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Analytical Modeling of the Optical Transfer Function of a Segmented Telescope with/without Adaptive Optics Correction of the Telescope's Dynamical Aberrations

Laurent Jolissaint[a], Brent Ellerbroek[b], George Angeli[b].

[a] Herzberg Institute of Astrophysics, 5071 W. Saanich Rd., Victoria, BC, V9E2E7, Canada
[b] Thirty Meter Telescope, 1200 E. California Blvd., MCode 102-8, Pasadena, CA, 91125

ABSTRACT

An all-analytic optical transfer function (OTF) tool for characterizing the performance of a large segmented telescope with/without adaptive optics (AO) correction of the telescope dynamical aberrations is presented. This tool is to be applied to the determination of the Thirty Meter Telescope (TMT) optical budget error, for both telescope aberrations and AO systems specifications. It takes into account the effect of the dynamical aberrations of all optical surfaces from all the hexagonal segments to the tertiary mirror, and includes as an option AO correction of these errors. Here we present the mathematical development of the method, and give an example of application to a 73 segments 10-m telescope, without AO correction.

Keywords: TMT, telescope aberrations, adaptive optics, segmented telescopes

1. INTRODUCTION

Because manufacturing imperfections, thermal effects, positioning errors, etc, a telescope optic is never perfect, even with active optics control. There is always a minimal amount of residual aberrations, that can be static and/or dynamic, and the figure error might be rather complex when considering segmentation. On the other hand, the next generation of extremely large segmented telescope (ELT) will be equipped with a zoo of adaptive optics systems (AO) modes - from ground layer to extreme AO, and these systems will correct for a certain amount of telescope errors, as these will be seen by the AO systems wavefront sensors. Therefore, a certain amount of the AO systems deformable mirror(s) (DM) will have to be dedicated to the correction of these unavoidable telescope aberrations, and the questions to ask are numerous. For instance: (1) for a given DM technology, how much telescope aberration on top of the atmospheric ones can the AO system tolerate, (2) if the DM stroke budget is not a critical issue, does it means that the telescope aberrations specifications can be somewhat relaxed ? (3) which telescope vibration modes are the more critical from the point of view of AO systems performance ? (4) and finally how does this telescope-AO interaction evolves from a given AO mode to an other ? are some AO modes more forgiving than others ?

To answer these questions, there is a need of a complete integrated model (IM) of the couple telescope plus AO systems. Such an IM is being developed for the TMT project, and takes into account the full mechanical & optical behavior of the telescope. While extremely accurate, such end-to-end (E2E) computer models have the common drawback of being very demanding in CPU and memory resources, and if one wants to assess statistical properties of the system's optical performance - for instance the long exposure point spread function (PSF) associated to a given telescope vibration mode, the IM has to be run for a long equivalent period, which can translate into hours or more, depending on parameters like wavelength, field size, etc. There is therefore a need for faster - and simpler - modeling tool of the system telescope plus AO, that can be used for a rapid exploration of the system's parameter space, and that can also serve as a sanity check of the IM results. In this paper, we present the development of an analytic model of the system telescope+AO's long exposure optical transfer function (OTF), for the case of dynamic aberrations, the static aberrations case having been presented previously in Jolissaint & Lavigne.[1] The method has been coded in a MATLAB code - OPTICA (Optical Performance of a Telescope Including Correction of the Aberrations by an AO system), and at the end of the paper we show some examples of application of the method on TMT.

Send correspondence to laurent.jolissaint@nrc-cnrc.gc.ca, telephone: +1 (250) 363-0071

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2. LONG EXPOSURE OPTICAL TRANSFER FUNCTION

Telescope aberrations can be separated into static ($\overline{\varphi}$) and dynamic ones ($\delta\varphi$):

$$\varphi(\mathbf{r},t) = \overline{\varphi}(\mathbf{r}) + \delta\varphi(\mathbf{r},t) \quad \text{where} \quad \langle \delta\varphi(\mathbf{r},t)\rangle_t = 0 \tag{1}$$

Static	Dynamic
polishing errors	wind pressure
positioning errors	telescope vibrations
mirrors print-through	thermal effects

Table 1. Possible sources of static and dynamic telescope aberrations - non exhaustive.

and we give in table 1 a list of possible sources of aberrations. One of the most convenient ways of characterizing the effect of these aberrations is via the system's optical transfer function (OTF) - Fourier transform of the PSF, and here as we are dealing with dynamical aberrations, the time average, or long exposure OTF is of particular interest. For an incoherent optical system, the long exposure OTF can be calculated from the pupil plane phasor auto-correlation,

$$\overline{\mathrm{OTF}}(\mathbf{f}) = \frac{1}{S_{\mathrm{P}}} \iint_{\mathbb{R}^2} P(\mathbf{r})\, P(\mathbf{r}+\lambda\mathbf{f})\, \exp\left\{-i\left[\overline{\varphi}(\mathbf{r}) - \overline{\varphi}(\mathbf{r}+\lambda\mathbf{f})\right]\right\} \left\langle \exp\left\{-i\left[\delta\varphi(\mathbf{r},t) - \delta\varphi(\mathbf{r}+\lambda\mathbf{f},t)\right]\right\}\right\rangle_t \mathrm{d}^2 r \tag{2}$$

where S_{P} is the pupil area, P the pupil transmission (1 inside, 0 outside), λ the optical wavelength, and $\mathbf{f}$ the focal plane angular frequency. The time average of the dynamic phasor in Eq. (2) can be developed further, considering the following: let us ξ be a random variable of probability distribution p_ξ; $\exp(-i\xi)$ is also a random variable, and its average is given by

$$\langle \exp(-i\xi)\rangle = \int p_\xi(\xi)\, \exp(-i\xi)\, \mathrm{d}\xi = \mathcal{F}\left\{p_\xi(\xi)\right\}(\nu = 1/2\pi) \tag{3}$$

which is simply the Fourier transform of p_ξ at the frequency $\nu = 1/2\pi$, In our case, ξ represents the phase difference between two points in the pupil plane, and with the assumption that the perturbed phase has a centered Gaussian statistics, it comes

$$p_\xi(\xi) = \frac{\exp\left[-\xi^2/(2\sigma_\xi^2)\right]}{\sigma_\xi\sqrt{2\pi}} \ \xrightarrow{\mathrm{FT}}\ \tilde{p_\xi}(\nu) = \exp(-2\pi^2\sigma_\xi^2\nu^2) \quad \textit{therefore} \quad \langle \exp(-i\xi)\rangle = \exp(-\sigma_\xi^2/2) \tag{4}$$

