Mon. Not. R. astr. Soc. (1985) **215**, 667–683

On the stability and evolution of relativistic radiation tori: equations and speculations

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Accepted 1985 February 26. Received 1985 January 28

Summary. Equations describing the general relativistic evolution of hypothetical radiation-supported tori in active galactic nuclei are presented and discussed. Equations describing global, non-axisymmetric perturbations of radiation tori are also given. The role of instabilities in transporting angular momentum and heat is examined and some simple prescriptions for use in numerical calculation of the secular evolution of these structures are suggested. In particular, if a sufficiently optically thick torus evolves to become dynamically unstable then a gyrotropic, marginally stable convection zone with angular momentum constant on isentropes may be established. Goldreich–Schubert–Fricke instability should be adequate to maintain a radiative zone in a barotropic state.

1 Introduction

In recent years there has been considerable interest in models of active galactic nuclei involving radiation-supported accretion tori orbiting massive black holes (Fishbone & Moncrief 1976; Kozłowski, Jaroszyński & Abramowicz 1978; Abramowicz, Jaroszyński & Sikora 1978; Begelman, Blandford & Rees 1984; Lynden-Bell 1978; Rees 1980; Rees, Begelman & Blandford 1981; Frank 1979).

In many respects these models are similar to the massive star models proposed originally by Hoyle & Fowler (1963). They comprise quasi-spherical, radiation-dominated bodies with radii in the range $\sim 10^{15}$–10^{17} cm emitting ultraviolet radiation at the Eddington limit. They are supposedly surrounded by hyperactive coronae from which may emanate outflowing winds and which may be the origin of the remainder of the infrared to γ-ray spectrum. They presumably energize the more distant emission line and radio-emitting regions and in some instances may provide the collimating mechanism for radio jets through their central funnels.

Unlike the case with massive stars, the gas in radiation tori has generally been assumed to possess negligible self-gravitation and instead to orbit in the non-spherical gravitational field of the central black hole in a hydrostatic balance between pressure gradients, gravitational and rotational forces. In this way it has been supposed (at least implicitly) that the dynamical instability of massive stars can be avoided.

Studies of radiation tori have apparently recapitulated the development of the modern theory of stars. Demonstrations of the existence of equilibrium models with *ad hoc* equations of state (e.g. Paczyński 1980; Paczyński & Abramowicz 1982; Begelman & Meier 1982; Wiita 1982) have been followed by attempts to describe internal energy generation in the absence of a detailed mechanism (Paczyński & Wiita 1980; Abramowicz, Calvani & Nobili 1980; Wiita 1982). More recently, there have been evolutionary calculations in which the torus is assumed to maintain a particular simple structure (isentropic with constant angular momentum). Global time evolution of radiation tori as a 4-parameter family has been studied by Abramowicz, Henderson & Ghosh (1983). These parameters are functions of time only, and are determined by the global conservation laws of mass, angular momentum, mechanical energy and enthalpy. The fluid is barotropic, so by the von Zeipel's theorem, the angular momentum depends only on the axial distance. This dependence is given by a power-law relation. Letting the exponent change with time amounts to angular momentum redistribution. Mass loss by a Roche-lobe type of overflow through the inner radius is also treated. The models have three external parameters which have to be explicitly specified in the conservation equations. These are α, the usual viscosity parameter; the rate at which mass is added to the torus at the outer radius and the torque applied at the outer radius. Using the pseudo-Newtonian potential (Paczyński & Wiita 1980) and assuming that the dynamical time-scale is much smaller than the secular time-scale so that hydrostatic equilibrium is maintained at each time step, they compute the evolutionary sequences of these tori. However, it is not clear that the internal evolution will follow this prescription and a more general formalism is called for (Hawley, Smarr & Wilson 1984).

