

# The absence of energy currents in an equilibrium state and chiral anomalies

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## Abstract

A long time ago F. Bloch showed that in a system of interacting non-relativistic particles the net particle-number current must vanish in any equilibrium state. Bloch's argument, while very simple, does not generalize easily to the energy current. We devise an alternative argument which proves the vanishing of the net energy currents in equilibrium states of lattice systems as well as systems of non-relativistic particles with finite-range potential interactions. We discuss some applications of these results. In particular, we show that neither a 1d lattice system nor a 1d system of non-relativistic particles with finite-range potential interactions can flow to a Conformal Field Theory with unequal left-moving and right-moving central charges.

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## I. INTRODUCTION

An old argument of F. Bloch explained in detail by D. Bohm [1] shows that in an equilibrium thermodynamic state of a quasi-1d system of non-relativistic particles the net particle number current through any section is zero.<sup>1</sup> Very recently H. Watanabe extended Bloch's argument to lattice systems [2]. This result appears very general and likely to apply to currents of other conserved quantities. For example, a non-vanishing energy current in an equilibrium state would conflict with Fourier's law. However, the Bloch-Bohm argument does not immediately apply to the energy current, since it relies in an essential way on the quantization of the particle number which does not have an analog in the case of energy.

There are also examples of systems where currents do not vanish in certain equilibrium states. Consider a Conformal Field Theory in 1+1d with unequal central charges for left-moving and right-moving sectors. If these central charges are  $c_L$  and  $c_R$ , then the energy current at temperature  $T$  can be shown to be [3, 4]

$$\langle j_E \rangle = \frac{\pi T^2}{12} (c_R - c_L). \quad (1)$$

Similarly, if we consider a 1+1d CFT with a  $U(1)$  current algebra with a left-moving level  $k_L$  and a right-moving level  $k_R$ , then one can show [5] that the net  $U(1)$  current at a chemical potential  $\mu$  and an arbitrary temperature is

$$\langle j_Q \rangle = \pi \mu (k_R - k_L). \quad (2)$$

This raises the question about the precise conditions under which equilibrium-state currents vanish. Note that in both examples symmetry anomalies (i.e. obstructions to gauging a symmetry) are present:  $k_R - k_L$  measures the anomaly of the  $U(1)$  symmetry, while  $c_R - c_L$  measures the anomaly of the diffeomorphism symmetry. In the case of the  $U(1)$  symmetry, this can be used to argue that CFTs with non-zero  $k_R - k_L$  cannot appear as the long-wavelength limit of a wide variety of systems, including lattice systems with an on-site  $U(1)$  symmetry and systems of non-relativistic particles. Indeed, if a system can be consistently coupled to a  $U(1)$  gauge field, the same must hold for its long-wavelength limit, ruling out effective field theory descriptions with nontrivial  $U(1)$  anomalies. This argument does not

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<sup>1</sup> Here by an equilibrium thermodynamic state we mean either a ground state or a Gibbs state. A quasi-1d system is a system which is infinitely-extended in only one direction.

work for energy currents since most microscopic Hamiltonians cannot be coupled to gravity in any natural way. Nevertheless, it is widely believed that a quasi-1d system of particles with short-range interactions or a quasi-1d lattice system with short-range interactions cannot flow to a CFT with a non-zero  $c_R - c_L$ .

In this letter, we prove the absence of equilibrium-state energy currents for quasi-1d lattice systems with finite-range interactions as well as for systems of non-relativistic particles with finite-range potential interactions. An immediate corollary is that such systems cannot flow to a 1+1d CFT with a non-zero  $c_R - c_L$ . We make only very modest assumptions, which roughly amount to the absence of phase transitions in quasi-1d systems at positive temperatures.<sup>2</sup> We also give an alternative derivation of the vanishing of the  $U(1)$  current in certain continuous systems and explain how 1+1d chiral CFTs perturbed by a chemical potential manage to evade this conclusion.

