# Existence of Complex Poles and Oscillatory Average Multiplicity in Multiperipheral Models*

Marvin L. Goldberger†
*Joseph Henry Laboratory, Princeton University, Princeton, New Jersey 08540*

and

Dennis Silverman
*Physics Department, University of California, San Diego, La Jolla, California 92037*

and

Chung-I Tan
*Physics Department, Brown University, Providence, Rhode Island 02912*
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Using only the general factorization and nondegenerate threshold properties of multiperipheral models, we demonstrate the existence of damped oscillatory components for both the total cross section and the rate of growth of average multiplicity, with identical frequencies and relative amplitudes. For weak oscillation, however, they are shown to be 180° out of phase. Relevant experimental information on this fine structure is also discussed.

A nonleading complex conjugate pair of Regge poles in the vacuum channel at the forward direction is well known to correspond to a damped oscillatory component of the high-energy total cross sections. Motivated by the multiperipheral nature of high-energy production processes, Chew and Snider[1] (CS) have recently conjectured the actual existence of such pairs of poles. They verified their assertion by carrying out a model calculation using a simplified version of the multi-Regge equations. Their conjecture enables them to relate possible features of data on total cross sections to the rate of growth of average multiplicity with energy.

We present here a further consequence of the CS conjecture, which is independent of the detailed dynamical origin of the complex poles. Using only the general factorization property of multiperiheral models, we demonstrate the existence of a damped oscillatory component for the rate of growth of average multiplicity. This oscillation has the same frequency, $2\pi/\mathrm{Im}\alpha$, in $\ln s$ as that of the total cross section; but they are, in general, not in phase with each other. In particular, for weak oscillation, they are 180° out of phase.

By way of introduction, we shall first demonstrate the generality of the CS conjecture by exhibiting an analytic solution of the Amati-Bertocchi-Fubini-Stanghellini-Tonin (ABFST) integral equation in the limit of large exchange mass. This also serves to establish our convention.[2] An exception to this general consequence of multiperipheralism is pointed out but is discarded on physical grounds.

The absorptive part $A(s;u,v)$ of the forward scattering amplitude for the process $p+k\to p+k$ in the ABFST model satisfies a linear integral equation.[2] In the simplest case of a single-particle rung of mass $\mu$, the kernel is $\pi g^2\delta^{(+)}((p-p')^2-\mu^2)$, and the internal propagator has a mass $m$. The usual procedure of solving this integral equation is first to reduce it to a one-dimensional equation[2,3] by performing a partial-wave analysis

$$a_l(u,v)\equiv\int_{\mu^2}^{\infty}\frac{ds}{2(uv)^{1/2}}[X(s,u,v)]^{-(l+1)}A(s;u,v), \tag{1}$$

where $p^2=u$, $k^2=v$, $(p+k)^2=s$ and

$$X\equiv[s-u-v+\Delta^{1/2}(s,u,v)]/2(uv)^{1/2}.$$

This analysis is performed with $u$, $v$ kept negative. Poles of $a_l$ are Lorentz poles, and the leading ones will control the asymptotic behavior of $A(s;u,v)$. The inversion formula, with the contour to the right of any singularity of the integrand, is

$$A(s;u,v)=\int_{c-i\infty}^{c+i\infty}(dl/2\pi i)X^{l+1}(X^2-1)^{-1/2}a_l(u,v). \tag{2}$$

The locations of poles will depend on the dynamics of the model used. In particular, there may only be real poles. This is indeed the case for the model with zero linkage mass, $\mu^2=0$, where the exact pole spectrum[4] at $t=0$ is known. Evidently, the presence of complex poles is associated with the finiteness of linkage mass. The study of the integral equation for finite values of $\mu^2$, however, has used either numerical methods[5] or the trace approximation, which in general is only a weak-coupling approximation. Although qualitative features of the solution have been found, an analytic solution is highly desirable. We shall provide such a solution[3] in the limit of large $\mu^2$.

