

EQUIVALENCE OF SOBOLEV NORMS INVOLVING GENERALIZED HARDY OPERATORS

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ABSTRACT. We consider the fractional Schrödinger operator with Hardy potential and critical or subcritical coupling constant. This operator generates a natural scale of homogeneous Sobolev spaces which we compare with the ordinary homogeneous Sobolev spaces. As a byproduct, we obtain generalized and reversed Hardy inequalities for this operator. Our results extend those obtained recently for ordinary (non-fractional) Schrödinger operators and have an important application in the treatment of large relativistic atoms.

1. INTRODUCTION AND MAIN RESULT

The Hardy inequality is of fundamental importance in many questions in harmonic analysis, partial differential equations, spectral theory and mathematical physics. This inequality comes in various forms and the form that we will be concerned with here is that if $0 < \alpha < d$, then

$$\| |p|^{\alpha/2} f \|_{L^2(\mathbb{R}^d)}^2 \geq \mathcal{H}_{d,\alpha} \| |x|^{-\alpha/2} f \|_{L^2(\mathbb{R}^d)}^2 \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d) \quad (1.1)$$

with a positive constant $\mathcal{H}_{d,\alpha}$. Here and in the following we use the notation

$$|p| = \sqrt{-\Delta}.$$

The optimal, that is, largest possible, value of the constant on the right side was found in [Her77, Theorem 2.5] and is given by

$$\mathcal{H}_{d,\alpha} = \frac{2^\alpha \Gamma((d + \alpha)/4)^2}{\Gamma((d - \alpha)/4)^2}.$$

Of course, the special case $\alpha = 2$, where the left side is $\|\nabla f\|_{L^2(\mathbb{R}^d)}^2$, had been known much longer and also the physically important special case $d = 3$ and $\alpha = 1$ appeared before [Kat66, Chapter 5, Equation (5.33)]. For alternative proofs of the inequality with the sharp constant we refer to [KPS81, Yaf99, FLS08, FS08].

Setting

$$a_* := -\mathcal{H}_{d,\alpha} = -\frac{2^\alpha \Gamma((d + \alpha)/4)^2}{\Gamma((d - \alpha)/4)^2},$$

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we infer from Hardy's inequality with optimal constant that the operator

$$\mathcal{L}_{a,\alpha} := |p|^\alpha + a|x|^{-\alpha} \quad \text{in } L^2(\mathbb{R}^d)$$

is non-negative for $a \geq a_*$. More precisely, the operator $\mathcal{L}_{a,\alpha}$ is defined as the Friedrichs extension of the corresponding expression on $C_c^\infty(\mathbb{R}^d)$. In some applications and, in particular, in our paper [FMSS], the norms

$$\left\| \mathcal{L}_{a,\alpha}^{s/2} f \right\|_{L^2(\mathbb{R}^d)}$$

for $a \geq a_*$ and $s > 0$ appear naturally and one needs to understand the relation between these norm and those in the special case $a = 0$, i.e.,

$$\left\| |p|^{\alpha s/2} f \right\|_{L^2(\mathbb{R}^d)}.$$

While it is relatively straightforward to see that the norms are equivalent for $s \in (0, 1]$ if $a < a_*$ (see Remark 1.2), the question becomes non-trivial for $s > 1$ and was only recently solved in the remarkable paper [KMV⁺18] in the local case $\alpha = 2$. An important consequence of their result is that for any $a > a_*$ there is an $s_{a,d,\alpha} > 1$ such that the norms are equivalent for all $0 < s < s_{a,d,\alpha}$. In fact, [KMV⁺18] also treat a more general problem with the $L^2(\mathbb{R}^d)$ norms replaced by those in $L^p(\mathbb{R}^d)$ for $1 < p < \infty$ and discuss the dependence on p .

The main result of our paper here will be the generalization of the results from [KMV⁺18] to the case $0 < \alpha < 2$. Our methods can probably be extended to cover the case of $L^p(\mathbb{R}^d)$ norms for $1 < p < \infty$, but we restrict ourselves to the case $p = 2$ which is the one relevant for the application we have in mind.

In order to state our main result precisely, for $0 < \alpha < d$ we define

$$\Psi_{\alpha,d}(\delta) := -2^\alpha \frac{\Gamma(\frac{\delta+\alpha}{2}) \Gamma(\frac{d-\delta}{2})}{\Gamma(\frac{d-\delta-\alpha}{2}) \Gamma(\frac{\delta}{2})} \quad \text{if } \delta \in (0, (d-\alpha)/2].$$

According to [FLS08, Lemma 3.2] the function $\delta \mapsto \Psi_{\alpha,d}(\delta)$ is continuous and strictly decreasing in $(0, (d-\alpha)/2]$ with

$$\lim_{\delta \rightarrow 0} \Psi_{\alpha,d}(\delta) = 0 \quad \text{and} \quad \Psi_{\alpha,d}\left(\frac{d-\alpha}{2}\right) = a^*.$$

As a consequence, we can define

$$\delta := \begin{cases} \Psi_{\alpha,d}^{-1}(a) & \text{if } a \in [a_*, 0], \\ 0 & \text{if } a \in (0, \infty). \end{cases} \quad (1.2)$$

The following is our main result.

Theorem 1.1 (Equivalence of Sobolev norms). *Let $\alpha \in (0, 2 \wedge d)$, $a \in [a_*, \infty)$ and let δ be defined by (1.2). Let $s \in (0, 2)$.*

(1) *If $\alpha \frac{s}{2} + \delta < \frac{d}{2}$, then*

$$\left\| |p|^{\alpha \frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} \lesssim_{d,\alpha,a,s} \left\| \mathcal{L}_{a,\alpha}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \quad (1.3)$$

(2) If $\alpha \frac{s}{2} < \frac{d}{2}$, then

$$\left\| \mathcal{L}_{a,\alpha}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} \lesssim_{d,\alpha,a,s} \left\| |p|^{\alpha \frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \quad (1.4)$$

Here and in what follows, we write $X \lesssim Y$ for non-negative quantities X and Y , whenever there is a constant A such that $X \leq A \cdot Y$. In order to emphasize that A depends on some parameter r , we will sometimes write $X \lesssim_r Y$. Moreover, $X \sim Y$ means $Y \lesssim X \lesssim Y$. In this case, we say X is *equivalent* to Y . Moreover, here and in what follows we use the notation

$$X \wedge Y := \min\{X, Y\} \quad \text{and} \quad X \vee Y := \max\{X, Y\}.$$

Remark 1.2. The crucial point of this theorem is that it allows for some $s > 1$ (unless $a = a_*$ in case of (1.3)). The inequalities for $s \leq 1$ can be easily seen as follows. By Hardy's inequality (1.1) we have the operator inequalities

$$\left(1 - \frac{a}{a_*}\right) |p|^\alpha \leq \mathcal{L}_{a,\alpha} \leq |p|^\alpha \quad \text{if } a < 0, \quad |p|^\alpha \leq \mathcal{L}_{a,\alpha} \leq \left(1 + \frac{a}{a_*}\right) |p|^\alpha \quad \text{if } a > 0,$$

and therefore, by operator monotonicity of roots (see, for instance, [Car10, Theorem 2.6]), for $0 < s < 1$,

$$\left(1 - \frac{a}{a_*}\right)^s |p|^{\alpha s} \leq \mathcal{L}_{a,\alpha}^s \leq |p|^{\alpha s} \quad \text{if } a < 0, \quad |p|^{\alpha s} \leq \mathcal{L}_{a,\alpha}^s \leq \left(1 + \frac{a}{a_*}\right)^s |p|^{\alpha s} \quad \text{if } a > 0.$$

This proves (1.3) and (1.4) for $0 < s \leq 1$.