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## II. LATTICE SYSTEMS

A lattice system in  $d$  spatial dimensions has a Hilbert space  $V = \otimes_{p \in \Lambda} V_p$ , where  $\Lambda$  (“the lattice”) is a discrete subset of  $\mathbb{R}^d$ , and  $V_p$  is finite-dimensional. An observable is localized at a point  $p \in \Lambda$  if it has the form  $A \otimes 1_{\Lambda \setminus p}$  for some  $A : V_p \rightarrow V_p$ . An observable is localized at a subset  $Q \subset \Lambda$  if it commutes with all observables localized at any  $p \in \Lambda \setminus Q$ . For a compact  $Q$ , this implies that an observable localized at  $Q$  is a sum of products of observables localized at all  $p \in Q$ . An observable localized in a compact set  $Q$  will be called a local observable with support  $Q$ .

The Hamiltonian of a lattice system has the form

$$H = \sum_{p \in \Lambda} H_p, \tag{3}$$

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<sup>2</sup> Many-body localization transitions are not accompanied by divergent susceptibilities and are not regarded as phase transitions for our purposes.

where the operators  $H_p : V \rightarrow V$  are Hermitian. We assume that the Hamiltonian has a finite range  $\delta$ , which means that there exists  $\delta > 0$  such that  $[H_p, A] = 0$  if  $A$  is an observable localized at  $q \in \Lambda$  and  $|p - q| > \delta$ . In other words, each  $H_p$  is a local observable supported in a ball of radius  $\delta$  centered at  $p$ . Therefore  $[H_p, H_q] = 0$  if  $|p - q| > 2\delta$ . We also assume that the operators  $H_p$  are uniformly bounded: there exists  $C > 0$  such that  $\|H_p\| < C$  for all  $p \in \Lambda$ .

Since we are interested in the thermodynamic limit, the subset  $\Lambda$  is assumed infinite. Still, we want the Hilbert space of every finite piece of the system to be finite-dimensional, so we assume that  $\Lambda \cap K$  is finite for any compact  $K \subset \mathbb{R}^d$ . We also assume a certain uniformity of the lattice  $\Lambda$ , in that the distance between points of  $\Lambda$  is uniformly bounded from below. On the other hand, since we will be studying the net energy current through a section of a system, we assume that  $\Lambda$  is compact in all but one direction. More precisely, we assume that  $\Lambda$  is contained in a subset  $\mathbb{R} \times W$ , where  $W \subset \mathbb{R}^{d-1}$  is compact. Then we may lump together all points at a particular  $x \in \mathbb{R}$  and regard our system as one-dimensional. From now on we focus on 1d lattice systems, with the lattice  $\Lambda \subset \mathbb{R}$  and a finite-range Hamiltonian  $H = \sum_{x \in \Lambda} H_x$ .

We are interested in Gibbs states at temperature  $T = 1/\beta$ . We assume that the state is clustering, i.e. correlators of local operators  $\langle AB \rangle - \langle A \rangle \langle B \rangle$  approach zero as  $L_{AB} = \text{dist}(\text{supp}(A), \text{supp}(B)) \rightarrow \infty$ . We also assume that the Kubo pairing

$$\langle\langle A; B \rangle\rangle = \frac{1}{\beta} \int_0^\beta du \langle A(-iu)B \rangle - \langle A \rangle \langle B \rangle \quad (4)$$

of any two local operators  $A$  and  $B$  decays at least as  $L_{AB}^{-(1+\epsilon)}$  for some  $\epsilon > 0$ . Here  $A(t) = e^{iHt} A e^{-iHt}$ . The Kubo pairing arises when studying the response of the system to an infinitesimal perturbation  $H \rightarrow H + \lambda B$  [6]. Then the change in the expectation value of  $A$  is

$$\delta \langle A \rangle = -\lambda \beta \langle\langle A; B \rangle\rangle. \quad (5)$$

Thus, up to a factor  $\beta$ , the Kubo pairing of local operators is the same as a generalized susceptibility for local perturbations. The assumption that the Kubo pairing decays faster than  $1/L_{AB}$  ensures that perturbations of the form  $\sum_x B_x$  where  $B_x$  is a local operator supported in a ball of fixed size centered at  $x$ , and  $\|B_x\|$  is uniformly bounded, lead to a well-defined change in the expectation values of all local observables.