Applying this result to Eq. (2), it comes

$$\overline{\mathrm{OTF}}(\mathbf{f}) = \frac{1}{S_{\mathrm{P}}} \iint_{\mathbb{R}^2} P(\mathbf{r})\, P(\mathbf{r}+\lambda\mathbf{f})\, \exp\left\{-i\left[\overline{\varphi}(\mathbf{r}) - \overline{\varphi}(\mathbf{r}+\lambda\mathbf{f})\right]\right\}\, \exp\left[-\tfrac{1}{2} D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})\right] \mathrm{d}^2 r \tag{5}$$

where $D_{\delta\varphi}$ is the phase structure function, defined by

$$D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f}) = \left\langle \left[\delta\varphi(\mathbf{r},t) - \delta\varphi(\mathbf{r}+\lambda\mathbf{f},t)\right]^2 \right\rangle_t \tag{6}$$

3. DEVELOPMENT OF THE STRUCTURE FUNCTION IN MIRROR MODES

We assume here that dynamic aberrations are not correlated between the primary, secondary and tertiary mirrors. The validity of this assumption is not discussed here. In this case, the total dynamic phase structure function is simply given by the sum of each mirror structure function

$$D_{\delta\varphi}^{\mathrm{tot}}(\mathbf{r},\lambda\mathbf{f}) = \sum_{i=1}^{3} D_{\delta\varphi}^{\mathrm{M}_i}(\mathbf{r},\lambda\mathbf{f}) \tag{7}$$

In this section, our objective is to develop each of the three mirrors structure function into their modal basis, for instance the hexagonal Zernike basis for the segments, and the classical Zernike basis for the secondary and tertiary mirrors. These expressions will be used later to derive the stationary structure functions.

3.1. Primary mirror structure function

Let us expand the equation of the structure function definition

$$D_{\delta\varphi}^{\mathrm{M}_1}(\mathbf{r},\lambda\mathbf{f}) = \left\langle \delta\varphi_{\mathrm{M1}}^2(\mathbf{r},t)\right\rangle_t - 2\left\langle \delta\varphi_{\mathrm{M1}}(\mathbf{r},t)\delta\varphi_{\mathrm{M1}}(\mathbf{r}+\lambda\mathbf{f},t)\right\rangle_t + \left\langle \delta\varphi_{\mathrm{M1}}^2(\mathbf{r}+\lambda\mathbf{f},t)\right\rangle_t \tag{8}$$

Now we write the mirror phase as the sum of the phase of each of the N_S segments, shifted by the segment position vector $\mathbf{r}_i$,

$$\delta\varphi_{\mathrm{M1}}(\mathbf{r},t) = \sum_{i=1}^{N_S} \delta\varphi_i(\mathbf{r}-\mathbf{r}_i,t) \tag{9}$$

so

$$\delta\varphi_{\mathrm{M1}}^2(\mathbf{r},t) = \sum_{i=1}^{N_S} \delta\varphi_i^2(\mathbf{r}-\mathbf{r}_i,t) \;\;;\;\; \delta\varphi_{\mathrm{M1}}(\mathbf{r},t)\,\delta\varphi_{\mathrm{M1}}(\mathbf{r}+\lambda\mathbf{f},t) = \sum_{i,j=1}^{N_S} \delta\varphi_i(\mathbf{r}-\mathbf{r}_i,t)\,\delta\varphi_j(\mathbf{r}+\lambda\mathbf{f}-\mathbf{r}_j,t) \tag{10}$$

then we get

$$\begin{aligned} D_{\delta\varphi}^{\mathrm{M}_1}(\mathbf{r},\lambda\mathbf{f}) =& \sum_{i=1}^{N_S}\left[\left\langle \delta\varphi_i^2(\mathbf{r}-\mathbf{r}_i,t)\right\rangle_t + \left\langle \delta\varphi_i^2(\mathbf{r}-\mathbf{r}_i+\lambda\mathbf{f},t)\right\rangle_t\right] \\ & - 2\sum_{i,j=1}^{N_S} \left\langle \delta\varphi_i(\mathbf{r}-\mathbf{r}_i,t)\,\delta\varphi_j(\mathbf{r}-\mathbf{r}_j+\lambda\mathbf{f},t)\right\rangle_t \end{aligned} \tag{11}$$

Let us introduce now the modal aberrations of the segments. We write the phase on a given segment i as the sum over N_H hexagonal modes H_k, the time dependence being now on the modes coefficients $a_{i,k}(t)$,

$$\delta\varphi_i(\mathbf{r},t) = \sum_{k=1}^{N_H} a_{i,k}(t)\,H_k(\mathbf{r}) \tag{12}$$

Note that any kind of basis can be used here, as long as it is defined on a hexagonal support, and in particular, orthogonality on the support is not required. We can therefore use the Zernike basis - which is orthogonal on a disc support only - a disc including the segment transmission H (disc diameter equals the segment diameter point-to-point), so $H_k \equiv HZ_k$. The components of the structure function can now be written as

$$\delta\varphi_i^2(\mathbf{r},t) = \sum_{k,l=1}^{N_H} a_{i,k}(t)\,a_{i,l}(t)\,H_k(\mathbf{r})\,H_l(\mathbf{r}) \tag{13}$$

$$\delta\varphi_i(\mathbf{r})\,\delta\varphi_j(\mathbf{r}+\lambda\mathbf{f},t) = \sum_{k,l=1}^{N_H} a_{i,k}(t)\,a_{j,l}(t)\,H_k(\mathbf{r})\,H_l(\mathbf{r}+\lambda\mathbf{f}) \tag{14}$$

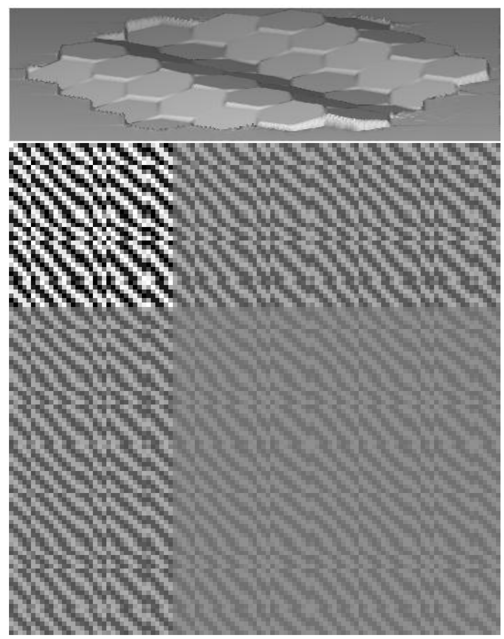

Figure 1. Top: Cosine wave across a 37 hexagonal segments mirror; Bottom: Segment-to-segment aberration covariance "matrix" associated to the wave, for piston and tip-tilt. Top blocks: covariance H_1 with $H_{1,2,3}$, second line H_2 with $H_{1,2,3}$ etc.