Remark 1.3. Another crucial point of this theorem is that it covers the full range $a \geq a_*$ and $\alpha \leq 2 \wedge d$. In fact, as we explain now, if $\alpha < d/2$, then there is a simple proof of a stronger bound than (1.4) and if, in addition, a belongs to a restricted range, then there is also a simple proof of a stronger bound than (1.3).

Our first claim is that if $\alpha < d/2$, then (1.4) holds for all $s \leq 2$. By the same argument as in the previous remark it suffices to prove this for $s = 2$. In that case, we have, by (1.1),

$$\left\| \mathcal{L}_{a,\alpha} f \right\|_{L^2(\mathbb{R}^d)} \leq \left\| |p|^\alpha f \right\|_{L^2(\mathbb{R}^d)} + |a| \left\| |x|^\alpha f \right\|_{L^2(\mathbb{R}^d)} \leq \left(1 + |a| \mathcal{H}_{d,2\alpha}^{-1/2}\right) \left\| |p|^\alpha f \right\|_{L^2(\mathbb{R}^d)},$$

as claimed.

To state our second claim precisely, let $a_{**} := -\mathcal{H}_{d,2\alpha}^{1/2}$ (which is well-defined for $\alpha < d/2$). We will show below that $a_{**} > a_*$. We claim that for $|a| < |a_{**}|$ inequality (1.4) holds for all $s \leq 2$. Again it suffices to prove this for $s = 2$ and then we have, similarly as before,

$$\left\| \mathcal{L}_{a,\alpha} f \right\|_{L^2(\mathbb{R}^d)} \geq \left\| |p|^\alpha f \right\|_{L^2(\mathbb{R}^d)} - |a| \left\| |x|^\alpha f \right\|_{L^2(\mathbb{R}^d)} \geq \left(1 - |a| \mathcal{H}_{d,2\alpha}^{-1/2}\right) \left\| |p|^\alpha f \right\|_{L^2(\mathbb{R}^d)},$$

as claimed.

Let us finally show that $a_{**} > a_*$. We have, in terms of $\psi = \Gamma'/\Gamma$,

$$\begin{aligned} \ln |a_*| - \ln |a_{**}| &= 2 \ln \Gamma\left(\frac{d+\alpha}{4}\right) - 2 \ln \Gamma\left(\frac{d-\alpha}{4}\right) - \ln \Gamma\left(\frac{d+2\alpha}{4}\right) + \ln \Gamma\left(\frac{d-2\alpha}{4}\right) \\ &= 2 \int_{\frac{d-\alpha}{4}}^{\frac{d+\alpha}{4}} \psi(t) dt - \int_{\frac{d-2\alpha}{4}}^{\frac{d+2\alpha}{4}} \psi(t) dt \\ &= \int_{\frac{d}{4}}^{\frac{d+\alpha}{4}} (\psi(t) - \psi(t + \frac{\alpha}{4})) dt - \int_{\frac{d-2\alpha}{4}}^{\frac{d-\alpha}{4}} (\psi(t) - \psi(t + \frac{\alpha}{4})) dt \\ &> 0. \end{aligned}$$

The last inequality comes from the fact that both integration intervals are of length $\alpha/4$ and that the integrand is an increasing function, which in turn follows from the fact that ψ' is decreasing [Dav65, (6.4.1)].

As a consequence of (1.3) and Hardy's inequality (1.1) we obtain immediately a Hardy inequality for fractional powers of $\mathcal{L}_{a,\alpha}$.

Proposition 1.4 (Generalized Hardy inequality). *Let $\alpha \in (0, 2 \wedge d)$, $a \in [a_*, +\infty)$ and let δ be defined by (1.2). Then, for any $s > 0$ with $\alpha \frac{s}{2} + \delta < \frac{d}{2}$,*

$$\left\| |x|^{-\alpha s/2} f \right\|_2 \lesssim_{d,\alpha,a,s} \left\| \mathcal{L}_{a,\alpha}^{s/2} f \right\|_2 \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \quad (1.5)$$

Conversely, if $\alpha \frac{s}{2} \in (0, d-2\delta)$, if $a < 0$ and if the above estimate holds, then $\alpha \frac{s}{2} + \delta < \frac{d}{2}$.

In fact, our method of proof will be different. We will not deduce (1.5) from Theorem 1.1, but rather (1.5) will be proved first and will be an important ingredient in the proof of Theorem 1.1. Another ingredient in the proof of Theorem 1.1 will be the following inequality which gives a *lower* bound on the norm of $|x|^{-\alpha s/2} f$ in terms of the difference $(\mathcal{L}_{a,\alpha}^{s/2} - |p|^{\alpha s/2}) f$. We have not seen similar inequalities before, not even in the case $\alpha = 2$ (although they can probably be extracted from the bounds on the difference of square functions in [KMV⁺18]).

Proposition 1.5 (Reverse Hardy inequality for differences). *Let $\alpha \in (0, 2 \wedge d)$, $a \in [a_*, +\infty)$. Then, for any $s \in (0, 2)$,*

$$\left\| (\mathcal{L}_{a,\alpha}^{s/2} - |p|^{\alpha s/2}) f \right\|_2 \lesssim_{d,\alpha,a,s} \left\| |x|^{-\alpha s/2} f \right\|_2 \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d).$$

Before proceeding, let us show that Theorem 1.1 is an immediate consequence of Propositions 1.4 and 1.5.