Conditions for local dynamical instability (essentially the relativistic Høiland criterion) have been given by Seguin (1975). More recently Papaloizou & Pringle (1985) have studied non-axisymmetric global modes. They have demonstrated that isentropic, constant angular momentum tori, which are known to be marginally stable to local perturbations (*cf.* Section 4 below), exhibit unstable global modes. Furthermore, isentropic tori are not generally stabilized by a finite angular momentum gradient (Papaloizou & Pringle 1985). Thin discs are also subject to similar instabilities (Goldreich & Narayan 1984). There appear to be two separate mechanisms at work. First, in the WKB approximation at least, there exist regions of trapped negative energy density from which energy can tunnel into regions of positive energy density. Good reflection at the surface of the torus provides enough feedback to sustain growth. Secondly, Kelvin–Helmholtz modes can be found localized near extrema in the ratio of vorticity to density. The conditions under which tori and discs are dynamically stable (if indeed they exist at all) are still unknown and to determine them will probably require extensive numerical study.

So, despite their attraction from the standpoint of the interpretation of quasar and Seyfert spectra, it must be shown that radiation tori are physically viable. The aims of this paper are limited and twofold. First, we present the general relativistic equations for the growth of linear perturbations and for handling secular evolution in a form that is useful for further analytical work and makes clear their physical content and the relationship to the more familiar non-relativistic equations. Secondly, we suggest prescriptions for handling the internal transport of heat and angular momentum within a numerical calculation of an evolving torus. This part of the paper is clearly predicated on the existence of dynamically stable rotating configurations.

In the following section, we present a compact summary of the assumptions, definitions and notation used in this paper. We specialize immediately to a radiation-dominated, electron-scattering torus as this is the case of pressing interest and it allows some simplification. In Section 3, we give the full dynamical equations for the adiabatic and inviscid evolution of a torus. These

Table 1. Definition of quantities introduced in the text.

Variable	Definition
$g_{\alpha\beta}$	Metric tensor for Boyer–Lindquist coordinates; $g_{rr}=\left(\dfrac{r^2+a^2\cos^2\theta}{r^2-2r+a^2}\right)$; $g_{\theta\theta}=r^2+a^2\cos^2\theta$
g	Determinant $g=-\lvert g_{\alpha\beta}\rvert=(r^2+a^2\cos^2\theta)\sin^2\theta$
g^*	Reduced determinant $g^*=-\lvert g_{ij}\rvert=(r^2-2r+a^2)\sin^2\theta$
∇	2D differential operator, $\nabla\equiv\partial_a$, $\nabla=g^{-1/2}\partial_a g^{1/2}$
u^α	4-velocity of fluid element; $u^\alpha u_\alpha=-1$
A	Redshift factor, $A=u^0$
e	Binding energy, $e=-u_0$
Ω	Fluid angular velocity measured at infinity, $\Omega=u^\varphi/u^0$
l	Geometrical angular momentum, $l=-u_\varphi/u_0$
$\mathbf{u}$	Poloidal velocity $u^a=A(dr/dt,\ d\theta/dt)$
$P_{\alpha\beta}$	Projection tensor, $P_{\alpha\beta}=g_{\alpha\beta}+u_\alpha u_\beta$
a_α	Acceleration, $a_\alpha=u^\beta u_{\alpha;\beta}$
G	$G=e^2/g^*$
γ	$\gamma=e^2(G\nabla l-A^2\nabla\Omega)$
$\varkappa$	Epicyclic frequency. $\varkappa^2=\gamma\cdot\nabla l$
ϱ	Rest mass density
p	Radiation pressure
w	Enthalpy density, $w=\varrho+4p$
S	Normalized 'entropy' per baryon, $S=p^{3/4}\varrho^{-1}$
$\bar{\varkappa}$	Opacity; $\bar{\varkappa}_{\mathrm{T}}$ is the Thomson opacity
η	Microscopic viscosity; $\eta_p=\dfrac{8}{9}\dfrac{p}{\bar{\varkappa}_{\mathrm{T}}\varrho}$ is the radiative viscosity
q_α	Heat flux vector; $q_\alpha=-\dfrac{1}{\bar{\varkappa}\varrho}P_\alpha^\beta(p_{,\beta}+4pa_\beta)=a_\alpha/\bar{\varkappa}$ in the diffusion approximation
$\sigma_{\alpha\beta}$	Shear tensor, $\sigma_{\alpha\beta}=½(u_{\alpha;\mu}P^{\mu\beta}+u_{\beta;\mu}P^{\mu\alpha})-1/3u^\gamma{}_{;\gamma}P_{\alpha\beta}$; $\sigma_{\alpha 0}=-½g^*A^3\Omega\nabla\Omega$; $\sigma_{\alpha\varphi}=½g^*A^3\nabla\Omega$
$T_{\alpha\beta}$	Stress tensor; $T_{\alpha\beta}=wu_\alpha u_\beta+pg_{\alpha\beta}-2\eta\sigma_{\alpha\beta}+q_\alpha u_\beta+u_\alpha q_\beta$