and after substitution in Eq. (11), it comes

$$\begin{aligned} D_{\delta\varphi}^{\mathrm{M}_1}(\mathbf{r},\lambda\mathbf{f}) =& \sum_{i=1}^{N_S}\sum_{k,l=1}^{N_H} \gamma_{i,k;i,l}\left[H_k(\mathbf{r}-\mathbf{r}_i)\,H_l(\mathbf{r}-\mathbf{r}_i) + H_k(\mathbf{r}-\mathbf{r}_i+\lambda\mathbf{f})\,H_l(\mathbf{r}-\mathbf{r}_i+\lambda\mathbf{f})\right] \\ & - 2\sum_{i,j=1}^{N_S}\sum_{k,l=1}^{N_H} \gamma_{i,k;j,l}\,H_k(\mathbf{r}-\mathbf{r}_i)\,H_l(\mathbf{r}-\mathbf{r}_j+\lambda\mathbf{f}) \end{aligned} \tag{15}$$

where

$$\gamma_{i,k;j,l} \equiv \left\langle a_{i,k}(t)\, a_{j,l}(t) \right\rangle_t \tag{16}$$

defines the aberration covariance - over time - between segments i & j and modes H_k & H_l. For visualization purpose, these covariances can be arranged into a square matrix of $N_H \times N_H$ square blocks of dimension $N_S \times N_S$. An example of such a covariance "matrix" is shown in figure 1. It corresponds to the piston & tip-tilt aberrations generated by a cosine wave, $W(\mathbf{r}, t) = A \cos(\mathbf{k} \cdot \mathbf{r} + \omega t)$, traveling across a 37 segments hexagonal mirror at a constant velocity and direction. The covariance indexes symmetry $\gamma_{i,k;j,l} = \gamma_{j,k;i,l} = \gamma_{i,l;j,k} = \gamma_{j,l;i,k}$ is particularly obvious in this case.

3.2. Secondary & tertiary mirrors structure function

The secondary and tertiary mirrors structure functions have the same expressions,

$$D_{\delta\varphi}^{\mathrm{M}_{2,3}}(\mathbf{r}, \lambda\mathbf{f}) = \left\langle \delta\varphi^2_{\mathrm{M2,3}}(\mathbf{r}, t) \right\rangle_t - 2\left\langle \delta\varphi_{\mathrm{M2,3}}(\mathbf{r}, t)\, \delta\varphi_{\mathrm{M2,3}}(\mathbf{r} + \lambda\mathbf{f}, t) \right\rangle_t + \left\langle \delta\varphi^2_{\mathrm{M2,3}}(\mathbf{r} + \lambda\mathbf{f}, t) \right\rangle_t \tag{17}$$

We now develop the mirror phase in N_Z Zernike modes:

$$\delta\varphi_{\mathrm{M2,3}}(\mathbf{r}, t) = \sum_{k=1}^{N_Z} a_k(t)\, Z_k(\mathbf{r}) \tag{18}$$

therefore

$$\delta\varphi^2_{\mathrm{M2,3}}(\mathbf{r}, t) = \sum_{k,l=1}^{N_Z} a_k(t)\, a_l(t)\, Z_k(\mathbf{r})\, Z_l(\mathbf{r}) \;;\; \delta\varphi_{\mathrm{M2,3}}(\mathbf{r}, t)\, \delta\varphi_{\mathrm{M2,3}}(\mathbf{r} + \lambda\mathbf{f}, t) = \sum_{k,l=1}^{N_Z} a_k(t)\, a_l(t)\, Z_k(\mathbf{r})\, Z_l(\mathbf{r} + \lambda\mathbf{f}) \tag{19}$$

so we get

$$D_{\delta\varphi}^{\mathrm{M}_{2,3}}(\mathbf{r}, \lambda\mathbf{f}) = \sum_{k,l=1}^{N_Z} \zeta_{k,l} \left[Z_k(\mathbf{r}) Z_l(\mathbf{r}) - 2 Z_k(\mathbf{r}) Z_l(\mathbf{r} + \lambda\mathbf{f}) + Z_k(\mathbf{r} + \lambda\mathbf{f}) Z_l(\mathbf{r} + \lambda\mathbf{f}) \right] \tag{20}$$

where

$$\zeta_{k,l} \equiv \left\langle a_k(t)\, a_l(t) \right\rangle_t \tag{21}$$

defines the covariance of the secondary or tertiary mirrors aberrations. Using the index symmetry, $\zeta_{k,l} = \zeta_{l,k}$, Eq. (21) can be arranged as follows, to minimize the number of summations,

$$D_{\delta\varphi}^{\mathrm{M}_{2,3}}(\mathbf{r}, \lambda\mathbf{f}) = \sum_{k=1}^{N_Z} \zeta_{k,k} \left[Z_k(\mathbf{r}) - Z_k(\mathbf{r} + \lambda\mathbf{f}) \right]^2 + 2 \sum_{k=1}^{N_Z - 1} \sum_{l=k+1}^{N_Z} \zeta_{k,l} \left[Z_k(\mathbf{r}) - Z_k(\mathbf{r} + \lambda\mathbf{f}) \right] \left[Z_l(\mathbf{r}) - Z_l(\mathbf{r} + \lambda\mathbf{f}) \right] \tag{22}$$

4. STATIONARITY APPROXIMATION AND MODEL SIMPLIFICATION TO $\mathcal{O}(\mathbf{N}^2)$

The exact calculation of the long exposure OTF necessitates, for each focal plane angular frequency $\mathbf{f}$, a numerical integration over the pupil position $\mathbf{r}$ - see Eq. (5). This takes time, and does not match with the requirement of as fast OTF modeling tool. Fortunately, we know from previous developments in astronomical adaptive optics (AO) analytical modeling[2,3] that the complexity of the calculation can be greatly reduced by the use of the so-called stationary approximation. The idea is that if the dynamical phase statistics would not be a function of the position $\mathbf{r}$ in the telescope pupil, i.e. if the phase was stationary in the pupil, then it would be possible to write its structure function only as a function of the separation distance $\boldsymbol{\rho} = \lambda\mathbf{f}$ in the pupil plane, and consequently extract the structure function exponential $\exp\left[-D_{\delta\varphi}/2\right]$ from the OTF integral - Eq. (5). This would reduce the OTF calculation to a simple evaluation of an expression for each focal plane angular frequency.