Proof of Theorem 1.1. If $\alpha \frac{s}{2} + \delta < \frac{d}{2}$, we obtain by Propositions 1.5 and 1.4

$$\begin{aligned} \left\| |p|^{\alpha \frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} &\leq \left\| \mathcal{L}_{a,\alpha}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} + \left\| (\mathcal{L}_{a,\alpha}^{s/2} - |p|^{\alpha s/2}) f \right\|_2 \\ &\lesssim \left\| \mathcal{L}_{a,\alpha}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} + \left\| |x|^{-\alpha s/2} f \right\|_2 \\ &\lesssim \left\| \mathcal{L}_{a,\alpha}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)}. \end{aligned}$$

This proves (1). If $\alpha \frac{s}{2} < \frac{d}{2}$, we argue similarly, but with the Hardy inequality (1.1) (with αs instead of α) instead of Proposition 1.4. This proves (2). \square

Thus, we have reduced the proof of Theorem 1.1 to that of Propositions 1.4 and 1.5. Let us explain some ideas involved in those proofs. The starting point of the analysis in [KMV⁺18] for the case $\alpha = 2$ (and that in [KVZ16]) are two-sided bounds on the heat kernel of $\mathcal{L}_{a,2}$. Correspondingly, we rely here on very recent two-sided bounds on the heat kernel of $\mathcal{L}_{a,\alpha}$ for $0 < \alpha < 2$ and $a < 0$ from [BGJP17]. We mention in passing that the corresponding results for $a > 0$ seem to be unknown and therefore our Theorem 1.1 and Proposition 1.4 in the range $a > 0$ are slightly weaker than the corresponding analogs for $\alpha = 2$. This, however, is of no relevance in our application [FMSS] where we deal exclusively with the case $a < 0$. Given these bounds on the heat kernel of $\mathcal{L}_{a,\alpha}$ it is a simple matter to obtain bounds on the corresponding Riesz kernels, i.e., the integral kernels of the operators $\mathcal{L}_{a,\alpha}^{-s/2}$. We state them explicitly as follows.

Theorem 1.6 (Riesz kernels of generalized Hardy operators). *Let $\alpha \in (0, 2 \wedge d)$, $a \in [a_*, \infty)$ and let δ be defined by (1.2). If $a \leq 0$, then for any $s > 0$ with $0 < \alpha \frac{s}{2} < d - 2\delta$,*

$$\mathcal{L}_{a,\alpha}^{-s/2}(x, y) \sim_{d,\alpha,a,s} |x - y|^{\alpha \frac{s}{2} - d} \left(1 \wedge \frac{|x|}{|x - y|} \wedge \frac{|y|}{|x - y|} \right)^{-\delta}, \quad (1.6)$$

and if $a > 0$, then for any $s > 0$ with $0 < \alpha \frac{s}{2} < d$,

$$0 \leq \mathcal{L}_{a,\alpha}^{-s/2}(x, y) \lesssim_{d,\alpha,s} |x - y|^{\alpha \frac{s}{2} - d}. \quad (1.7)$$

One of the applications of this theorem is, for instance, a proof of Proposition 1.4.

Proof of Proposition 1.4. Clearly, (1.5) is equivalent to the L^2 boundedness of the operator $|x|^{-\alpha s/2} \mathcal{L}_{a,\alpha}^{-s/2}$. Using the upper bounds from Theorem 1.6 this follows from the L^2 -boundedness of the operator with integral kernel

$$|x|^{-\alpha \frac{s}{2}} |x - y|^{\alpha \frac{s}{2} - d} \left(1 \wedge \frac{|x|}{|x - y|} \wedge \frac{|y|}{|x - y|} \right)^{-\delta}.$$

This follows in a straightforward way by a Schur test. Since the same argument appears already in [KMV⁺18, Proposition 3.2] (with s in place of $\alpha s/2$ and σ in place of δ) and since we will perform similar Schur tests also later on in this paper, we omit the details here.

The fact that (1.5) fails for $\frac{d}{2} \leq \alpha \frac{s}{2} + \delta < d - \delta$ and $a < 0$ follows from the lower bound in Theorem 1.6 by the same argument as in [KMV⁺18, Proposition 3.2]. \square

This reduces the proof of Proposition 1.4 to that of Theorem 1.6, which will be given in Section 2.

While the proof of Proposition 1.4 and that of Theorem 1.6 rely on bounds on the individual heat kernel of $\mathcal{L}_{a,\alpha}$, the proof of Proposition 1.5 relies on bounds on the difference between the heat kernels of $\mathcal{L}_{a,\alpha}$ and $|p|^\alpha$ and, in particular, on cancellations

between both. We do not discuss these bounds here, but refer to Section 3 and, in particular, to Lemma 3.1. We would like to stress here, however, that the Gaussian off-diagonal decay of the heat kernel for $\alpha = 2$ is replaced by some algebraic decay for $0 < \alpha < 2$. Therefore the corresponding bounds are significantly more involved than those in [KMV⁺18].

We would like to end this introduction with several applications of our main result, Theorem 1.1. The most straightforward one is the following Sobolev inequality for the operator $\mathcal{L}_{a,\alpha}$. It is an immediate consequence of Theorem 1.1 and the usual Sobolev inequality.

Corollary 1.7. *Let $\alpha \in (0, 2 \wedge d)$, $a \in [a_*, \infty)$ and let δ be defined by (1.2). Then, for any $s \in (0, 2)$ with $\alpha \frac{s}{2} + \delta < \frac{d}{2}$,*

$$\|f\|_{L^{\frac{2d}{d-\alpha s}}(\mathbb{R}^d)} \lesssim_{d,\alpha,a,s} \left\| \mathcal{L}_{a,\alpha}^{\frac{s}{2}} f \right\|_{L^2(\mathbb{R}^d)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \quad (1.8)$$

Our main motivation for Theorem 1.1 comes from our forthcoming paper [FMSS] where we describe the ground state density of a large atom with relativistic effects taken into account near the nucleus. This problem was solved before in the non-relativistic case in [ILS96]. The relativistic case is substantially more difficult since the kinetic energy $\sqrt{|p|^2 + m^2} - m$ and the potential energy $|x|^{-1}$ show for large $|p|$ the same scaling behavior. Therefore the latter one cannot be treated as a perturbation and the Sobolev spaces associated to $\mathcal{L}_{a,1}$ appear naturally. While we will eventually show that the problem under consideration is dominated by the small $|p|$ behavior, Theorem 1.1 yields the crucial a-priori bound which controls the worst case behavior coming from large $|p|$. On a more technical level, since we are dealing with a many-body problem, we need a certain trace ideal inequality for the operator $\mathcal{L}_{a,1}$. Theorem 1.1 allows us to deduce such an inequality from the corresponding inequality for $|p|$. For further details, we refer to [FMSS].

Our third application comes from an alternative description of an atom in the presence of relativistic effects, namely by the Coulomb–Dirac operator

$$D^\nu := -i\boldsymbol{\alpha} \cdot \nabla - \frac{\nu}{|x|} \quad (1.9)$$

in $L^2(\mathbb{R}^3; \mathbb{C}^4)$ with $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ being the triplet of Dirac matrices. This operator is essentially self-adjoint on $C_c^\infty(\mathbb{R}^3 \setminus \{0\}; \mathbb{C}^4)$ for $|\nu| \leq \frac{\sqrt{3}}{2}$, while for $\nu \in (\frac{\sqrt{3}}{2}, 1]$ there is a distinguished choice of a self-adjoint extension. We refer to [EL07] for more on this and for references. In the following we always mean this operator when we write D^ν . A fundamental difference between D^ν and $\mathcal{L}_{a,1}$ is that the former is not bounded from below. Nevertheless, based on results of [MM17], we will be able to prove bounds analogous to those in Theorem 1.1.