three-(space) dimensional equations must be solved numerically in order to follow the dynamical evolution of a torus from some initial configuration. In Section 4 we discuss the dynamical stability of equilibrium models, generalizing the analysis of Seguin (1975) to global, non-axisymmetric perturbations. In Section 5, we introduce the principal heat conduction and viscous terms and summarize their influence on the stability criteria for short-wavelength modes. One possible prescription for evolving a torus is suggested in Section 6. Our conclusions are collected in the final section.

2 Definitions and assumptions

We are concerned with the evolution of a radiation-supported, electron scattering torus (whose self-gravitation can be neglected) disposed symmetrically about a spinning black hole. We therefore assume *ab initio* that the spacetime is that of a Kerr black hole described by Boyer–Lindquist coordinates ($G=c=1$) with signature $(-+++)$ (e.g. Misner, Thorne & Wheeler 1973, chapter 33). All lengths are measured in units of the gravitational radius of the hole. Indices i,j refer to coordinates $0,\varphi$ and a,b to r,θ. Greek letters denote all four coordinates. Poloidal vectors are denoted by bold type and can be either covariant or contravariant. They are transformed using the metric components g_{ab} in the usual fashion as the equations require.

We also assume that the electron scattering photon mean free path is short enough to justify treating the combined radiation and gas motion hydrodynamically and radiation transport in the diffusion approximation. Note that we do not require that there be local thermal source equilibrium, although in practice for the conditions envisaged within an active galactic nucleus a Thomson optical depth in excess of ~ 1000 will guarantee this (Blandford 1985). Thermodynamic quantities are defined in the rest frame of the fluid which is assumed to be chemically homogeneous and to have negligible gas pressure (see Table 1). Viscosity and heat conduction are handled using the Eckart (1940) representation of the stress energy tensor (e.g. Misner *et al.* 1973, chapter 22). Generalization to alternative stationary, axisymmetric metrics and equations of state is straightforward.

3 Dynamical equations for a perfect fluid

The dynamical equations for a perfect fluid are obtained by setting the divergence of the non-dissipative part of the stress energy tensor to zero and projecting parallel and orthogonal to the 4-velocity. They have been given by several authors (e.g. Bardeen 1970, 1973; Novikov & Thorne 1973).

In our notation and in a form which will prove convenient they are:

$$\frac{\partial}{\partial t}(\varrho A)+\frac{\partial}{\partial \varphi}(\varrho A\Omega)+\nabla\cdot(\varrho \mathbf{u})=0 \qquad \text{(baryon conservation)} \tag{3.1}$$

$$\frac{\partial}{\partial t}(wA)+\frac{\partial}{\partial \varphi}(wA\Omega)+\nabla\cdot(w\mathbf{u})-A\frac{\partial p}{\partial t}-A\Omega\frac{\partial p}{\partial \varphi}-\mathbf{u}\cdot\nabla p=0 \qquad \text{(energy conservation)} \tag{3.2}$$

$$w\left[\frac{\partial \mathbf{u}}{\partial t}+\Omega\frac{\partial \mathbf{u}}{\partial \varphi}+\nabla e-\Omega\nabla(le)+A^{-1}\mathbf{u}\cdot\nabla\mathbf{u}-A^{-1}u^a\nabla u_a\right]$$
$$+\mathbf{u}\left(\frac{\partial p}{\partial t}+\Omega\frac{\partial p}{\partial \varphi}\right)A^{-1}\nabla p+A^{-1}\mathbf{u}(\mathbf{u}\cdot\nabla p)=0 \qquad \text{(Euler equation)} \tag{3.3}$$

$$wAe\left[\frac{\partial l}{\partial t}+\Omega\frac{\partial l}{\partial \varphi}\right]+we(\mathbf{u}\cdot\nabla)l+l\frac{\partial p}{\partial t}+\frac{\partial p}{\partial \varphi}=0 \qquad \text{(angular momentum equation).} \tag{3.4}$$