These decay assumptions are likely true for any positive temperature. Correlators of local observables decay exponentially away from phase transitions. One also expects the generalized susceptibilities for uniform perturbations to be finite away from phase transitions, although we are not aware of a proof. Since we are considering 1d systems, we do not expect any phase transitions at positive temperatures. Zero-temperature states can then be treated as  $T \rightarrow 0$  limits of Gibbs states.

### III. ENERGY CURRENTS IN LATTICE SYSTEMS

By definition, the energy at a site  $x \in \Lambda$  is  $H_x$ , and its time derivative is

$$\frac{dH_x}{dt} = i[H, H_x] = i \sum_{y \in \Lambda} [H_y, H_x]. \quad (6)$$

Hence we define the energy current from site  $y$  to site  $x$  to be

$$J_{xy}^E = i[H_y, H_x]. \quad (7)$$

Any  $a \in \mathbb{R} \setminus \Lambda$  divides  $\Lambda$  into two parts:  $\Lambda = \Lambda_+(a) \cup \Lambda_-(a)$ , where  $\Lambda_+(a)$  (resp.  $\Lambda_-(a)$ ) is defined by the condition  $x > a$  (resp.  $x < a$ ). The net current from  $\Lambda_-(a)$  to  $\Lambda_+(a)$  is

$$J^E(a) = \sum_{x > a, y < a} J_{xy}^E. \quad (8)$$

For any  $a, b \notin \Lambda$  and  $b > a$  we have

$$J^E(b) - J^E(a) = - \sum_{a < x < b} i[H, H_x]. \quad (9)$$

Since  $\langle [H, A] \rangle = 0$  for any local observable  $A$ , we get that  $\langle J^E(a) \rangle$  is independent of  $a$ .

Consider an infinitesimal variation of the Hamiltonian  $\delta H$  such that  $\delta H_x = 0$  for sufficiently large positive  $x$ . Then

$$\delta \langle J^E(a) \rangle = \langle \delta J^E(a) \rangle - \beta \langle \langle J^E(a); \delta H \rangle \rangle. \quad (10)$$

Pick an  $R > 0$  such that  $a + R \notin \Lambda$ . Using the equation (9) and the property of the Kubo pairing

$$\langle \langle [H, A]; B \rangle \rangle = \frac{1}{\beta} \langle [B, A] \rangle, \quad (11)$$

the second term in (10) can be written as

$$-\beta \langle \langle J^E(a); \delta H \rangle \rangle = -\beta \langle \langle J^E(a+R); \delta H \rangle \rangle - \sum_{a < x < a+R} \langle i[\delta H, H_x] \rangle. \quad (12)$$

On the other hand, varying eq. (9) we can re-write the first term in eq. (10) as

$$\langle \delta J^E(a) \rangle = \langle \delta J^E(a+R) \rangle + \sum_{a < x < a+R} \langle i[\delta H, H_x] \rangle. \quad (13)$$

Hence

$$\delta \langle J^E(a) \rangle = \langle \delta J^E(a+R) \rangle - \beta \langle \langle J^E(a+R); \delta H \rangle \rangle. \quad (14)$$

Now let us take the limit  $R \rightarrow +\infty$ . The first term is zero for sufficiently large  $R$  since  $\delta H_x = 0$  for sufficiently large positive  $x$ , and both  $H_x$  and  $\delta H_x$  are assumed to have finite support, for all  $x \in \Lambda$ . Using the assumed decay of the Kubo pairing, the second term can be estimated to be no larger than  $C/R^\epsilon$  for some  $C > 0$  and thus goes to zero for  $R \rightarrow +\infty$ . Hence  $\delta \langle J^E(a) \rangle = 0$ .

A similar argument shows that  $\delta \langle J^E(a) \rangle = 0$  for variations of  $H$  which vanish for sufficiently large negative  $x$ . Therefore  $\delta \langle J^E(a) \rangle = 0$  for arbitrary variations of  $H$  within the allowed class.