Corollary 1.8. *Let $\nu \in (0, 1]$ and let δ be defined by (1.2) with $d = 3$, $\alpha = 1$ and*

$$a = -\frac{\tan\left(\frac{\pi}{2}\sqrt{1-\nu^2}\right)}{\sqrt{1-\nu^2}} \text{ if } \nu < 1, \quad a = -\frac{\pi}{2} \text{ if } \nu = 1.$$

Let $s \in (0, 2)$.

(1) *If $\frac{s}{2} + \delta < \frac{3}{2}$, then*

$$\left\| |p|^{\frac{s}{2}} f \right\|_{L^2} \lesssim_{\nu, s} \left\| |D^\nu|^{\frac{s}{2}} f \right\|_{L^2} \text{ for all } f \in C_c^\infty(\mathbb{R}^3 : \mathbb{C}^4). \quad (1.10)$$

(2) *One has*

$$\left\| |D^\nu|^{\frac{s}{2}} f \right\|_{L^2} \lesssim_{\nu, s} \left\| |p|^{\frac{s}{2}} f \right\|_{L^2} \text{ for all } f \in C_c^\infty(\mathbb{R}^3 : \mathbb{C}^4). \quad (1.11)$$

Proof. We begin with the proof of the simpler second part, which does not rely on our main theorem here and uses the same idea as in Remark 1.3. We recall that $(-i\alpha \cdot \nabla)^2 = -\Delta \otimes \mathbf{1}_{\mathbb{C}^4}$. Hence by the Cauchy–Schwarz inequality and by Hardy’s inequality (1.1) with $d = 3$ and $\alpha = 2$, we find

$$(D^\nu)^2 \leq A_\nu (-\Delta) \otimes \mathbf{1}_{\mathbb{C}^4}.$$

By operator monotonicity as in Remark 1.2 we conclude that for any $0 < t \leq 1$

$$|D^\nu|^{2t} \leq A_\nu^t |p|^{2t} \otimes \mathbf{1}_{\mathbb{C}^4}.$$

This is (1.11) with $s = 2t$.

We now turn to the significantly more difficult part of proving (1.10). It is shown in [MM17] that

$$(D^\nu)^2 \geq A_\nu \mathcal{L}_{a,1}^2 \otimes \mathbf{1}_{\mathbb{C}^4}.$$

with a depending on ν as in the statement of the theorem. This bound is not stated explicitly in [MM17], but can be easily obtained from their results. Namely, this bound restricted to the zeroth angular momentum channel is equivalent to their Lemma IV.4, whereas their Lemma IV.5 says that on the orthogonal complement of this channel, $(D^\nu)^2$ can, in fact, be bounded from below by a constant times $-\Delta \otimes \mathbf{1}_{\mathbb{C}^4}$, which in turn (by Cauchy–Schwarz and Hardy, as before) can be bounded from below by $\mathcal{L}_{a,1}^2 \otimes \mathbf{1}_{\mathbb{C}^4}$. Again by operator monotonicity we conclude that for any $0 < t < 1$

$$|D^\nu|^{2t} \geq A_\nu^t \mathcal{L}_{a,1}^{2t} \otimes \mathbf{1}_{\mathbb{C}^4}.$$

Theorem 1.1 with $s = 2t$ now implies the assertion. \square

2. HEAT AND RIESZ KERNELS

Our goal in this section is to prove Theorem 1.6. We begin by recalling the recent two-sided bounds on the heat kernel of $\mathcal{L}_{a,\alpha}$ for $a \in [a_*, 0)$ from [BGJP17]. The results in the (simpler) case $a = 0$ can be found, for instance, in [BBK⁺09]. We also note that in the special case $a = 0$ and $\alpha = 1/2$ an explicit formula for $e^{-t|p|^\alpha}$ is available, see, for instance, [SW71, Theorem 1.14].

Theorem 2.1 (Heat kernels of generalized Hardy operators). *Let $\alpha \in (0, 2 \wedge d)$, $a \in [a_*, 0]$ and let δ be defined by (1.2). Then the heat kernel of $\mathcal{L}_{a,\alpha}$ satisfies for all $x, y \in \mathbb{R}^d$ and $t > 0$,*

$$e^{-t\mathcal{L}_{a,\alpha}}(x, y) \sim \left(1 \vee \frac{t^{1/\alpha}}{|x|}\right)^\delta \left(1 \vee \frac{t^{1/\alpha}}{|y|}\right)^\delta t^{-d/\alpha} \left(1 \wedge \frac{t^{1+d/\alpha}}{|x-y|^{d+\alpha}}\right). \quad (2.1)$$

For $a > 0$ we use the maximum principle for the heat equation to obtain the following bound.

Lemma 2.2. *Let $\alpha \in (0, 2 \wedge d)$ and $a \in (0, \infty)$. Then the heat kernel of $\mathcal{L}_{a,\alpha}$ satisfies for all $x, y \in \mathbb{R}^d$ and $t > 0$,*

$$0 \leq e^{-t\mathcal{L}_{a,\alpha}}(x, y) \leq e^{-t|p|^\alpha}(x, y) \quad (2.2)$$

Combining this bound with that from Theorem 2.1 for $a = 0$ we obtain

$$0 \leq e^{-t\mathcal{L}_{a,\alpha}}(x, y) \lesssim t^{-d/\alpha} \left(1 \wedge \frac{t^{1+d/\alpha}}{|x-y|^{d+\alpha}}\right). \quad (2.3)$$

The upper bound in this lemma is probably not optimal, but it is enough for our purposes.

From these heat kernel bounds we can now derive the claimed Riesz kernel bounds.

Proof of Theorem 1.6. By the spectral theorem the Riesz kernel can be represented as

$$\mathcal{L}_{a,\alpha}^{-s/2}(x, y) = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-t\mathcal{L}_{a,\alpha}}(x, y) t^{s/2} \frac{dt}{t}. \quad (2.4)$$

For $a > 0$ we insert the bounds from Lemma 2.2 and obtain

$$0 \leq \mathcal{L}_{a,\alpha}^{-s/2}(x, y) \leq |p|^{-\alpha s/2}(x, y) = \text{const} |x-y|^{\alpha \frac{s}{2} - d}.$$

This proves the claim for $a \geq 0$.

From now on, we assume that $a < 0$ (and therefore $\delta > 0$). Inserting the two-sided bound on $e^{-t\mathcal{L}_{a,\alpha}}(x, y)$ from Theorem 2.1 and changing variables we see that the right side of (2.4) is equivalent to

$$\begin{aligned} & \int_0^\infty t^{-\frac{d}{\alpha} + \frac{s}{2}} \left(1 \vee \frac{t^{1/\alpha}}{|x|}\right)^\delta \left(1 \vee \frac{t^{1/\alpha}}{|y|}\right)^\delta \left(1 \wedge \frac{t^{1+d/\alpha}}{|x-y|^{d+\alpha}}\right) \frac{dt}{t} \\ &= |x-y|^{\alpha \frac{s}{2} - d} \int_0^\infty dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha} - 1}\right) \left(1 \vee \frac{|x-y|}{|x|} t^{\frac{1}{\alpha}}\right)^\delta \left(1 \vee \frac{|x-y|}{|y|} t^{\frac{1}{\alpha}}\right)^\delta. \end{aligned} \quad (2.5)$$

Since $\mathcal{L}_{a,\alpha}^{-s/2}(x, y)$ is symmetric in x and y , we may and will assume that $|x| \leq |y|$. Therefore, we need to show that the integral on the right side of (2.5) is equivalent to

$$\left(1 \wedge \frac{|x|}{|x-y|}\right)^{-\delta} \quad (2.6)$$

for $|x| \leq |y|$. To do this, we distinguish whether $|x-y| \leq 4|x|$ or not.

