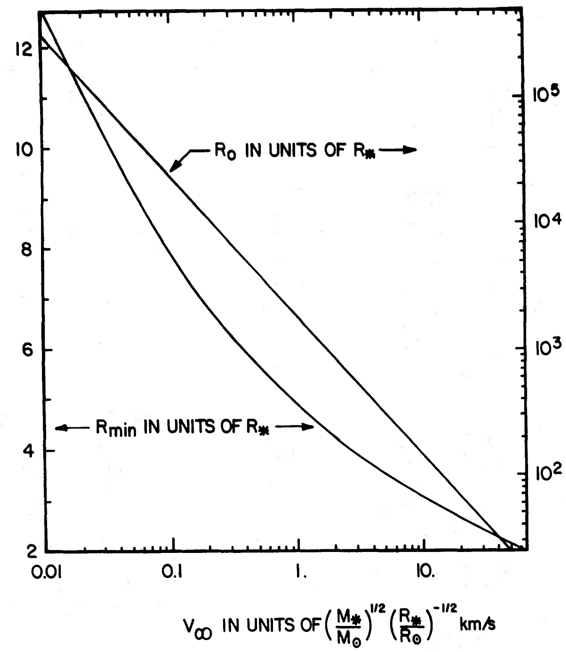

FIG. 2.—Capture impact parameters R_0 and corresponding periastron distances R_{min} for identical stars as a function of relative velocity at infinity.

With these modes, we compute the functions $T_2(\eta)$ and $T_3(\eta)$ according to equation (41), using the rational approximations and recurrence relations of § VI. The results are shown in Figure 1, subject to one remark: The figure (which is on log-log axes) shows straight-line power-law behavior for larger values of η. In fact, strictly the modes of Table 1 give a straight line only over a finite range of η, followed by an exponential falloff. However, when more and more modes are included in order of decreasing eigenfrequency, the straight line extends to higher and higher values of η before the exponential sets in. Physically, larger η means slower tidal perturbations. All stellar models have an *accumulation point* of modes near zero frequency, corresponding to almost-neutral advective modes (g modes). As η increases, more and more of these modes are excited (albeit slightly). This results in the power-law behavior which is shown (extrapolated) in the figure. (This point was overlooked in FPR, where an exponential falloff, correct for a finite set of modes, was conjectured.)

The functions $T_2(\eta)$ and $T_3(\eta)$ are "universal" in the sense that they hold for stars of arbitrary mass and radius (as long as an $n = 3$ polytrope is an adequate approximation to their envelope structure). Given two stars of particular masses M_1 and M_2, and radii R_1 and R_2, one can obtain their particular capture cross section σ or impact parameter R_0 as a function of their relative velocity,

$$\sigma(v) \equiv \pi R_0(v)^2 . \qquad (51)$$

Using R_{min} as a parametrization, one computes in succession (here $M_T = M_1 + M_2$, $\mu = M_1M_2/M_T$)

$$\eta_1 = \left(\frac{M_1}{M_T}\right)^{1/2}\left(\frac{R_{min}}{R_1}\right)^{3/2}, \qquad \eta_2 = \left(\frac{M_2}{M_T}\right)^{1/2}\left(\frac{R_{min}}{R_2}\right)^{3/2}, \qquad (52)$$

$$\Delta E = \frac{GM_2^2}{R_1}\left[\left(\frac{R_1}{R_{min}}\right)^6 T_2(\eta_1) + \left(\frac{R_1}{R_{min}}\right)^8 T_3(\eta_1)\right] + \frac{GM_1^2}{R_2}\left[\left(\frac{R_2}{R_{min}}\right)^6 T_2(\eta_2) + \left(\frac{R_2}{R_{min}}\right)^8 T_3(\eta_2)\right], \qquad (53)$$

$$v = (2\Delta E/\mu)^{1/2} , \qquad (54)$$

$$R_0(v) = (2GM_TR_{min}/v^2)^{1/2} . \qquad (55)$$

For the case of identical stars, $M_1 = M_2$ and $R_1 = R_2$, the resulting capture impact parameters and values of R_{min} are shown in Figure 2. If all distances are measured in units of the stellar radii R_1 and velocities in units of $(M_1/M_\odot)^{1/2}(R_1/R_\odot)^{-1/2}$ km s^{-1}, then Figure 2 holds for any specific values of M_1 and R_1. Note that the unit of velocity does not differ much from 1 km s^{-1} over most of the main sequence.

The function $R_0(v)$ is very nearly a power law:

$$R_0 \approx 1940v^{-1.1} \quad \text{(in above units)} . \qquad (56)$$

This is because for identical stars η is never less than 2, and $T_2(\eta)$ is always in its power-law regime (see Fig. 1). For stars of unequal mass, η can be as small as ~ 1 (eq. [31]), and a power law would not be so exact.

Let us complete the calculation for identical stars by computing the capture rate per unit volume of an isothermal star cluster with rms velocity dispersion v_0 and number density of stars N. The distribution of *relative* velocities is

$$n(v)d^3v = (4\pi v_0^2/3)^{-3/2} \exp(-3v^2/4v_0^2) 4\pi v^2 dv\,, \tag{57}$$

so the capture rate for a particular star is (cf. Clayton 1968, chap. 4)

$$\Gamma = N\langle\sigma v\rangle = N(4\pi v_0^2/3)^{-3/2} \int_0^\infty v\sigma(v) \exp(-3v^2/4v_0^2) 4\pi v^2 dv\,, \tag{58}$$

which, using equations (51) and (56) and doing the integral, gives

$$\Gamma = 1.25 \times 10^{-19} \left(\frac{R_1}{R_\odot}\right)^{0.9} \left(\frac{M_1}{M_\odot}\right)^{1.1} \left(\frac{v_0}{10\ \mathrm{km\ s^{-1}}}\right)^{-1.2} \left(\frac{N}{10^4\ \mathrm{pc^{-3}}}\right) \mathrm{s^{-1}}\,. \tag{59}$$