Now we consider the temperature dependence of the net energy current. Re-scaling simultaneously the temperature  $T \mapsto \lambda T$  and the Hamiltonian  $H \mapsto \lambda H$  leaves the state unchanged, thus for any observable  $A$  which does not depend explicitly on  $T$  or  $H$  we have

$$\left( T \frac{d}{dT} + \lambda \frac{d}{d\lambda} \right) \langle A \rangle_\lambda = 0, \quad (15)$$

where  $\langle A \rangle_\lambda$  denotes the average over a Gibbs state with a Hamiltonian  $\lambda H$  and temperature  $T$ . More generally, if  $A$  is multiplied by  $\lambda^p$  under  $H \mapsto \lambda H$ , then

$$\left( T \frac{d}{dT} + \lambda \frac{d}{d\lambda} \right) \left\langle \frac{A}{T^p} \right\rangle_\lambda = 0. \quad (16)$$

The energy current  $J^E$  has  $p = 2$ . On the other hand, since re-scaling the Hamiltonian by a constant factor is an allowed deformation, we have

$$\frac{d}{d\lambda} \langle J^E(a) \rangle_\lambda = 0. \quad (17)$$

Therefore

$$\langle J^E(a) \rangle = CT^2, \quad (18)$$

where  $C$  is some constant which is unchanged under all allowed variations of the Hamiltonian.

Finally, let us assume that our state can be continuously connected to the maximally mixed state  $T = \infty$ . Then the above temperature dependence is incompatible with the fact that the operators  $J^E(a)$  are bounded, unless  $C = 0$ . Thus the net energy current vanishes.

#### IV. ENERGY CURRENTS IN PARTICLE SYSTEMS

There is a well-known difficulty with defining an energy current in systems of particles with a potential interaction. It is related to the non-locality of the potential interaction. The standard way of dealing with this difficulty involves a formal expansion of the potential  $V(\mathbf{x} - \mathbf{y})$  into an infinite sum of zero-range potentials (the Dirac delta-function  $\delta(\mathbf{x} - \mathbf{y})$  and its derivatives) [7]. For 1d or quasi-1d systems with a finite-range potential there is an alternative approach: one can define the energy density and the energy current which are local only in one dimension. This is sufficient for our purposes. To simplify notation, we will only discuss the strictly 1d case, but the modifications to the quasi-1d case are minor.

The second-quantized Hamiltonian has the form

$$H = \int dx k(x) + \int dx \rho(x)W(x) + \frac{1}{2} \int dx dy \rho(x)\rho(y)V(x, y), \quad (19)$$

where  $k(x)$  is the usual kinetic energy density operator,

$$k(x) = \frac{1}{2m} \partial_x \psi^\dagger(x) \partial_x \psi(x), \quad (20)$$

$\rho(x) = \psi^\dagger(x)\psi(x)$  is the particle density operator,  $W(x)$  is the external potential, and  $V(x, y) = V(y, x)$  is the pairwise interaction potential. We define the potential energy density as

$$\pi(x) = W(x)\rho(x) + \frac{1}{2}\rho(x) \int V(x, y)\rho(y)dy, \quad (21)$$

and the total energy density as  $h(x) = k(x) + \pi(x)$ . To find the energy current  $j^E(x)$ , we need to solve the conservation equation

$$i[H, h(x)] = -\partial_x j^E(x). \quad (22)$$

When computing the commutator on the left, the following identities are useful:

$$[\rho(x), \rho(y)] = 0, \quad [\rho(x), j^Q(y)] = \frac{i}{m} \rho(y) \partial_y \delta(x - y), \quad i[k(x), \rho(y)] = j^Q(x) \partial_x \delta(x - y), \quad (23)$$

where  $j^Q = \frac{-i}{2m}(\psi^\dagger \partial_x \psi - (\partial_x \psi^\dagger) \psi)$  is the particle-number current. A solution has the form  $j^E(x) = j_k^E(x) + j_\pi^E(x)$ , where

$$j_k^E(x) = \frac{-i}{4m^2} (\partial_x \psi^\dagger(x) \partial_x^2 \psi(x) - \partial_x^2 \psi^\dagger(x) \partial_x \psi(x)), \quad (24)$$

and

$$j_\pi^E(x) = j^Q(x)W(x) + j^Q(x) \int V(x, y)\rho(y)dy + \frac{i}{2m}\rho(x) (\partial_x V(x, y))|_{y=x} + \frac{1}{2} \int_{z < x < y} (\partial_y j^Q(y)\rho(z) - \partial_z j^Q(z)\rho(y)) V(y, z)dydz. \quad (25)$$

One can check that the energy current is Hermitean. Note that if the potential  $V(x, y)$  has range  $\delta$ , i.e. vanishes whenever  $|x - y| \geq \delta$ , all terms in  $j_\pi^E(x)$  are quasi-local: they commute with all local observables whose support is farther from  $x$  than  $\delta$ . It is important for what follows that a quasi-local energy current can be constructed for an arbitrary symmetric finite-range potential  $V(x, y)$ .