Case $|x - y| \leq 4|x|$. In this case we have $|y| \leq |x| + |x - y| \leq 5|x|$ and therefore

$$|x - y| \lesssim |x| \sim |y|.$$

Thus, the integral in (2.5) is equivalent to

$$\int_0^\infty dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \left(1 \vee \lambda^{-1} t^{\frac{1}{\alpha}}\right)^{2\delta}$$

with $\lambda = |x|/|x - y| \geq 1/4$, and we need to show that this is equivalent to 1. We have

$$\int_0^{\lambda^\alpha} dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \left(1 \vee \lambda^{-1} t^{\frac{1}{\alpha}}\right)^{2\delta} = \int_0^{\lambda^\alpha} dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \sim 1,$$

since $\lambda \geq 1/4$ and since the integral converges according to the assumption $\frac{s}{2} < \frac{d}{\alpha}$.

On the other hand, using again $\lambda \geq 1/4$,

$$\begin{aligned} \int_{\lambda^\alpha}^\infty dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \left(1 \vee \lambda^{-1} t^{\frac{1}{\alpha}}\right)^{2\delta} &= \int_{\lambda^\alpha}^\infty dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \lambda^{-2\delta} t^{\frac{2\delta}{\alpha}} \\ &\lesssim \int_{\lambda^\alpha}^\infty dt t^{\frac{s}{2}-\frac{d}{\alpha}-1} \lambda^{-2\delta} t^{\frac{2\delta}{\alpha}} \sim \lambda^{\frac{s\alpha}{2}-d} \lesssim 1, \end{aligned}$$

where we used the assumption $\frac{s}{2} + \frac{2\delta}{\alpha} < \frac{d}{\alpha}$ for the convergence of the integral. This proves the claimed upper bound. Since the integral from λ^α to infinity is non-negative, we also obtain the claimed lower bound.

Case $|x - y| \geq 4|x|$. In this case we have both $|x - y| \leq |x| + |y| \leq 2|y|$ and $|y| \leq |x| + |x - y| \leq \frac{5}{4}|x - y|$ and therefore

$$|x| \leq |y| \sim |x - y|.$$

Thus, the integral in (2.5) is equivalent to

$$\int_0^\infty dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \left(1 \vee \lambda^{-1} t^{\frac{1}{\alpha}}\right)^\delta \left(1 \vee t^{\frac{1}{\alpha}}\right)^\delta$$

with $\lambda = |x|/|x - y| \leq 1/4$, and we need to show that this is equivalent to $\lambda^{-\delta}$. We have, using $\lambda \leq 1/4$,

$$\int_{\lambda^\alpha}^1 dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \left(1 \vee \lambda^{-1} t^{\frac{1}{\alpha}}\right)^\delta \left(1 \vee t^{\frac{1}{\alpha}}\right)^\delta = \int_{\lambda^\alpha}^1 dt t^{\frac{s}{2}} \lambda^{-\delta} t^{\frac{\delta}{\alpha}} \sim \lambda^{-\delta}$$

and, using $\frac{s}{2} + \frac{2\delta}{\alpha} < \frac{d}{\alpha}$,

$$\int_1^\infty dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \left(1 \vee \lambda^{-1} t^{\frac{1}{\alpha}}\right)^\delta \left(1 \vee t^{\frac{1}{\alpha}}\right)^\delta = \int_1^\infty dt t^{\frac{s}{2}-\frac{d}{\alpha}-1} \lambda^{-\delta} t^{\frac{\delta}{\alpha}} \sim \lambda^{-\delta}.$$

On the other hand,

$$\int_0^{\lambda^\alpha} dt t^{\frac{s}{2}} \left(1 \wedge t^{-\frac{d}{\alpha}-1}\right) \left(1 \vee \lambda^{-1} t^{\frac{1}{\alpha}}\right)^\delta \left(1 \vee t^{\frac{1}{\alpha}}\right)^\delta = \int_0^{\lambda^\alpha} dt t^{\frac{s}{2}} \sim \lambda^{\frac{s\alpha}{2}+\alpha} \lesssim \lambda^{-\delta}.$$

This yields the claimed upper bound and since the latter integral is non-negative, we also obtain the claimed lower bound. This completes the proof of the theorem. \square

3. DIFFERENCE OF HEAT KERNELS

Our goal in this section is to prove Proposition 1.5. A key tool for this will be a bound on the difference of the heat kernels of $\mathcal{L}_{a,\alpha}$ and $|p|^\alpha$, namely,

$$K_t^\alpha(x, y) := e^{-t|p|^\alpha}(x, y) - e^{-t\mathcal{L}_{a,\alpha}}(x, y).$$

We recall from the previous section that we have bounds on the individual kernels $e^{-t|p|^\alpha}(x, y)$ and $e^{-t\mathcal{L}_{a,\alpha}}(x, y)$. The following lemma shows that there is a cancellation, coming from taking the difference, in the region $(|x| \vee |y|)^\alpha \geq t$ and $|x| \sim |y|$.

We will formulate the bound in terms of the functions

$$L_t^\alpha(x, y) := \mathbb{1}_{\{(|x| \vee |y|)^\alpha \leq t\}} t^{-\frac{d}{\alpha}} \left(\frac{t^{2/\alpha}}{|x||y|} \right)^\delta + \mathbb{1}_{\{(|x| \vee |y|)^\alpha \geq t\}} \frac{t}{(|x| \vee |y|)^{d+\alpha}} \left(1 \vee \frac{t^{1/\alpha}}{|x| \wedge |y|} \right)^\delta$$

and

$$M_t^\alpha(x, y) := \mathbb{1}_{\{(|x| \vee |y|)^\alpha \geq t\}} \mathbb{1}_{\{\frac{1}{2}|x| \leq |y| \leq 2|x|\}} \frac{t^{1-\frac{d}{\alpha}}}{(|x| \wedge |y|)^\alpha} \left(1 \wedge \frac{t^{1+\frac{d}{\alpha}}}{|x-y|^{d+\alpha}} \right).$$

Lemma 3.1 (Difference of kernels). *Let $\alpha \in (0, 2 \wedge d)$, $a \in [a_*, \infty)$ and let δ be defined by (1.2). Then for all $x, y \in \mathbb{R}^d$ and $t > 0$*

$$|K_t^\alpha(x, y)| \lesssim L_t^\alpha(x, y) + M_t^\alpha(x, y). \quad (3.1)$$

Proof. By scaling it suffices to consider $t = 1$ and by symmetry it suffices to consider $|x| \leq |y|$. We will drop the subscript t in K_t^α , L_t^α and M_t^α .