Here we have converted all quantities to physical units. The capture rate per unit volume is

$$r = \frac{1}{2} N\Gamma = 6.24 \times 10^{-16} \left(\frac{R_1}{R_\odot}\right)^{0.9} \left(\frac{M_1}{M_\odot}\right)^{1.1} \left(\frac{v_0}{10\ \mathrm{km\ s^{-1}}}\right)^{-1.2} \left(\frac{N}{10^4\ \mathrm{pc^{-3}}}\right)^2 \mathrm{pc^{-3}\ s^{-1}}\,. \tag{60}$$

The factor $\frac{1}{2}$ appears because we are dealing with identical stars.

VIII. DISCUSSION

To compare our results with the estimates of FPR, we take their parameters $R_1 \approx 5 \times 10^{10}$ cm, $M_1 \approx 0.5\ M_\odot$, $N \approx 10^4\ \mathrm{pc^{-3}}$, $v_0 \approx 10\ \mathrm{km\ s^{-1}}$, and obtain from equation (59) $\Gamma \approx 4.36 \times 10^{-20}\ \mathrm{s^{-1}}$. Setting their parameter $x = 3$ in the second equation after their equation (2) gives $\Gamma(\mathrm{FPR}) \approx 4.8 \times 10^{-20}\ \mathrm{s^{-1}}$, a remarkable agreement. Thus the FPR proposal to explain globular cluster X-ray sources remains viable. For a volume of $\frac{4}{3}\pi(0.5\ \mathrm{pc})^3$, we obtain from equation (60) at total rate of $\sim 1 \times 10^{-16}\ \mathrm{s^{-1}}$, and so we expect a globular cluster now (at an age $\sim 5 \times 10^{17}$ s) to have formed ~ 50 close binary systems.

It is of interest to compare the tidal capture rate with the rate for direct collisions. This can easily be computed from equations (58), (51), and (55) if $R_{\min}$ in equation (55) is replaced by $R_1 + R_2 = 2R_1$. We find

$$\Gamma_{\mathrm{coll}} = 7.73 \times 10^{-20} \left(\frac{R_1}{R_\odot}\right) \left(\frac{M_1}{M_\odot}\right) \left(\frac{v_0}{10\ \mathrm{km\ s^{-1}}}\right)^{-1} \left(\frac{N}{10^4\ \mathrm{pc^{-3}}}\right) \mathrm{s^{-1}}\,, \tag{61}$$

a result consistent with that of Hills and Day (1976). The dependence on R_1, M_1, and v_0 is very similar to that of the capture rate, but the numerical coefficient is of course smaller.

The rate of formation of close binaries by tidal capture should increase as cores relax to higher densities. Why we do not see globular clusters which have relaxed all the way to a singular state is not well understood (cf. Lynden-Bell 1975). If core densities substantially larger than $N \sim 10^4\ \mathrm{pc^{-3}}$ exist, then the influence of tidal capture (as well as collisions) may be dramatic.

It remains to be seen whether the curious physical process of two-body tidal capture has application to other astronomical situations.

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APPENDIX

RECURRENCE RELATIONS FOR $I_{lm}(y)$

From equation (43) we have

$$I_{l,m\pm 1} = \int_0^\infty dx (1 + x^2)^{-l} \cos[A(x) \pm 2 \tan^{-1} x]\,, \tag{A1}$$

where

$$A(x) = 2^{1/2} y(x + x^3/3) + 2m \tan^{-1} x\,. \tag{A2}$$

Using the expansion for cos $(A \pm B)$ and the relations

$$\cos(2\tan^{-1}x) = 2/(1+x^2) - 1\,, \qquad \sin(2\tan^{-1}x) = 2x/(1+x^2)\,, \tag{A3}$$

we find

$$I_{l,m\pm1} = 2I_{l+1,m} - I_{lm} \mp 2\int_0^\infty dx x(1+x^2)^{-l-1}\sin[A(x)]\,. \tag{A4}$$

Now integrate the last term by parts, differentiating $\sin[A(x)]$ in the integrand. The result is equation (44) of the text.

To obtain a recurrence relation on l for $m = 0$, start with the relation

$$\frac{1}{(1+x^2)^l} = \frac{1}{2l-2}\left\{\frac{2l-3}{(1+x^2)^{l-1}} + \frac{d}{dx}\left[\frac{x}{(1+x^2)^{l-1}}\right]\right\}. \tag{A5}$$

Then equation (43) gives

$$I_{l0} = \frac{2l-3}{2l-2}I_{l-1,0} + \frac{1}{2l-2}\int_0^\infty dx\cos[2^{1/2}y(x+x^3/3)]\frac{d}{dx}\left[\frac{x}{(1+x^2)^{l-1}}\right]. \tag{A6}$$

Integrating the second term twice by parts gives equation (45) of the text.

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Note added in proof.—Heggie (*M.N.R.A.S.*, **173**, 729 [1975]) has emphasized that the effect of binaries on the evolution of a cluster depends crucially on whether they are "hard" (orbital velocity larger than cluster dispersion velocity) or "soft" (the reverse). The tidal capture impact parameter $R_0(v)$ of equation (55) includes both hard and soft captures; however, it is readily shown [eqs. (52)–(55) and using the power-law regime of $T_2(\eta)$] that the capture impact parameter for hard captures alone is $\sim R_0(v)/2^{1/10}$, which is smaller by only $\sim 7\%$. For all practical purposes, then, tidal capture yields binaries which are hard in the Heggie sense.

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