For any bounded function  $\phi : \mathbb{R} \rightarrow \mathbb{R}$  we can consider a modified potential  $V_\phi(x, y) = \phi(x)\phi(y)V(x, y)$ , which is also symmetric and finite-range. If  $\phi(x)$  is small in some region of space, particle interactions are suppressed there. We claim that the energy current  $\langle j^E(a) \rangle$  does not change as one varies  $\phi$ , provided the Kubo pairings of local operators decay at least as  $1/L^{1+\epsilon}$ . Indeed, consider an arbitrary infinitesimal variation of  $\phi(x)$ . It can be decomposed into a sum of two contributions: one vanishing for  $x \ll 0$  and another one vanishing for  $x \gg 0$ . It is sufficient to show that the the energy current is unchanged under the two separately. Let us consider a variation of  $\phi$  which vanishes for  $x \gg 0$ . As in the previous section, using the conservation equation and its variation we find:

$$\delta \langle j^E(a) \rangle = \langle \delta j^E(a + R) \rangle - \beta \langle \langle j^E(a + R); \delta H \rangle \rangle, \quad (26)$$

where  $R$  is arbitrary. Taking the limit  $R \rightarrow +\infty$ , we conclude that  $\langle j^E(a) \rangle$  is unchanged under arbitrary infinitesimal variations of  $\phi$  which vanish for  $x \gg 0$ . An identical argument shows that  $\langle j^E(a) \rangle$  is unchanged under arbitrary infinitesimal variations of  $\phi$  which vanish for  $x \ll 0$ .

Now let us take a constant  $\phi = 1$  and decrease it to 0 (while keeping the temperature fixed). Unless one passes through a phase transition with divergent susceptibilities,  $\langle j^E(a) \rangle$  is unchanged. Since it vanishes when  $V(x, y) = 0$ , it must also be zero for the initial

potential  $V(x, y)$ . It is widely believed that finite-temperature phase transitions cannot occur in systems of 1d particles with finite-range potential interactions. Assuming this, we proved that the equilibrium energy current vanishes for all  $T > 0$  and therefore also for  $T = 0$ .

## V. $U(1)$ CURRENTS IN CONTINUOUS SYSTEMS

In this section we discuss why Bloch's result does not apply to some continuous systems, like chiral 1+1d CFTs, but does apply to others, like systems of non-relativistic particles.

Consider a continuous system with a Hamiltonian  $H = \int h(\mathbf{x}) d^n x$ . We assume time-translation symmetry but not necessarily spatial translation symmetry. The space is an  $n$ -dimensional manifold equipped with a Riemannian metric. Local coordinates are denoted  $x^M$ ,  $M = 1, \dots, n$ . The energy density  $h(\mathbf{x})$  is assumed to be quasi-local, in the sense that there exists a  $\delta > 0$  such that for any strictly local observable  $A(\mathbf{x})$  (i.e. a function of fields and their derivatives at the point  $\mathbf{x}$ ) we have  $[h(\mathbf{y}), A(\mathbf{x})] = 0$  whenever  $|\mathbf{x} - \mathbf{y}| > \delta$ . Both local field theories (whether Lorentz-invariant or not) and non-relativistic particles interacting via a finite-range potential are examples of such systems.

In the presence of a  $U(1)$  symmetry, we have a local charge density operator  $\rho(\mathbf{x})$  such that the generator of  $U(1)$  is  $Q = \int \rho(\mathbf{x}) d^n x$ . We also assume that there exists a quasi-local  $U(1)$  current  $j_Q^M(\mathbf{x})$  satisfying

$$i[H, \rho(\mathbf{x})] = -\partial_M j_Q^M(\mathbf{x}). \quad (27)$$

This condition is satisfied for field theories as well as for systems of non-relativistic particles if  $\rho$  is the particle-number density.