(3.1), (3.2) combine to give the 'entropy' conservation equation

$$\frac{\partial S}{\partial t}+\Omega\frac{\partial S}{\partial \varphi}+A^{-1}\mathbf{u}\cdot\nabla S=0. \tag{3.5}$$

Note that the entropy equation is valid even if the radiation field is not in thermal equilibrium.

Equations (3.1), (3.2), (3.3), (3.4) constitute five equations in the five unknowns ϱ, p, Ω, $\mathbf{u}$, (N.B. only one of the four quantities A, Ω, l, e is independent). They can, in principle, be used to perform full numerical dynamical calculations though in practice only axisymmetric calculations have hitherto been attempted (e.g. Hawley & Smarr 1985; Hawley, Smarr & Wilson 1984). Equations (3.1)–(3.5) and subsequent equations in this paper are written in a form that removes or minimizes references to the metric tensor, either directly or indirectly through Christoffel symbols. It is in this form that their physical content and the relationship to the corresponding non-relativistic equations becomes most clear. They are not necessarily in the most useful form for numerical work.

4 Equilibrium models of radiation tori and dynamical stability

4.1 EQUILIBRIUM MODELS

Relativistic equilibrium models can be constructed by setting **u** and ∂_i to zero in equations (3.1)–(3.5) (Bardeen 1970, 1973; Fishbone & Moncrief 1976; Kozlowski *et al.* 1978; Jaroszyński, Abramowicz & Paczyński 1980; Abramowicz *et al.* 1978). This leaves the equation of hydrostatic equilibrium

$$\frac{-\nabla\varrho}{w}=\nabla\ln e-\frac{\Omega\nabla l}{1-\Omega l}=-\nabla\ln A+\frac{l\nabla\Omega}{1-\Omega l}. \tag{4.1}$$

Once $l(r,\,\theta)$ is specified, the equipotential surfaces are determined. The pressure distribution $p,(r,\,\theta)$ follows after specifying the density distribution $\varrho(r,\,\theta)$. If the equation of state is barotropic, i.e. $p=p(\varrho)$ then $w=w(p)$ which in turn implies that $\Omega=\Omega(l)$. In a given geometry this defines the relativistic von Zeipel cylinders on which the angular velocity is constant (Abramowicz 1974).

The equilibria defined by equation (4.1) can only be stationary on time-scales shorter than the dissipative (viscous or heat transport) time-scales. They may also be unstable to the exponential growth of linearized perturbations on dynamical time-scales to which possibility we now turn.

4.2 DYNAMICAL STABILITY

Dynamical equations for small perturbations are obtained by linearizing equations (3.1), (3.3), (3.4), (3.5). it is convenient to replace the independent variables φ,t by ξ,θ defined by

$$\xi=e(t-l\varphi) \tag{4.2}$$

$$\eta=Ae^2(\varphi-\Omega t). \tag{4.3}$$

We introduce Eulerian perturbations to ϱ,p, etc. denoted $\delta\varrho$, δp etc. The linearized continuity equation becomes

$$\frac{\partial\delta\varrho}{\partial\xi}+\varrho G\frac{\partial\delta l}{\partial\eta}+\nabla\cdot(\varrho\delta\mathbf{u})=0 \tag{4.4}$$

where $G=e^2/g^*$ and we use the relations $\partial_\xi=A[\partial_t+\Omega\partial_\varphi]$, $\partial_\theta=[l\partial_t+\partial_\varphi]/e$, and $l\delta u^\varphi=\delta u^0=(le/G)\delta l$. The linearized entropy equation is

$$\frac{\partial\delta S}{\partial\xi}+(\delta\mathbf{u}\cdot\nabla)S=0. \tag{4.5}$$