If $a \geq 0$ we combine the maximum principle as in the proof of Lemma 2.2 and the bound from Theorem 2.1 for $a = 0$ and obtain

$$0 \leq K^\alpha(x, y) \leq e^{-|p|^\alpha}(x, y) \sim 1 \wedge |x - y|^{-d-\alpha}.$$

This proves the bound $K^\alpha(x, y) \lesssim L^\alpha(x, y)$ if $|y| \leq 1$ (by bounding the minimum by 1) or if $|y| \geq 1$ and $|x| \leq (1/2)|y|$ (by bounding the minimum by $|x - y|^{-d-\alpha} \lesssim |y|^{-d-\alpha}$).

If $a < 0$, then again by combining the maximum principle and Theorem 2.1 we find

$$0 \leq -K^\alpha(x, y) \leq e^{-\mathcal{L}_{a,\alpha}}(x, y) \lesssim \left(1 \vee |x|^{-\delta} \right) \left(1 \vee |y|^{-\delta} \right) \left(1 \wedge |x - y|^{-d-\alpha} \right).$$

Again this gives the bound $-K^\alpha(x, y) \lesssim L^\alpha(x, y)$ if $|y| \leq 1$ (because then the product of the first two factors on the right side is $\leq (|x||y|)^{-\delta}$) or if $|y| \geq 1$ and $|x| \leq (1/2)|y|$ (because then the smaller one of the first two factors equals 1).

Thus, from now on we assume that $|y| \geq 1$ and $(1/2)|x| \leq |y| \leq |x|$. By Duhamel's formula we have

$$e^{-|p|^\alpha} - e^{-\mathcal{L}_{a,\alpha}} = a \int_0^1 ds e^{-(1-s)|p|^\alpha} |x|^{-\alpha} e^{-s\mathcal{L}_{a,\alpha}}. \quad (3.2)$$

If $a \geq 0$ we use Lemma 2.2 and Theorem 2.1 (with $a = 0$) to conclude that

$$K^\alpha(x, y) \lesssim \int_0^1 ds \int_{\mathbb{R}^d} dz |z|^{-\alpha} s^{-\frac{d}{\alpha}} (t-s)^{-\frac{d}{\alpha}} \left(1 \wedge \frac{(1-s)^{1+\frac{d}{\alpha}}}{|x-z|^{d+\alpha}} \right) \left(1 \wedge \frac{s^{1+\frac{d}{\alpha}}}{|y-z|^{d+\alpha}} \right),$$

and if $a < 0$ we use Theorem 2.1 to conclude that

$$\begin{aligned}
 -K^\alpha(x, y) &\lesssim \int_0^1 ds \int_{\mathbb{R}^d} dz s^{-\frac{d}{\alpha}} (1-s)^{-\frac{d}{\alpha}} |z|^{-\alpha} \left(1 \wedge \frac{(1-s)^{1+\frac{d}{\alpha}}}{|x-z|^{d+\alpha}} \right) \\
 &\quad \times \left(1 + \frac{s^{\frac{\delta}{\alpha}}}{|y|^\delta} \right) \left(1 + \frac{s^{\frac{\delta}{\alpha}}}{|z|^\delta} \right) \left(1 \wedge \frac{s^{1+\frac{d}{\alpha}}}{|y-z|^{d+\alpha}} \right) \\
 &\lesssim \int_0^1 ds \int_{\mathbb{R}^d} dz s^{-\frac{d}{\alpha}} (1-s)^{-\frac{d}{\alpha}} |z|^{-\alpha} \left(1 \wedge \frac{(1-s)^{1+\frac{d}{\alpha}}}{|x-z|^{d+\alpha}} \right) \\
 &\quad \times \left(1 + \frac{s^{\frac{\delta}{\alpha}}}{|z|^\delta} \right) \left(1 \wedge \frac{s^{1+\frac{d}{\alpha}}}{|y-z|^{d+\alpha}} \right).
 \end{aligned}$$

The second inequality here comes from the fact that $s^{1/\alpha} \leq 1 \leq |y|$.

Thus, it remains to bound

$$\int_0^1 ds \int_{\mathbb{R}^d} dz s^{-\frac{d}{\alpha}} (1-s)^{-\frac{d}{\alpha}} |z|^{-\alpha} \left(1 \wedge \frac{(1-s)^{1+\frac{d}{\alpha}}}{|x-z|^{d+\alpha}} \right) \left(1 + \frac{s^{\frac{\delta}{\alpha}}}{|z|^\delta} \right) \left(1 \wedge \frac{s^{1+\frac{d}{\alpha}}}{|y-z|^{d+\alpha}} \right). \quad (3.3)$$

We do this by dividing the z integration into two parts and we begin with the part where $|z| \geq \frac{1}{2}|x|$. In this case, we can bound

$$|z|^{-\alpha} \lesssim |x|^{-\alpha} \quad \text{and} \quad s^{\frac{1}{\alpha}} \leq 1 \leq |y| \lesssim |x| \lesssim |z|.$$

Applying these bounds and then enlarging the z integration to all of \mathbb{R}^d we find that the integral in question is bounded by a constant times

$$\begin{aligned}
 &\frac{1}{|x|^\alpha} \int_0^1 ds \int_{\mathbb{R}^d} dz s^{-\frac{d}{\alpha}} (1-s)^{-\frac{d}{\alpha}} \left(1 \wedge \frac{(1-s)^{1+\frac{d}{\alpha}}}{|x-z|^{d+\alpha}} \right) \left(1 \wedge \frac{s^{1+\frac{d}{\alpha}}}{|y-z|^{d+\alpha}} \right) \\
 &\lesssim \frac{1}{|x|^\alpha} \int_0^1 ds \int_{\mathbb{R}^d} dz e^{-(1-s)|p|^\alpha}(x, z) e^{-s|p|^\alpha}(z, y) \\
 &= \frac{1}{|x|^\alpha} \int_0^1 ds e^{-|p|^\alpha}(x, y) \\
 &\lesssim \frac{1}{|x|^\alpha} (1 \wedge |x-y|^{-d-\alpha}) = M^\alpha(x, y),
 \end{aligned}$$

Here we used Theorem 2.1 twice (with $a = 0$) together with the semi-group property.