Stationarity is realized for instance for the non-corrected atmospheric turbulent phase, at least in a local sense (not too close from complex or sharp structures), and it is shown for instance in Roddier[2] how the total telescope+atmosphere OTF can be written as a product of OTFs. In the case of the dynamical aberrations of a telescope - whether segmented or not - it depends on the type of aberrations, so stationarity is not generally valid. Consider for instance the two first simplest aberrations: piston and tilt. For a segmented aperture, if all the segments are identical and subject to the same aberration amplitude fluctuation, the phase statistics would be indeed stationary for the piston aberration, but not for tilt, as tilt is always null at the segment rotation center. Therefore, in general, the telescope phase is not stationary.

In order to simplify the calculation complexity, then, we have no choice but to assume that the phase *is* stationary, which can be realized by replacing the phase structure function by its spatial average - over $\mathbf{r}$ - in the pupil. Let us examine how this average must be defined, and what is the consequence of this stationarity approximation. We start by realizing that the long exposure OTF can be seen as proportional to a weighted average of $\exp\left[-D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})/2\right]$ in the pupil, i.e. more precisely we can define

$$\begin{aligned}\left\langle \exp\left[-\tfrac{1}{2}D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})\right]\right\rangle_{\mathbf{r}} &\equiv \frac{\overline{\mathrm{OTF}}(\mathbf{f})}{\mathrm{OTF}_{\mathrm{st}}(\mathbf{f})} \\ &= \frac{\iint_{\mathbb{R}^2} P(\mathbf{r})\,P(\mathbf{r}+\lambda\mathbf{f})\,\exp\left\{-i\left[\overline{\varphi}(\mathbf{r})-\overline{\varphi}(\mathbf{r}+\lambda\mathbf{f})\right]\right\}\,\exp\left[-\tfrac{1}{2}D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})\right]\mathrm{d}^2r}{\iint_{\mathbb{R}^2} P(\mathbf{r})\,P(\mathbf{r}+\lambda\mathbf{f})\,\exp\left\{-i\left[\overline{\varphi}(\mathbf{r})-\overline{\varphi}(\mathbf{r}+\lambda\mathbf{f})\right]\right\}\mathrm{d}^2r}\end{aligned} \tag{23}$$

where $\mathrm{OTF}_{\mathrm{st}}$ indicates the telescope static OTF. Now, from Jensen's inequality (see for instance[3]), we know that as the function $\exp(x)$ is a concave down function, the exponential of the average of a range $x\in[a,b]$ is smaller than the average of the exponential over $[\exp(a),\exp(b)]$, i.e. $\exp\langle x\rangle \le \langle\exp x\rangle$, so we must have

$$\exp\left[-\tfrac{1}{2}\left\langle D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})\right\rangle_{\mathbf{r}}\right] \le \left\langle \exp\left[-\tfrac{1}{2}D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})\right]\right\rangle_{\mathbf{r}} \tag{24}$$

Therefore, the average structure function would have to be computed using

$$\overline{D}_{\delta\varphi}(\lambda\mathbf{f}) = \frac{\iint_{\mathbb{R}^2} P(\mathbf{r})\,P(\mathbf{r}+\lambda\mathbf{f})\,\exp\left\{-i\left[\overline{\varphi}(\mathbf{r})-\overline{\varphi}(\mathbf{r}+\lambda\mathbf{f})\right]\right\}D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})\,\mathrm{d}^2r}{\iint_{\mathbb{R}^2} P(\mathbf{r})\,P(\mathbf{r}+\lambda\mathbf{f})\,\exp\left\{-i\left[\overline{\varphi}(\mathbf{r})-\overline{\varphi}(\mathbf{r}+\lambda\mathbf{f})\right]\right\}\mathrm{d}^2r} \tag{25}$$

but this is a complex quantity, yet by definition a structure function is positive and real. To move on, we therefore have no choice but to neglect the complex exponential term in Eq. (25), and define our average structure function with

$${}_{\mathbb{R}}\overline{D}_{\delta\varphi}(\lambda\mathbf{f}) \equiv \frac{\iint_{\mathbb{R}^2} P(\mathbf{r})\,P(\mathbf{r}+\lambda\mathbf{f})\,D_{\delta\varphi}(\mathbf{r},\lambda\mathbf{f})\,\mathrm{d}^2r}{\iint_{\mathbb{R}^2} P(\mathbf{r})\,P(\mathbf{r}+\lambda\mathbf{f})\,\mathrm{d}^2r} \tag{26}$$

where the symbol $\mathbb{R}$ is here to differentiate with the complex structure function definition. We expect this approximation to be of minor consequence, as the telescope static phase amplitude should be in principle small, so $\exp(i\varphi)\lesssim 1$. Now, thanks to the above stationarity approximation, each mirror structure function exponential can be extracted from the total OTF integral Eq. (5), and the global OTF can now be written as a product of individual OTFs,

$$\overline{\mathrm{OTF}}_{\{\approx\}}(\mathbf{f}) = \mathrm{OTF}_{\mathrm{st}}(\mathbf{f})\,\overline{\mathrm{OTF}}_{\mathrm{M}_1}(\mathbf{f})\,\overline{\mathrm{OTF}}_{\mathrm{M}_2}(\mathbf{f})\,\overline{\mathrm{OTF}}_{\mathrm{M}_3}(\mathbf{f}) \le \overline{\mathrm{OTF}}(\mathbf{f}) \tag{27}$$

where the symbol $\{\approx\}$ is there to recall the approximation, and $\overline{\mathrm{OTF}}_{\mathrm{M}_i} \equiv \exp(-\frac{1}{2}{}_{\mathbb{R}}\overline{D}_{\delta\varphi}^{\mathrm{M}_i})$. It is worth noting that as the atmospheric turbulence aberrations are independent of the telescope ones, it is possible to take them into account and simply multiply the telescope global OTF with the atmospheric OTF - including AO filtering or not. Such an AO corrected atmospheric OTF can be calculated using an all analytical modeling as well, as shown by Ellerbroek[4] or Jolissaint et al..[5]