Likewise the linearized angular momentum equation gives

$$w\left[\frac{\partial\delta l}{\partial\xi}+(\delta\mathbf{u}\cdot\nabla)l\right]+\frac{\partial\delta p}{\partial\eta}=0. \tag{4.6}$$

Linearizing the Euler equation gives

$$\mathbf{a}\delta w+w\delta\mathbf{a}+\nabla\delta p=0. \tag{4.7}$$

Following Seguin (1975) we define

$$\gamma=e^2(G\nabla l-A^2\nabla\Omega) \tag{4.8}$$

so that

$$\delta\mathbf{a}=\frac{\partial\delta\mathbf{u}}{\partial\xi}-\gamma\delta l \tag{4.9}$$

where we have used $\delta u_0 = -e^2 A\Omega\delta l$. (In the case of the Schwarzschild metric where $g_{0\varphi}=0$, we have $G=A^2\Omega/l$ and $\gamma=\Omega(1-\Omega l)^{-2}\nabla\ln(l/\Omega)$.)

We substitute equation (4.9) in equation (4.7) to obtain

$$w\left[\frac{\partial\delta\mathbf{u}}{\partial\xi}-\gamma\delta l\right]-\frac{\nabla p\delta w}{w}+\nabla\delta p=0. \tag{4.10}$$

Equations (4.4), (4.5). (4.6) and (4.10) supplemented by the definitions of w and S describe the evolution of small perturbations about a given equilibrium structure. They revert to their familiar non-relativistic form when A, $e\to 1$, $c\to\infty$, $G\to 1/(r^2\sin^2\theta)$. In general, they must be solved numerically. However, it is possible to understand some limiting cases analytically.

4.3 SHORT WAVELENGTH LIMIT

We assume that all perturbed quantities vary as $\exp[i(\sigma t+m\varphi-k_a x^a)]$ with $k_r r$, $k_\theta \gg 1$. We also adopt *ab initio* the ordering: $\sigma=O(\Omega)$, $(\delta p/p)=O(\delta\varrho/\varrho k_r r)$ and assume that all scale lengths are comparable with the radius. The leading terms in the perturbation equations are

$$\mathbf{k}\cdot\delta\mathbf{u}=0 \tag{4.11}$$

$$\delta\mathbf{u}\cdot\nabla\ln S-i\bar{\sigma}\delta\varrho/\varrho=0 \tag{4.12}$$

$$i\bar{\sigma}\delta l+\delta\mathbf{u}\cdot\nabla l=0 \tag{4.13}$$

$$w[i\bar{\sigma}\delta\mathbf{u}-\gamma\delta l]-\frac{\nabla p}{w}\delta\varrho-i\mathbf{k}\delta p=0 \tag{4.14}$$

where

$$\bar{\sigma}=A(\sigma+m\Omega). \tag{4.15}$$

This leads to the dispersion relation

$$\bar{\sigma}^2=\frac{(\mathbf{k}\times\nabla l)\cdot(\mathbf{k}\times\gamma)}{k^2}-\frac{\varrho(\mathbf{k}\times\nabla\ln S)\cdot(\mathbf{k}\times\nabla p)}{w^2k^2} \tag{4.16}$$

which is a simple generalization of Seguin's (1975) dispersion relation to non-axisymmetric modes. Necessary and sufficient conditions for stability to these modes can be derived by regarding the right-hand side of equation (4.16) as a quadratic in k_r/k_θ and requiring it to be positive definite. If we use the relation

$$\varrho(\nabla p\times\nabla\ln S)=w^2(\gamma\times\nabla l) \tag{4.17}$$

[which is derived by taking the 'curl' of the equilibrium equation (4.1)] then these conditions for local stability can be written as a relativistic generalization of the Høiland criteria

$$(\nabla p\times\gamma)\cdot(\nabla l\times\nabla\ln S)>0$$

and

$$w^2(\gamma\cdot\nabla l)-\varrho\nabla p\cdot\nabla\ln S>0 \tag{4.18}$$

correcting some misprints in Seguin (1975). Note that an angular momentum gradient can stabilize a Schwarzschild-unstable entropy gradient and conversely an entropy gradient can stabilize a configuration that is unstable by the Rayleigh criterion. The 'local' or WKB approximations that have been used in deriving the dispersion relation (4.16) break down in the modes discussed by Papaloizou & Pringle (1985). Equation (4.18) imposes restrictions on the relative direction of ∇S, ∇l, γ, ∇p as discussed at greater length in Blandford (1985).