In the ABFST model, the propagator term serves as the damping factor in the momentum transfer squared, $u'$. The major contribution to the integral equation comes from small values of $-u'$. In the limit of large $\mu^2$, with $m^2$ fixed, the leading order terms of the partial-wave integral equation become

$$a_l(u,v)=\frac{\pi g^2}{2(uv)^{1/2}}\left[\frac{\mu^2}{(uv)^{1/2}}\right]^{-(l+1)}+\frac{g^2}{16\pi^2(l+1)}\int_{-\infty}^{0}\frac{du'}{(m^2-u')^2}\left(\frac{-u'}{-u}\right)^{1/2}\left[\frac{\mu^2}{(uu')^{1/2}}\right]^{-(l+1)}a_l(u',v). \tag{3}$$

We may also cut off the integral at a fixed large negative value $-\Delta$. Equation (3) now has a separable kernel and can be solved exactly to yield

$$a_l(u,v)=\frac{(\pi/2)g^2(uv)^{l/2}(\mu^2)^{-(l+1)}}{D(l)} \tag{4}$$

where

$$D(l)\equiv 1-D_1(l)=1-\frac{g^2}{16\pi^2}\frac{(\mu^2)^{-(l+1)}}{(l+1)}\int_{-\Delta}^{0}\frac{(-u)^{l+1}du}{(m^2-u)^2}. \tag{5}$$

$D(l)$ is meromorphic in $l$, with poles at negative integers. Restricting ourselves to the region $\mathrm{Re}l<0$, we find, for large $\mu^2$,

$$D(l)=1+\frac{g^2}{16\pi m^2}\left(\frac{\mu^2}{m^2}\right)^{-(l+1)}\frac{1}{\sin\pi l}. \tag{6}$$

The zero of $D(l)$ can then be found by examining the transcendental equation $D(\alpha_i)=0$. This equation is similiar to that obtained by CS using what is essentially a weak-coupling approximation to the multi-Regge integral equation. We, on the other hand, have arrived at Eq. (6) by considering the limit of large linkage mass. In this limit, the transcendental equation can further be approximated by $\alpha=-1+\gamma e^{-\omega(\alpha+1)}$ where we have introduced $\omega\equiv\ln(\mu^2/m^2)$, and $\gamma\equiv g^2/16\pi^2m^2$. This results in a spectrum of complex poles similiar to that obtained by CS. Its characteristic features are the position of the leading pair of complex poles with $\mathrm{Im}\alpha_1\approx 2\pi/\omega$, and the spacing between poles, $\Delta\mathrm{Im}\alpha_n\approx 2\pi/\omega$. The dependence of complex pole positions on coupling strength $\gamma$ is mostly contained in their real parts.

Our analysis is also applicable if, on average, several particles are emitted at each link. We interpret $\mu$ as an average linkage mass. In realistic models,[6] $m$ is the pion mass and $\mu^2$ is typically 1 $\mathrm{GeV}^2$. With $\mu^2/m^2\simeq 50$, our large-exchange-mass limit result may actually be physically relevant. The generalization of our method to the multi-Regge model can also be carried out without altering the relevant structure of $D(l)$. In particular, the spacing in $\mathrm{Im}\alpha$ is still determined by $\mu^2/m^2$.

As emphasized by CS, the oscillatory behavior is a reflection of the inability to increase the number of linkages without a corresponding increase in $\ln s$ by a finite amount, a characteristic shared by all multiperipheral models. This increment is determined by the ratio of the linkage mass and the average momentum transfer between links. In the case of large $\mu^2$, with the damping parameter $m^2$ in momentum transfer fixed, the increment in $\ln s$ will have to be large, leading to a large period of oscillation. This is reflected in the $l$ plane by the presence of complex poles with small imaginary parts. For small $\mu^2$, the situation is simply reversed. In the limit $\mu^2\to 0$, no limitation exists on the number of linkages at a finite value of $\ln s$. No oscillation is thus expected by these multiperipheral considerations. All complex poles will move away from the finite region of the $l$ plane, in agreement with the exact $\mu^2=0$ result. However, we shall exclude this possibility on physical grounds.

Our analysis thus far, together with the work of CS, strongly suggests the existence of complex conjugate poles if a multiperipheral description of high-energy production processes is correct. The exact pole spectrum cannot at this stage be determined, because it is highly model dependent. However, if it is known from other sources such as experiments on total cross sections, meaningful predictions on other experimental observables can be obtained, without having to specify how these poles are generated. One such observable is the single-particle distribution spectrum.[7] Another one is the average multiplicity at high energy.