Suppose we can promote  $U(1)$  symmetry to a local  $U(1)$  symmetry with generators

$$Q_f = \int \rho(\mathbf{x}) f(\mathbf{x}) d^n x, \quad (28)$$

where  $f(\mathbf{x})$  is an arbitrary compactly supported function. Requiring  $[Q_f, Q_g] = 0$  for all compactly supported  $f, g$ , we get

$$[\rho(\mathbf{x}), \rho(\mathbf{y})] = 0. \quad (29)$$

Note that this is violated in 1+1d CFTs with nonzero  $k_R - k_L$  thanks to the Schwinger term in the commutator. Using (29) we can deduce that the net  $U(1)$  current, if present, cannot

depend on the chemical potential  $\mu$ . To see this, consider an infinitesimal deformation of the Hamiltonian of the form

$$\delta H = \int f(\mathbf{x})\rho(\mathbf{x})d^n x. \quad (30)$$

The condition (29) assures us that the current is undeformed,  $\delta j_Q^M = 0$ , regardless of  $f(\mathbf{x})$ . Following the same procedure as in Section IV, we find the change in the expectation value of  $J_Q(a) = \int_W j_Q^x(a, w)d^{n-1}w$ :

$$\delta \langle J_Q(a) \rangle = -\beta \langle \langle J_Q(a+R); \int f(\mathbf{y})\rho(\mathbf{y})d^n y \rangle \rangle, \quad (31)$$

where  $R$  is arbitrary. Writing a general bounded  $f(\mathbf{x})$  as a sum of two functions vanishing for  $x \gg 0$  and  $x \ll 0$ , respectively, we argue as in Sections III, IV that the change in the net  $U(1)$  current vanishes. Then, taking  $f$  to be constant, we deduce that the net  $U(1)$  current is independent of the chemical potential, provided we stay away from phase transitions (which should not occur for 1d systems at positive temperature).

In 1+1d CFTs with a non-zero  $k_R - k_L$  the condition (29) is violated by Schwinger terms, hence the current can and does depend on the chemical potential. In fact, repeating the above argument but taking into account the Schwinger term, one recovers the dependence (2).

Finally let us discuss the possible dependence of the net  $U(1)$  current on the temperature. In our favorite class of examples (non-relativistic particles with a finite-range potential interaction) the following strengthening of Eq. (27) holds:

$$i[h(\mathbf{x}), \rho(\mathbf{y})] = -\partial_M^y (j_Q^M(\mathbf{y})\delta(\mathbf{x} - \mathbf{y})). \quad (32)$$

More generally, suppose one can gauge the time-translation symmetry, and that the generator of an infinitesimal transformation  $t \mapsto t + f(\mathbf{x})$  is

$$T_f = \int f(\mathbf{x})h(\mathbf{x})d^n x. \quad (33)$$

If we assume that this can be done without destroying the  $U(1)$  symmetry (this is akin to requiring the absence of a mixed gravity-gauge anomaly), then  $[T_f, Q]$  must be a total derivative, for all  $f$ . This implies

$$i[h(\mathbf{x}), \rho(\mathbf{y})] = -\partial_M^y A^M(\mathbf{x}, \mathbf{y}), \quad (34)$$

for some quasi-local operator  $A^M(\mathbf{x}, \mathbf{y})$ . Integrating over  $\mathbf{x}$ , we find the conservation equation (27) with  $j_Q^M(\mathbf{y}) = \int A^M(\mathbf{x}, \mathbf{y})d^n x$ .

Eq. (34) is a natural generalization of (32). Assuming it, we can constrain the dependence of the net  $U(1)$  current on the temperature as follows. Consider a modified Hamiltonian  $H_\phi = \int \phi(\mathbf{x})h(\mathbf{x}) d^n x$ , where  $\phi(\mathbf{x})$  is positive bounded function. The Hamiltonian  $H_\phi$  still admits a quasi-local conserved  $U(1)$  current given by

$$j_{\phi,Q}^M(\mathbf{y}) = \int \phi(\mathbf{x})A^M(\mathbf{x},\mathbf{y})d^n x. \quad (35)$$

Now consider varying  $\phi$ . The above formula and quasi-locality of  $A^M$  show that the variation of  $j_{\phi,Q}^M(\mathbf{y})$  depends only on the variation of  $\phi(\mathbf{x})$  in the  $\delta$ -neighborhood of  $\mathbf{y}$ . The by-now-familiar argument then shows that the net  $U(1)$  current does not depend on  $\phi(\mathbf{x})$  provided we stay away from phase transitions.