The physical nature of the instability criterion can be understood (Seguin 1975; Blandford 1984). If a small ribbon of fluid moving azimuthally, is displaced slowly and effectively isobarically in the poloidal direction by distance $\delta\mathbf{x}$, conserving both its entropy and its angular momentum, then it will have a different entropy and angular momentum from its surroundings and so will experience buoyancy and centrifugal forces. This effect is essentially the Eulerian perturbation to the acceleration as measured by a comoving observer, $\delta\mathbf{a}_c=\delta\mathbf{a}-\nabla p/w^2\delta w$, where we have used the fact that for these slow disturbances $\delta p=0$.

Now $\delta w=(\partial w/\partial S)_p(\nabla S\cdot\delta\mathbf{x})$ and $\delta\mathbf{a}=-\gamma dl=-(\partial\mathbf{a}/\partial l)\delta l$ since $\delta\mathbf{u}=0$ (the ribbon is held stationary before being released from its displaced position). Therefore the perturbation to the acceleration $\delta\mathbf{a}_c$ is $-\gamma(\nabla l\cdot\delta\mathbf{x})+1/w(\partial w/\partial S)_p(\nabla S\cdot\delta\mathbf{x})\mathbf{a}$. The virtual work done is then $-(\gamma\cdot\delta\mathbf{x})(\nabla l\cdot\delta\mathbf{x})+1/w(\partial w/\partial S)_p(\nabla S\cdot\delta\mathbf{x})(\mathbf{a}\cdot\delta\mathbf{x})$, which must be negative for stability. Now incompressibility (equation 4.11) dictates that the displacement $\delta\mathbf{x}$ be perpendicular to $\mathbf{k}$ and $(\partial w/\partial S)_p=-\varrho/S$ and so we recover the stability conditions obtained from equation (4.16).

4.4 ADIABATIC TORUS

Some simplification of the general perturbation equations ensues if we stipulate that the torus be adiabatic. In this case, $\delta(\nabla p/w)=\nabla(\delta p/w)$, $(\delta\varrho/\varrho)-3/4(\delta p/p)$ and we drop the entropy equation. The three remaining equations are

$$\frac{3\varrho w}{4p}\frac{\partial\delta h}{\partial\xi}+\varrho G\frac{\partial\delta l}{\partial\eta}+\nabla\cdot(\varrho\delta\mathbf{u})=0 \tag{4.19}$$

$$\frac{\partial\delta l}{\partial\xi}+(\delta\mathbf{u}\cdot\nabla)l+\frac{\partial\delta h}{\partial\eta}=0 \tag{4.20}$$

$$\frac{\partial}{\partial\xi}\delta\mathbf{u}-\gamma\delta l+\nabla\delta h=0 \tag{4.21}$$

where

$$\delta h=\delta p/w. \tag{4.22}$$

Next, let us assume a variation $\propto\exp[i(\sigma t+m\varphi)]$. We replace $\partial/\partial\xi$ with $i\bar{\sigma}$ (equation 4.15) and $\partial/\partial\eta$ with $i\mu$ where

$$\mu=(l\sigma+m)/e. \tag{4.23}$$

It also proves convenient to introduce the square of the local epicyclic frequency

$$\varkappa^2=\gamma\cdot\nabla l=e^2G|\nabla l|^2-\frac{\nabla l\cdot\nabla\Omega}{(1-\Omega l)^2}. \tag{4.24}$$

Note that from equation (4.18) ∇l is antiparallel to $\nabla\Omega$ if the torus is stable to short-wavelength perturbations. We eliminate δl between the angular momentum and Euler equations and solve for $\delta\mathbf{u}$

$$\delta\mathbf{u}=\frac{i}{\bar{\sigma}^2}\left[\frac{[\bar{\sigma}(\gamma\cdot\nabla)\delta h+\mu\gamma^2\delta h]\nabla l}{(\sigma^2-\varkappa^2)}+\bar{\sigma}\nabla\delta h+\mu\gamma\delta h\right]. \tag{4.25}$$