The average multiplicity for high-energy production processes is defined by

$$\langle n\rangle \equiv (\sum_{n=1}^{\infty} nA_n)/(\sum_{n=1}^{\infty} A_n) \equiv N/A, \tag{7}$$

where $A_n$ is the $n$-particle intermediate-state contribution to the forward absorptive part $A$. The knowledge of the pole spectrum in the $l$ plane yields $A$, and we shall demonstrate that information can also be extracted for $N$, and therefore also for $\langle n\rangle$. We make use of the recursive relation $A_n = A_j \otimes A_{n-j}$, for arbitrary $1 \leq j \leq n-1$. It then follows that, upon summing $n$ and $j$,

$$N(p,k) \equiv \langle n\rangle A = A(p,k) + 2\int \frac{d^4p'}{(2\pi)^4} \frac{A(p,-p')A(p',k)}{(m^2-p'^2)^2}. \tag{8}$$

To utilize the knowledge of the pole spectrum of $A$, we perform a partial-wave analysis. Defining partial wave $N(l;u,v)$ by Eq. (2), we find Eq. (8) can be diagonalized[8] exactly,

$$N(l;u,v) = a_l(u,v) + \frac{1}{(2\pi)^3(l+1)} \int_{-\infty}^{0} \frac{du'(-u')a_l(u,u')a_l(u',v)}{(m^2-u')^2}. \tag{9}$$

Using a spectral representation for $a_l(u,v)$,

$$a_l(u,v) = \sum_{i=0}^{\infty} \frac{\beta_i(u,v)}{l-\alpha_i}, \tag{10}$$

where $\mathrm{Re}\alpha_{i+1} \leq \mathrm{Re}\alpha_i$, we find that $N(l;u,v)$ contains both double and single poles at $\alpha_i$, $i=0,1,2,\cdots$. Using the inversion formula, Eq. (2), we find that each double pole will lead to an asymptotic behavior of the form $c_i s^{\alpha_i} \ln s$, where the single poles will lead to $d_i s^{\alpha_i}$.

If two of the secondary poles are complex conjugate poles, it follows that the average multiplicity $\langle n\rangle$ will also have a damped component, with an $s^{-(\alpha_0 - \mathrm{Re}\alpha_1)}$ damping and a period $2\pi/\mathrm{Im}\alpha_1$ in $\ln s$, $\alpha_0$ being the Pomeranchukon. What is not determined at this stage is the phase of this oscillation relative to that of the total cross section. To obtain this information, we shall make use of our earlier result. As we have mentioned, the large link mass limit may actually represent the real situation very well.

The major advantage of studying the large $\mu^2$ limit is the simplification on the $u$, $v$ dependence of $a_l(u,v)$. It follows from Eq. (4) that

$$a_l(u,v) = \frac{(uv)^{l/2}}{(m_1 m_2)^l} a_l(m_1^2, m_2^2). \tag{11}$$

Substituting Eq. (11) into Eq. (9), we find

$$N(l;m_1^2,m_2^2) = a_l(m_1^2,m_2^2) + \left(\frac{2}{\pi}\right)\left(\frac{m^2}{g^2}\right)\left(\frac{\mu^2}{m^2}\right)^{l+1} D_1(l) a_l(m_1^2,m^2) a_l(m^2,m_2^2), \tag{12}$$

where we have distinguished between incoming particles $m_1$, $m_2$ and the "propagating" particles $m$. $D_1(l)$ is given by Eq. (5), satisfying the condition $D_1(\alpha_i)=1$, $i=0,1,\cdots$.

Let us denote the Regge residues of $a_l(m_1^2,m_2^2)$, $a_l(m_1^2,m^2)$, and $a_l(m^2,m_2^2)$, at $\alpha_i$, as $\beta_i^{(1,2)}$, $\beta_i^{(1,0)}$, and $\beta_i^{(0,2)}$, respectively. We again find that $N(l,m_1^2,m_2^2)$ has the following $l$-plane pole structure:

$$N_l(m_1^2,m_2^2) = \sum_{i=0} \frac{c_i^{(1,2)}}{(l-\alpha_i)^2} + \sum_{i\neq j} \frac{e_{ij}^{(1,2)}}{(l-\alpha_i)(l-\alpha_j)} + \sum_{i=0} \frac{\beta_i^{(1,2)}}{(l-\alpha_i)}, \tag{13}$$

where

$$c_i^{(1,2)} = \left(\frac{2}{\pi}\right)\left(\frac{m^2}{g^2}\right)\left(\frac{\mu^2}{m^2}\right)^{\alpha_i+1} \beta_i^{(1,0)}\beta_i^{(0,2)}, \quad e_{ij}^{(1,2)} = \left(\frac{2}{\pi}\right)\left(\frac{m^2}{g^2}\right)\left(\frac{\mu^2}{m^2}\right)^{l+1} \beta_i^{(1,0)}\beta_j^{(0,2)}. \tag{14}$$