4.1. Primary mirror stationary structure function

As $D_{\delta\varphi}^{\mathrm{M}_1}$ is a sum of products of segments modes, the pupil averaged structure function is basically a sum of correlation products. Using the notation $\mathcal{C}\{f(\mathbf{u});g(\mathbf{u})\}$ for the correlation product $\iint f(\mathbf{u})g(\mathbf{u}+\mathbf{v})d^2u$, we get, for the numerator of ${}_{\mathbb{R}}\overline{D}_{\delta\varphi}^{\mathrm{M}_1}$,

$$\begin{aligned}\mathrm{NUM}\left\{{}_{\mathbb{R}}\overline{D}_{\delta\varphi}^{\mathrm{M}_1}(\lambda\mathbf{f})\right\} = &\sum_{i=1}^{N_S}\sum_{k,l=1}^{N_H}\gamma_{i,k;i,l}\Big[\mathcal{C}\left\{H_k(\mathbf{r}-\mathbf{r}_i)\,H_l(\mathbf{r}-\mathbf{r}_i);P(\mathbf{r})\right\}+\mathcal{C}\left\{P(\mathbf{r});H_k(\mathbf{r}-\mathbf{r}_i)\,H_l(\mathbf{r}-\mathbf{r}_i)\right\}\Big] \\ &-2\sum_{i,j=1}^{N_S}\sum_{k,l=1}^{N_H}\gamma_{i,k;j,l}\,\mathcal{C}\left\{H_k(\mathbf{r}-\mathbf{r}_i);H_l(\mathbf{r}-\mathbf{r}_j)\right\}\end{aligned} \tag{28}$$

We will compute these correlations products via their Fourier transforms, using the property

$$\mathcal{F}\Big\{\mathcal{C}\big\{f(\mathbf{r}-\mathbf{r}_i);g(\mathbf{r}-\mathbf{r}_j)\big\}\Big\} = \exp\left[i2\pi\boldsymbol{\nu}\cdot(\mathbf{r}_i-\mathbf{r}_j)\right]\tilde{f}^\star(\boldsymbol{\nu})\,\tilde{g}(\boldsymbol{\nu}) \tag{29}$$

where the symbol $\star$ indicates complex conjugation, and $\boldsymbol{\nu}$ is the pupil plane spatial frequency vector. Applying Eq. (29) to Eq. (28), we find

$$\begin{aligned}\mathcal{F}\Big\{\mathrm{NUM}\big\{{}_{\mathbb{R}}\overline{D}^{\mathrm{M}_1}_{\delta\varphi}(\lambda\mathbf{f})\big\}\Big\} = {} & 2\widetilde{P}(\boldsymbol{\nu})\sum_{i=1}^{N_S}\sum_{k,l=1}^{N_H}\gamma_{i,k;i,l}\,\Re\big\{\exp(-i2\pi\boldsymbol{\nu}\cdot\mathbf{r}_i)\,\mathcal{F}\{H_k(\mathbf{r})\,H_l(\mathbf{r})\}\big\} \\ & -2\sum_{i,j=1}^{N_S}\sum_{k,l=1}^{N_H}\gamma_{i,k;j,l}\widetilde{H}^\star_k(\boldsymbol{\nu})\,\widetilde{H}_l(\boldsymbol{\nu})\exp\left[i2\pi\boldsymbol{\nu}\cdot(\mathbf{r}_i-\mathbf{r}_j)\right]\end{aligned} \tag{30}$$

where $\Re$ is the real part operator, and we made use of the fact that the pupil is symmetric, so $\widetilde{P}^\star = \widetilde{P}$. If we use the indexes symmetries, i.e. $(k,l)\equiv(l,k)$ and $(i,j)\equiv(j,i)$, we can restrict the number of summations, and using the relation $-\sin(x)=\cos(x+\pi/2)$, we find

$$\begin{aligned}&\mathcal{F}\Big\{\mathrm{NUM}\big\{{}_{\mathbb{R}}\overline{D}^{\mathrm{M}_1}_{\delta\varphi}(\lambda\mathbf{f})\big\}\Big\} = 2\widetilde{P}(\boldsymbol{\nu})\sum_{k=1}^{N_H}\mathcal{F}\big\{H_k^2(\mathbf{r})\big\}\sum_{i=1}^{N_S}\gamma_{i,k;i,k}\cos(2\pi\boldsymbol{\nu}\cdot\mathbf{r}_i) \\ &+4\widetilde{P}(\boldsymbol{\nu})\sum_{k=1}^{N_H-1}\sum_{l=k+1}^{N_H}\Re\Big\{i^{\,p_{k,l}}\mathcal{F}\big\{H_k(\mathbf{r})\,H_l(\mathbf{r})\big\}\Big\}\sum_{i=1}^{N_S}\gamma_{i,k;i,l}\cos(2\pi\boldsymbol{\nu}\cdot\mathbf{r}_i+p_{k,l}\,\pi/2) \\ &-2\sum_{i=1}^{N_S}\Big[\sum_{k=1}^{N_H}\gamma_{i,k;i,k}|\widetilde{H}_k(\boldsymbol{\nu})|^2+2\sum_{k=1}^{N_H-1}\sum_{l=k+1}^{N_H}\gamma_{i,k;i,l}(1-p_{k,l})\widetilde{H}^\star_k(\boldsymbol{\nu})\,\widetilde{H}_l(\boldsymbol{\nu})\Big] \\ &-4\sum_{i=1}^{N_S-1}\sum_{j=i+1}^{N_S}\cos[2\pi\boldsymbol{\nu}\cdot(\mathbf{r}_i-\mathbf{r}_j)]\Big[\sum_{k=1}^{N_H}\gamma_{i,k;j,k}|\widetilde{H}_k(\boldsymbol{\nu})|^2+2\sum_{k=1}^{N_H-1}\sum_{l=k+1}^{N_H}\gamma_{i,k;j,l}(1-p_{k,l})\widetilde{H}^\star_k(\boldsymbol{\nu})\,\widetilde{H}_l(\boldsymbol{\nu})\Big]\end{aligned} \tag{31}$$

where $p_{k,l}$ is a parity factor defined by the segment modes azimuthal indexes (following Zernike polynomials indexation),

$$p_{k,l}=p_{l,k}=[m(k)+m(l)] \mod 2 = \begin{cases}0 & \text{if the product } H_kH_l \text{ is even,}\\ 1 & \text{if the product } H_kH_l \text{ is odd.}\end{cases} \tag{32}$$