It remains to bound the part of the integral in (3.3) corresponding to $|z| \leq \frac{1}{2}|x|$. We bound

$$\begin{aligned}
& \int_{|z| \leq \frac{1}{2}|x|} dz |z|^{-\alpha} \frac{1}{|x-z|^{d+\alpha}} \left(1 + \frac{s^{\frac{\delta}{\alpha}}}{|z|^\delta}\right) \frac{1}{|y-z|^{d+\alpha}} \\
& \lesssim \frac{1}{|x|^{d+\alpha}|y|^{d+\alpha}} \int_{|z| \leq \frac{1}{2}|x|} dz |z|^{-\alpha} \left(1 + \frac{s^{\frac{\delta}{\alpha}}}{|z|^\delta}\right) \\
& \sim \frac{1}{|x|^{d+\alpha}|y|^{d+\alpha}} \left(|x|^{d-\alpha} + s^{\frac{\delta}{\alpha}}|x|^{d-\alpha-\delta}\right) \\
& \sim \frac{1}{|x|^{d+\alpha}|y|^{d+\alpha}} |x|^{d-\alpha}.
\end{aligned}$$

In the next to last step we used $\alpha + \delta < d$ (which follows from $\delta \leq (d - \alpha)/2$ and $\alpha < d$) and in the last step we used again $s^{1/\alpha} \leq 1 \leq |y| \lesssim |x|$. Inserting this into the above bound we obtain

$$\begin{aligned}
& \int_0^1 ds s^{-\frac{d}{\alpha}} (1-s)^{-\frac{d}{\alpha}} \int_{|z| \leq \frac{1}{2}|x|} dz |z|^{-\alpha} \left(1 \wedge \frac{(1-s)^{1+\frac{d}{\alpha}}}{|x-z|^{d+\alpha}}\right) \\
& \quad \times \left(1 + \frac{s^{\frac{\delta}{\alpha}}}{|z|^\delta}\right) \left(1 \wedge \frac{s^{1+\frac{d}{\alpha}}}{|y-z|^{d+\alpha}}\right) \\
& \lesssim \frac{|x|^{d-\alpha}}{|x|^{d+\alpha}|y|^{d+\alpha}} \int_0^1 ds s(1-s) \\
& \sim \frac{|x|^{d-\alpha}}{|x|^{d+\alpha}|y|^{d+\alpha}} \\
& \lesssim \frac{1}{|y|^{d+\alpha}} \leq L^\alpha(x, y).
\end{aligned}$$

In the next to last step we used again $1 \leq |y| \lesssim |x|$. This proves the claimed bound. \square

Finally, we can complete the proof of our main result by giving the

Proof of Proposition 1.5. Since for $0 < s < 2$,

$$\int_0^\infty \frac{dt}{t} t^{-s/2} (e^{-t} - 1) = \Gamma(-s/2),$$

we have by the spectral theorem and by scaling

$$\begin{aligned}
 (\mathcal{L}_{a,\alpha}^{s/2} - |p|^{\alpha s/2}) f &= -\frac{1}{\Gamma(-s/2)} \int_0^\infty \frac{dt}{t} t^{-s/2} ((e^{-t|p|^\alpha} - 1) f - (e^{-t\mathcal{L}_{a,\alpha}} - 1) f) \\
 &= -\frac{1}{\Gamma(-s/2)} \int_0^\infty \frac{dt}{t} t^{-s/2} (e^{-t|p|^\alpha} - e^{-t\mathcal{L}_{a,\alpha}}) f \\
 &= -\frac{1}{\Gamma(-s/2)} \int_0^\infty \frac{dt}{t} t^{-s/2} \int_{\mathbb{R}^d} dy K_t^\alpha(\cdot, y) f(y).
 \end{aligned}$$

Therefore, by Lemma 3.1 we can bound

$$\begin{aligned}
 \|(\mathcal{L}_{a,\alpha}^{s/2} - |p|^{\alpha s/2}) f\|_2 &\lesssim \left\| \int_{\mathbb{R}^d} dy \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} L_t^\alpha(x, y) |y|^{\alpha \frac{s}{2}} g(y) \right\|_2 \\
 &\quad + \left\| \int_{\mathbb{R}^d} dy \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} M_t^\alpha(x, y) |y|^{\alpha \frac{s}{2}} g(y) \right\|_2, \quad (3.4)
 \end{aligned}$$

where we abbreviate $g(y) := |y|^{-\alpha s/2} |f(y)|$. Our goal is bound both terms on the right side of (3.4) by a constant times $\|g\|_2$.

We begin with the first term and compute

$$\begin{aligned}
 \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} L_t^\alpha(x, y) &= (|x||y|)^{-\delta} \int_{t \geq (|x| \vee |y|)^\alpha} \frac{dt}{t} t^{-\frac{s}{2} - \frac{d-2\delta}{\alpha}} \\
 &\quad + \int_{t \leq (|x| \vee |y|)^\alpha} \frac{dt}{t} t^{-\frac{s}{2}} \frac{t}{(|x| \vee |y|)^{d+\alpha}} \left(1 \vee \frac{t^{1/\alpha}}{|x| \wedge |y|}\right)^\delta \\
 &\sim \frac{1}{(|x| \vee |y|)^{\frac{s\alpha}{2} + d}} \left(\frac{|x| \vee |y|}{|x| \wedge |y|} \right)^\delta.
 \end{aligned}$$

Here we used the fact that $\frac{s}{2} + \frac{d-2\delta}{\alpha} > 0$ (which follows from $\delta \leq (d-\alpha)/2$.) Thus,

$$\left\| \int_{\mathbb{R}^d} dy \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} L_t^\alpha(x, y) |y|^{\alpha \frac{s}{2}} g(y) \right\|_2 \lesssim \left\| \int_{\mathbb{R}^d} dy \frac{1}{(|x| \vee |y|)^d} \left(\frac{|x| \vee |y|}{|x| \wedge |y|} \right)^\delta g(y) \right\|_2.$$

For any $\delta < \beta < d - \delta$ (such β exist since $\delta \leq (d-\alpha)/2 < d/2$) we have

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} dx \left(\frac{|y|}{|x|} \right)^\beta \frac{1}{(|x| \vee |y|)^d} \left(\frac{|x| \vee |y|}{|x| \wedge |y|} \right)^\delta = \int_{\mathbb{R}^d} \frac{dz}{|z|^\beta (|z| \vee 1)^d} \left(\frac{|z| \vee 1}{|z| \wedge 1} \right)^\delta < \infty,$$

and therefore by a Schur test with weights we conclude that

$$\left\| \int_{\mathbb{R}^d} dy \frac{1}{(|x| \vee |y|)^d} \left(\frac{|x| \vee |y|}{|x| \wedge |y|} \right)^\delta g(y) \right\|_2 \lesssim \|g\|_2.$$

This shows that the first term in (3.4) satisfies the claimed bound.