They can also be written slightly differently using the factorization property:

$$\beta_i^{(1,0)}\beta_i^{(0,2)} = \beta_i^{(0,0)}\beta_i^{(1,2)}. \tag{15}$$

The asymptotic behavior of $N$ and $A$ can now be found by Eq. (2). But instead, let us consider the idealized situation[9] where the $l$ plane only contains the Pomeranchukon $\alpha_0$, and a pair of complex conjugate poles $\alpha_1 = \alpha_2^*$. Letting $\Delta\alpha_1 \equiv \alpha_0 - \alpha_1$, we find, asymptotically, i.e., $(s/m_1m_2) \gg 1$,

$$\langle n\rangle \simeq \left(\frac{c_0^{(1,2)}}{\beta_0^{(1,2)}}\right)\ln s\left\{1+2\,\mathrm{Re}\left(\left[\frac{\beta_1^{(0,0)}}{\beta_0^{(0,0)}}\left(\frac{\mu^2}{m^2}\right)^{-\Delta\alpha_1}-1\right]\left(\frac{\beta_1^{(1,2)}}{\beta_0^{(1,2)}}\right)\left(\frac{s}{m_1m_2}\right)^{-\Delta\alpha_1}\right)\right\}+\mathrm{const}, \tag{16}$$

where Eq. (15) has been used. The presence of the term $(s/m_1m_2)^{-\Delta\alpha_1}$ leads to a damped oscillatory component with period $(2\pi/\mathrm{Im}\alpha_1)$ in $\ln s$. In general, all $\beta$'s are complex, so that the phase of the oscillation is still undertermined at this stage. However, for a weak oscillation, i.e., $|\beta_1^{(0,0)}/\beta_0^{(0,0)}| \ll 1$, using also the fact that $|(\mu^2/m^2)^{-\Delta\alpha_1}| \ll 1$, we obtain[10]

$$\langle n\rangle \simeq \left(\frac{c_0^{(0,0)}}{\beta_0^{(0,0)}}\right)(\ln s)\left\{1-2\,\mathrm{Re}\left[\left(\frac{\beta_1^{(1,2)}}{\beta_0^{(1,2)}}\right)\left(\frac{s}{m_1m_2}\right)^{-\Delta\alpha_1}\right]\right\}+\mathrm{const}. \tag{17}$$

We find that in this case the phase of $\langle n\rangle$ relative to that of total cross section is 180°. Furthermore, the amplitude of oscillation is the same as that of total cross section.[1]

Independent of the large-link-mass result, we expect this special phase relation to hold when the oscillations involved are weak. To reach this conclusion only the general factorization property of multiperipheral models is needed. This is because the contribution to the oscillation of $\langle n\rangle$ from $N$ in Eq. (7) is always of second order in $|\beta_1/\beta_0|$ and that from $A$ is of first order. In the ABFST model where only pions are kept as propagating particles, the former is smaller than the latter by a factor of the order $|\beta_1^{(0,0)}/\beta_0^{(0,0)}|$. The smallness of this factor corresponds to a weak oscillation in the $\pi$-$\pi$ amplitude and has been verified numerically for realistic couplings.[5] In more general multiperipheral models, for our conclusion to hold, we need all oscillations associated with internal propagating channels to be weak.

Experimental tests of multiperipheralism in the past have concentrated on the verification of its predictions of gross structures of high-energy reaction cross sections. Here on the other hand, we are looking at fine structures which follow from multiperipheral considerations. Existing accelerator experiments show that weak oscillatory total cross sections may exist in $\pi p$ and $\kappa p$ reactions. Cosmic-ray data on $pp$ average multiplicity, however, seem to indicate that it is free of oscillations.[11] Care must be taken in interpreting these observations because of the competing contributions from secondary Regge trajectories. Nevertheless, if we accept these tentative experimental indications, it follows from Eq. (17) that multiperipheralism suggests the observation of oscillation in $\langle n\rangle_{\pi p}$[12] and $\langle n\rangle_{\kappa p}$ but not in $\sigma_{pp}$. Verification of this fine structure requires high-precision experiments rather than higher energies.

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