What we need now is an expression for the denominator of ${}_{\mathbb{R}}\overline{D}^{\mathrm{M}_1}_{\delta\varphi}$, given by the autocorrelation of the telescope pupil (see Eq. (26)). Again, the pupil autocorrelation can be easily calculated in the Fourier space. We get

$$\iint_{\mathbb{R}^2}P(\mathbf{r})P(\mathbf{r}+\lambda\mathbf{f})\mathrm{d}^2r = \sum_{i=1}^{N_S}\sum_{j=1}^{N_S}\mathcal{C}\big\{S(\mathbf{r}-\mathbf{r}_i);S(\mathbf{r}-\mathbf{r}_j)\big\} \tag{33}$$

which becomes, with Eq. (29), and noting that the segment transmission can be written as the piston mode H_1,

$$\mathcal{F}\Big\{\mathrm{DEN}\big\{{}_{\mathbb{R}}\overline{D}^{\mathrm{M}_1}_{\delta\varphi}(\lambda\mathbf{f})\big\}\Big\} = |\widetilde{H_1}(\boldsymbol{\nu})|^2\sum_{i,j=1}^{N_S}\exp\left[i2\pi\boldsymbol{\nu}\cdot(\mathbf{r}_i-\mathbf{r}_j)\right] = |\widetilde{H_1}(\boldsymbol{\nu})|^2\Big\{N_S+2\sum_{i=1}^{N_S-1}\sum_{j=i+1}^{N_S}\cos\left[2\pi\boldsymbol{\nu}\cdot(\mathbf{r}_i-\mathbf{r}_j)\right]\Big\} \tag{34}$$

4.2. Secondary and tertiary mirrors stationary structure function

We now compute the M2 & M3 average structure functions. Applying Eq. (26) to Eq. (20), we get

$$\mathrm{NUM}\big\{{}_{\mathbb{R}}\overline{D}^{\mathrm{M}_{2,3}}_{\delta\varphi}(\lambda\mathbf{f})\big\} = \sum_{k,l=1}^{N_Z}\zeta_{k,l}\Big[\mathcal{C}\big\{Z_k(\mathbf{r})Z_l(\mathbf{r});P(\mathbf{r})\big\}-2\,\mathcal{C}\big\{Z_k(\mathbf{r});Z_l(\mathbf{r})\big\}+\mathcal{C}\big\{P(\mathbf{r});Z_k(\mathbf{r})Z_l(\mathbf{r})\big\}\Big] \tag{35}$$

Applying Eq. (29), and using the parity property $\zeta_{k,l} = \zeta_{l,k}$ to minimize the number of summations, we find

$$\mathcal{F}\Big\{\mathrm{NUM}\big\{{}_{\mathbb{R}}\overline{D}^{\mathrm{M}_{2,3}}_{\delta\varphi}(\lambda\mathbf{f})\big\}\Big\} = 2\sum_{k=1}^{N_Z}\zeta_{k,k}\Big[\widetilde{P}(\boldsymbol{\nu})\mathcal{F}\big\{Z_k^2(\mathbf{r})\big\} - |\widetilde{Z_k}(\boldsymbol{\nu})|^2\Big] + 4\sum_{k=1}^{N_Z-1}\sum_{l=k+1}^{N_Z}\zeta_{k,l}\,(1-p_{k,l})\Big[\widetilde{P}(\boldsymbol{\nu})\mathcal{F}\big\{Z_k(\mathbf{r})Z_l(\mathbf{r})\big\} - \widetilde{Z}_k^{\star}(\boldsymbol{\nu})\widetilde{Z}_l(\boldsymbol{\nu})\Big] \quad (36)$$

where $p_{k,l}$ is the parity factor, defined the same way as before - Eq. (32). As with the segmented structure function, Fourier transforms of Zernike polynomials can be calculated analytically or from FFTs, and stored for use. Note that the denominator of M2 & M3 structure functions is the same as for the segmented structure function, and is given by Eq. (34).

4.3. Geometrical matrices

At this point, we can see that the calculation of the long exposure OTF via the stationary structure function involves in one side the telescope mirrors aberrations covariances $\gamma_{i,k;j,l}$ and $\zeta_{i,j}$ and on the other side some geometrical objects - pupil Fourier transform, modes products Fourier transforms, etc - that do not depend on the aberration amplitudes, but only on the segment positions and modes indexes. We will call them the *geometrical matrices*, and if we examine carefully the stationary structure functions expressions Eq. (31) and Eq. (36), we can see that they can be restricted to three quantities: (1) the pupil Fourier transform $\widetilde{P}$, (2) the segments modes products Fourier transform $\mathcal{F}\{H_k H_l\}$ and (3) the secondary (tertiary) modes products Fourier transform $\mathcal{F}\{Z_k Z_l\}$. Now, these three Fourier transforms can be calculated either analytically, or via the application of an FFT algorithm on the pupil transmission and modes products. We show in Ref. 6 how the hexagonal modes Fourier transform can be calculated, and the pupil transmission Fourier transform is easily calculated from $\widetilde{P}(\boldsymbol{\nu}) = \widetilde{H_1}(\boldsymbol{\nu})\sum_j \exp(i2\pi\boldsymbol{\nu}\cdot\mathbf{r}_j)$. As the secondary and tertiary mirrors are non segmented plain disc mirrors, it is certainly easier to use in that case the classical method - via FFT of the Zernike modes - rather than a more complex analytical approach.

5. ADAPTIVE OPTICS DYNAMIC ABERRATIONS CORRECTION MODEL

In this paragraph we show how AO correction can be included in the calculation of the long exposure OTF. Basically, in the framework of the stationarity approximation, the AO system contribution can be seen as a spatial frequency filtering of the telescope phase; this AO filter is made up of different independent terms, and as far as only the dynamical aberrations of the telescope are concerned, we will have (1) the DM high-pass filter, (2) the wavefront sensor (WFS) spatial aliasing (not a real filter, but at least a linear operator) and (3) the servo-lag filter.