We now turn to the second term in (3.4). Since $|x| \sim |y|$ on the support of this kernel, we have

$$\left\| \int_{\mathbb{R}^d} dy \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} M_t^\alpha(x, y) |y|^{\frac{\alpha s}{2}} g(y) \right\| \lesssim \left\| \int_{\mathbb{R}^d} dy \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} M_t^\alpha(x, y) (|x||y|)^{\frac{\alpha s}{4}} g(y) \right\|.$$

This replaces the kernel by a symmetric one and we only have to perform a single Schur test (instead of two). We have

$$\begin{aligned} & \sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} dx \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} M_t^\alpha(x, y) (|x||y|)^{\frac{\alpha s}{4}} \\ &= \sup_{y \in \mathbb{R}^d} \int_{\frac{1}{2}|y| \leq |x| \leq 2|y|} dx \int_{t \leq (|x| \vee |y|)^\alpha} \frac{dt}{t} t^{-\frac{s}{2}} \frac{t^{1-\frac{d}{\alpha}}}{(|x| \wedge |y|)^\alpha} \left(1 \wedge \frac{t^{1+\frac{d}{\alpha}}}{|x-y|^{d+\alpha}} \right) (|x||y|)^{\frac{\alpha s}{4}} \\ &\lesssim \sup_{y \in \mathbb{R}^d} |y|^{\frac{\alpha s}{2}-\alpha} \int_{\frac{1}{2}|y| \leq |x| \leq 2|y|} dx \int_{t \leq (2|y|)^\alpha} \frac{dt}{t} t^{-\frac{s}{2}+1-\frac{d}{\alpha}} \left(1 \wedge \frac{t^{1+\frac{d}{\alpha}}}{|x-y|^{d+\alpha}} \right). \end{aligned}$$

We now interchange the order of integration and do the x integral first. We bound

$$\int_{\frac{1}{2}|y| \leq |x| \leq 2|y|} dx \left(1 \wedge \frac{t^{1+\frac{d}{\alpha}}}{|x-y|^{d+\alpha}} \right) \leq \int_{\mathbb{R}^d} dx \left(1 \wedge \frac{t^{1+\frac{d}{\alpha}}}{|x-y|^{d+\alpha}} \right) \sim t^{\frac{d}{\alpha}}.$$

Therefore, the supremum above is bounded by a constant times

$$\sup_{y \in \mathbb{R}^d} |y|^{\frac{\alpha s}{2}-\alpha} \int_{t \leq (2|y|)^\alpha} \frac{dt}{t} t^{-\frac{s}{2}+1} \sim 1.$$

Thus, the Schur test yields

$$\left\| \int_{\mathbb{R}^d} dy \int_0^\infty \frac{dt}{t} t^{-\frac{s}{2}} M_t^\alpha(x, y) (|x||y|)^{\frac{\alpha s}{4}} g(y) \right\| \lesssim \|g\|_2.$$

This concludes the proof of the proposition. \square

4. A GENERALIZATION

In our application in [FMSS] we will need a slight generalization of the bounds that we have derived so far to the case of not necessarily power-like potentials. More precisely, we consider functions V on \mathbb{R}^d satisfying

$$\frac{a}{|x|^\alpha} \leq V(x) \leq \frac{\tilde{a}}{|x|^\alpha} \quad (4.1)$$

with parameters $a_* \leq a \leq \tilde{a} < \infty$ and we prove the following result.

Theorem 4.1. *Let $\alpha \in (0, 2 \wedge d)$, $a_* \leq a \leq \tilde{a} < \infty$ and let δ be defined by (1.2). Let $s \in (0, 2)$.*

(1) If $\alpha \frac{s}{2} + \delta < \frac{d}{2}$, then for any V satisfying (4.1),

$$\left\| |p|^{\alpha s/2} f \right\|_{L^2(\mathbb{R}^d)} \lesssim_{d,\alpha,a,s} \left\| (|p|^\alpha + V)^{s/2} f \right\|_{L^2(\mathbb{R}^d)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \quad (4.2)$$

(2) If $\alpha \frac{s}{2} < \frac{d}{2}$, then for any V satisfying (4.1),

$$\left\| (|p|^\alpha + V)^{s/2} f \right\|_{L^2(\mathbb{R}^d)} \lesssim_{d,\alpha,a,s} \left\| |p|^{\alpha s/2} f \right\|_{L^2(\mathbb{R}^d)} \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d). \quad (4.3)$$

We emphasize that δ is defined with respect to a (and not with respect to \tilde{a}). It is interesting to note that the constants in Theorem 4.1 are independent of \tilde{a} . By an approximation argument this would allow to extend the theorem to a larger class of potentials, but we will not need this.

The proof of Theorem 4.1 follows immediately from the following two propositions in the same way as Theorem 1.1 followed from Propositions 1.4 and 1.5.

Proposition 4.2. *Let $\alpha \in (0, 2 \wedge d)$, $a_* \leq a \leq \tilde{a} < \infty$ and let δ be defined by (1.2). Then, for any $s > 0$ with $\alpha \frac{s}{2} + \delta < \frac{d}{2}$ and for any V satisfying (4.1),*

$$\left\| |x|^{-\alpha s/2} f \right\|_2 \lesssim_{d,\alpha,a,s} \left\| (|p|^\alpha + V)^{s/2} f \right\|_2 \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d).$$

Proof. According to the maximum principle, we have for all $x, y \in \mathbb{R}^d$ and $t > 0$,

$$0 \leq e^{-t(|p|^\alpha + V)}(x, y) \leq e^{-t\mathcal{L}_{a,\alpha}}(x, y), \quad (4.4)$$

and therefore, by the analogue of (2.4), also

$$(|p|^\alpha + V)^{-s/2}(x, y) \leq \mathcal{L}_{a,\alpha}^{-s/2}(x, y).$$

This implies that the upper bounds in Theorem 1.6 remain valid for $(|p|^\alpha + V)^{-s/2}$. The proposition now follows in the same way as Proposition 1.4. \square

Proposition 4.3. *Let $\alpha \in (0, 2 \wedge d)$, $a_* \leq a \leq \tilde{a} < \infty$ and $s \in (0, 2)$. For any V satisfying (4.1),*

$$\left\| \left((|p|^\alpha + V)^{s/2} - |p|^{\alpha s/2} \right) f \right\|_2 \lesssim_{d,\alpha,a,s} \left\| |x|^{-\alpha s/2} f \right\|_2 \quad \text{for all } f \in C_c^\infty(\mathbb{R}^d).$$

Proof. Let

$$\tilde{K}_t^\alpha(x, y) := e^{-t|p|^\alpha}(x, y) - e^{-t(|p|^\alpha + V)}(x, y).$$

According to the maximum principle, we have for any $x, y \in \mathbb{R}^d$ and $t > 0$,

$$e^{-t|p|^\alpha}(x, y) - e^{-t\mathcal{L}_{a,\alpha}}(x, y) \leq \tilde{K}_t^\alpha(x, y) \leq e^{-t|p|^\alpha}(x, y) - e^{-t\mathcal{L}_{\tilde{a},\alpha}}(x, y).$$

Therefore Lemma 3.1 with a and \tilde{a} implies that the corresponding statement also holds for \tilde{K}_t^α in place of K_t^α . (Here we use the fact that $\tilde{\delta}$, defined by (1.2) with \tilde{a} in place of a , satisfies $\tilde{\delta} \leq \delta$.) With the analogue of Lemma 3.1 at hand, the proof of the proposition follows in the same way as Proposition 1.5. \square

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