In particular, re-scaling  $h(\mathbf{x})$  by a positive constant  $\lambda$  does not change the expectation value of the  $U(1)$  current. On the other hand, re-scaling the Hamiltonian (and the chemical potential, if nonzero) by a constant can be absorbed into re-scaling the temperature. As before, this implies

$$\left(T \frac{d}{dT} + \mu \frac{d}{d\mu}\right) \left\langle \frac{J_Q(a)}{T} \right\rangle = 0, \quad (36)$$

therefore

$$\langle J_Q(a) \rangle = T f\left(\frac{\mu}{T}\right), \quad (37)$$

for some function  $f$ . If we also assume that the relation (29) holds, then  $\langle J_Q(a) \rangle$  is independent of  $\mu$ , and we get

$$\langle J_Q(a) \rangle = C'T, \quad (38)$$

where  $C'$  is a constant. For a system of non-relativistic particles with a finite-range bounded potential, the interactions become negligible at high temperatures. Since the  $U(1)$  current vanishes in the high-temperature limit, we must have  $C' = 0$ . Of course, for particle systems there is a much simpler proof of the vanishing of the net  $U(1)$  current [1]. In unitary 1+1d CFTs one also finds that  $C' = 0$ . We cannot exclude the possibility that there exist chiral 1+1d field theories without conformal or Lorentz invariance where the net  $U(1)$  current obeys (38) with a nonzero  $C'$ . To evade the Bloch-Bohm argument, such field theories would need to admit excitations with irrational charges.

## VI. APPLICATIONS

We have shown that the equilibrium energy current vanishes identically both for infinitely-extended 1d lattice systems with finite-range interactions and quasi-1d systems of non-relativistic particles with finite-range potential interactions. The only assumption was the absence of phase transitions at positive temperatures, which is expected to hold for quasi-1d systems like the ones we considered. More precisely, we assumed that generalized susceptibilities are finite and that equal-time correlators of local operators cluster for all positive temperatures. In view of eq. (1), our result implies that such system cannot flow to a 1+1d CFT with a nonzero  $c_R - c_L$ .

It is well-known that a nonzero  $c_R - c_L$  may appear in 1+1d CFTs describing the gapless edge of a gapped 2d system. The above result shows that  $c_R - c_L$  is determined by the bulk properties of the 2d material and does not depend on the edge. Indeed, we may consider a strip of the 2d phase bounded by two different edges (with opposite orientations) as a 1d material, and then the above result shows that  $c_R - c_L$  must be equal for the two edges. This is not surprising since  $c_R - c_L$  is related to the bulk thermal Hall conductance. The same comments apply, *mutatis mutandis*, to  $k_R - k_L$  and the Hall conductance.

The vanishing of the net  $U(1)$  current is implicitly assumed in the definition of magnetization. Usually, one says that since  $\nabla \cdot \langle \mathbf{j}_Q(\mathbf{x}) \rangle = 0$  in an equilibrium state, one can define the magnetization by the equation  $\nabla \times \mathbf{M}_Q(\mathbf{x}) = \langle \mathbf{j}_Q(\mathbf{x}) \rangle$ . If the net current in some direction were nonzero, the magnetization so defined would be either multi-valued (if the direction is periodically identified) or would grow linearly with distance. In either case, it could not be regarded as a local property of the material. Bloch's theorem shows that the magnetization is well-defined. An analogous quantity for energy currents ("energy magnetization") is of importance in the theory of the thermal Hall effect [8]. Our results on the vanishing of the net energy current justify the existence of energy magnetization in a wide variety of situations.

One final remark is that the vanishing of  $U(1)$  and energy currents strictly applies to infinite systems in equilibrium. In a large but finite quasi-1d system, like a macroscopic ring, there can be a non-vanishing  $U(1)$  or energy current in equilibrium. However, it must

go to zero when the size of the system goes to infinity.

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