Servo-lag filtering comes from the effect of the AO loop delay and WFS integration time, which average out the high temporal and spatial frequencies of the WFS measurement, then decrease the quality of the correction at these frequencies. This error term is of second order relative to the others, though, and for now will be neglected in our model, for the sake of simplicity. It can be included easily, if there are concerns that telescope dynamical aberrations might be present near or above the AO system temporal bandwidth. The DM filter is approximated here as a perfect square high-pass filter, correcting for all aberrations in the square domain $|\nu_x| < 1/(2\Delta_{\mathrm{DM}}); |\nu_y| < 1/(2\Delta_{\mathrm{DM}})$, where Δ_{DM} is the DM actuator pitch as seen from the primary mirror, and none outside this domain. Such a model, while very simple, gives reasonably accurate results when compared with more sophisticated DM filter models, and is certainly acceptable in the spirit of the first order approach we are pursuing in this work (sophisticated models can be used in our approach too, but we do not want to develop this discussion here). The effect of WFS aliasing is to fold back into the low spatial frequency (square) domain the uncorrected high spatial frequency fluctuations of the corrugated phase, with some weighting.

The spectrum (Fourier transform) of the residual phase of the telescope can therefore be written as

$$\widetilde{\delta\varphi_r}(\boldsymbol{\nu},t) = \Gamma_{\mathrm{ao}}(\boldsymbol{\nu})\widetilde{\delta\varphi}(\boldsymbol{\nu},t) - \widetilde{\mathcal{A}}\big\{\Gamma_{\mathrm{ao}}(\boldsymbol{\nu})\widetilde{\delta\varphi}(\boldsymbol{\nu},t)\big\} \equiv \widetilde{F}_{\mathrm{ao}}(\boldsymbol{\nu})\,\widetilde{\delta\varphi}(\boldsymbol{\nu},t) \quad (37)$$

where $\Gamma_{\rm ao}$ represents the DM high-pass filter, and $\mathcal{A}$ the aliasing operator. The aliased spectrum is subtracted from the telescope phase: indeed, the AO system tries to correct for this term, as it is wrongly seen as a low spatial frequency aberration by the WFS. The aliasing operator is given by[6]:

$$\widetilde{\mathcal{A}}\{\widetilde{\phi}\} = \frac{\nu_x\,\nu_y}{\nu_x^2+\nu_y^2} \sum_{k,l=-\infty,\,\neq 0}^{\infty} (-1)^{k+l} \left(\frac{\nu_x}{\nu_y - l/\Delta_{\rm DM}} + \frac{\nu_y}{\nu_x - k/\Delta_{\rm DM}} \right) \widetilde{\phi}(\nu_x - k/\Delta_{\rm DM}, \nu_y - l/\Delta_{\rm DM}) \tag{38}$$

In practice, the sum over k, l can be limited over a few indexes, up to a spatial frequency that includes most of the high spatial frequency phase spectrum ($\pm$ 3 indexes is generally sufficient).

This filter can now be applied to the telescope phase to modelize the AO correction. Practically, this is done by replacing the segments modes H_i by their filtered version, $H_i^{\rm ao} = F_{\rm ao} * H_i$, which becomes, in the Fourier space, $\widetilde{H}_i^{\rm ao} = \widetilde{F}_{\rm ao}\,\widetilde{H}_i$. In the stationary structure function Eq. (31), the telescope phase appears in the geometrical matrices $\mathcal{F}\{H_k(\mathbf{r})\,H_l(\mathbf{r})\}$ and $\widetilde{H}_k^\star(\boldsymbol{\nu})\,\widetilde{H}_l(\boldsymbol{\nu})$: with AO correction, these will have to be replaced with respectively

$$\mathcal{F}\{H_k^{\rm ao}(\mathbf{r})\,H_l^{\rm ao}(\mathbf{r})\} = \mathcal{F}\Big\{\mathcal{F}^{-1}\{\widetilde{F}_{\rm ao}(\boldsymbol{\nu})\widetilde{H}_k(\boldsymbol{\nu})\}\mathcal{F}^{-1}\{\widetilde{F}_{\rm ao}(\boldsymbol{\nu})\widetilde{H}_l(\boldsymbol{\nu})\}\Big\} \tag{39}$$

$$\widetilde{H}_k^{\star,\rm ao}(\boldsymbol{\nu})\,\widetilde{H}_l^{\rm ao}(\boldsymbol{\nu}) = |\widetilde{F}_{\rm ao}(\boldsymbol{\nu})|^2\,\widetilde{H}_k^\star(\boldsymbol{\nu})\,\widetilde{H}_l(\boldsymbol{\nu}) \tag{40}$$

6. EXAMPLE OF OTF CALCULATION ON A 73 SEGMENTS MIRROR

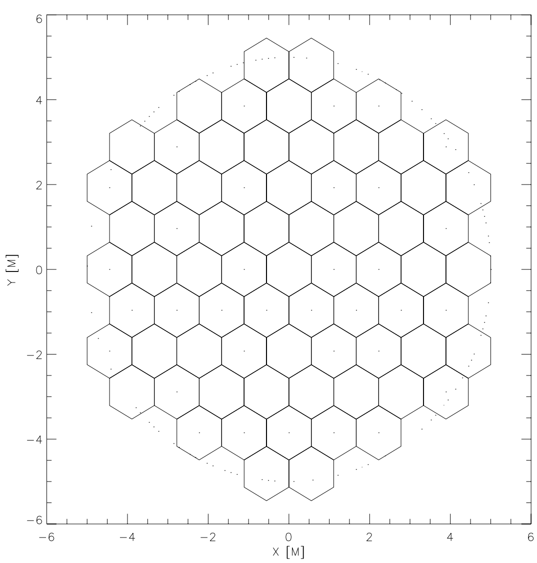

Figure 2. 73 segments telescope pupil. Mirror diameter 10.967 m, segments diameter 1.279 m, segment gaps 4 mm.

In this section, we demonstrate the capability of our method on a very simple case: the phase aberration is a cosine wave (figure 1) $W(\mathbf{r},t) = A\cos(\mathbf{k}\cdot\mathbf{r} + \omega t)$ in translation across the primary mirror of a 73 hexagonal segments telescope - figure 2. Please note that the simulation parameters have been chosen for illustration purpose only and do not reflect anything related to the TMT telescope design. We assume that the segments are infinitely stiff, therefore we only take into account the piston and tip-tilt hexagonal modes H_1, H_2 and H_3. The segment-to-segment aberration covariances can be derived easily for the cosine wave: we find, with $\theta = k_x(x_i - x_j) + k_y(y_i - y_j)$,

$$\gamma_{i,1;j,1} = 1/2\,A^2\,\cos\theta \tag{41}$$

$$\gamma_{i,1;j,2} = 2\,A^2\,(k_x/a)\,\sin\theta \tag{42}$$

$$\gamma_{i,1;j,3} = 2\,A^2\,(k_y/a)\,\sin\theta \tag{43}$$

$$\gamma_{i,2;j,2} = 8\,A^2\,(k_x/a)^2\,\cos\theta \tag{44}$$

$$\gamma_{i,2;j,3} = 8\,A^2\,(k_x k_y/a^2)\,\cos\theta \tag{45}$$

$$\gamma_{i,3;j,3} = 8\,A^2\,(k_y/a)^2\,\cos\theta \tag{46}$$

and the other covariances can be constructed using the symmetry properties of $\gamma_{i,k;j,l}$. The wave period has been set to 4 segment diameter, i.e. 5.116 m, and the propagation orientation angle to $+70.5°$ relative to the x-axis. As the pupil and focal plane are related via a Fourier transform, a periodic perturbation of period P in the pupil plane will generate replicas of the telescope PSF shifted by an amount λ/P. In our case, we find that this shift must be equal to 0.04", and this is exactly the distance between the consecutive dots we can measure in the telescope PSFs shown in the figure 3. In the same figure, we see how the PSF gets more and more spread along the wave propagation direction with the increase of the wave amplitude. Note that this example does not include any AO correction yet, as this option was not coded at the time of the writing of this report. More complete tests including a comparison with the full order N^4 will be published elsewhere.

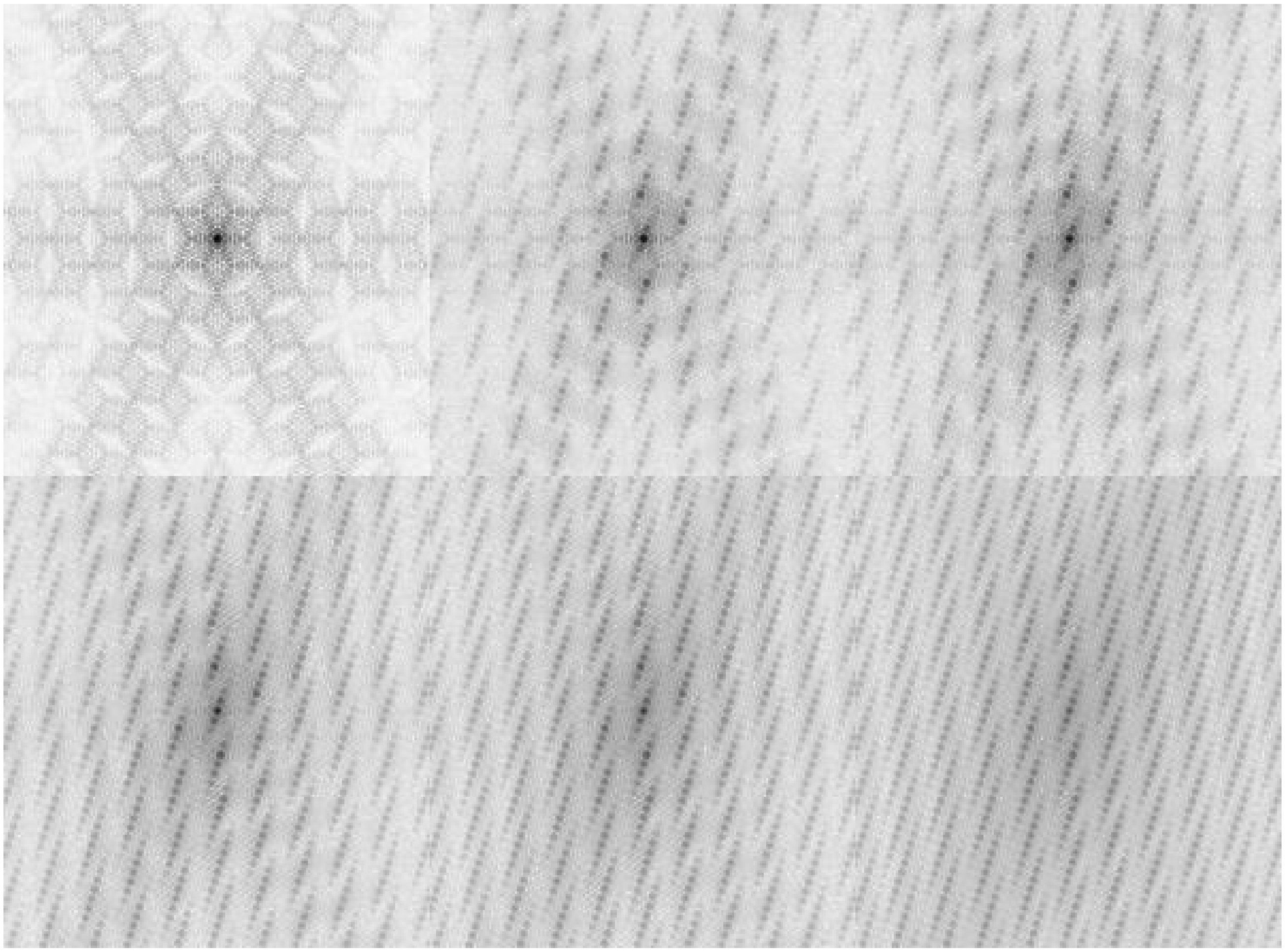

Figure 3. Telescope PSF and cosine aberration wave. Imaging wavelength 1 μm, field size 1.87". From top left to bottom right, wave amplitude 0, 80, 160, 320, 480 and 800 nm.

7. CONCLUSION

In this paper, we have shown how an all-analytic modeling of the long exposure OTF of a hexagonal segmented telescope with dynamical aberration can be built. In the same framework, adaptive optics correction can be included as well, using an equivalent spatial filter including both the deformable mirror filtering and the wavefront sensor aliasing. The model has been tested on a simple case - a 73 segments telescope with a cosine wave perturbation, and preliminary results shows the validity of the approach. Further tests are needed, though, to assess the limit of validity of the working assumptions, particularly the phase stationarity approximation. Our model provides a fast and first order tool that can be used to (1) evaluate the impact of a given telescope vibration mode, (2) define the dynamical specifications of the telescope aberrations (3) check the results of more sophisticated end-to-end models (integrated model) of the segmented telescope.

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