We next substitute in the continuity equation to obtain a single partial differential equation for δh

$$\nabla\cdot\left[\frac{\varrho}{\bar{\sigma}^2}\left\{\frac{\bar{\sigma}(\gamma\cdot\nabla)\delta h+\mu\gamma^2\delta h}{\bar{\sigma}^2-\varkappa^2}\nabla l+\bar{\sigma}\nabla\delta h+\mu\gamma\delta h\right\}\right]$$

$$-\frac{\mu\varrho G}{\bar{\sigma}^3}\left[\frac{\bar{\sigma}(\gamma\cdot\nabla)\delta h+\mu\gamma^2\delta h}{\bar{\sigma}^2-\varkappa^2}(\nabla l)^2+\bar{\sigma}\nabla l\cdot\nabla\delta h+(\bar{\sigma}^2+\varkappa^2)\mu\delta h\right]$$

$$+\frac{3}{4}\frac{\bar{\sigma}\varrho w\delta h}{p}=0. \tag{4.26}$$

Equation (4.26) agrees with the corresponding equation given by Papaloizou & Pringle (1985) in the non-relativistic limit. It can be solved in principle subject to suitable boundary conditions at the surface of the torus. (The most natural boundary condition is $\delta h\to 0$; the vanishing of $\nabla\delta h$ implies that the surface shape and velocity field is kept fixed.) However it may prove more practical to work with the three first-order equations (4.19)–(4.21) and to keep the time-dependence explicit.

Equation (4.26) displays regular singularities at the corotation and Lindblad resonances where $\bar{\sigma}=0, \pm\varkappa$; that is to say when the frequency of the perturbation measured by a comoving observer either vanishes or coincides with the epicyclic frequency.

4.5 CONSTANT ANGULAR MOMENTUM

Equation (4.26) for the perturbation δh becomes much simpler when both the angular momentum and the entropy are constant. In this case, it is convenient to change the dependent variable to δl. After some algebra, equation (4.26) reduces to

$$\nabla\cdot(\varrho e\nabla\delta l)-\mu^2 G\varrho e\delta l+\frac{3}{4}\frac{\bar{\sigma}^2 e\varrho w\delta l}{p}=0. \tag{4.27}$$

Again this is the relativistic generalization of the equation used by Papaloizou & Pringle (1985) in their demonstration of the dynamical instability of isentropic non-relativistic tori with constant angular momentum.

Following Paploizou & Pringle (1985), we introduce a fixed angular frequency Ω_c and define

$$\sigma_c=\sigma+m\Omega_c. \tag{4.28}$$

Multiplying equation (4.27) by δl^* and integrating with respect to $g^{1/2}dx^1dx^2$ we obtain a quadratic for the angular frequency σ_c

$$\sigma_c^2X+\sigma_cY-Z=0 \tag{4.29}$$

where

$$X=\int|\delta l|^2\left[\frac{3}{4}\frac{w}{p}A^2-\frac{l^2G}{e^2}\right]\varrho e g^{1/2}dx^1dx^2, \tag{4.30}$$

$$Y=2\int|\delta l|^2\left[\frac{3}{4}\frac{w}{p}A^2m(\Omega-\Omega_c)-\frac{Glm(1-\Omega_c l)}{e^2}\right]\varrho e g^{1/2}dx^1dx^2, \tag{4.31}$$

$$Z=\int\left\{|\nabla\delta l|^2+m^2|\delta l|^2\left[(1-\Omega_c l)^2\frac{G}{e^2}-\frac{3}{4}\frac{wA^2}{p}(\Omega-\Omega_c)^2\right]\right\}\varrho e g^{1/2}dx^1dx^2. \tag{4.32}$$

If $X>0$, as it must be for all cases of physical interest, then a necessary condition for the presence of growing modes is that Z be negative. It is the presence of the third term in equation

(4.32) associated with the compressibility of the gas which allows instability to occur. Further generalization of Papaloizou & Pringle (1985) is possible.

5 Thermal conductivity and viscosity

5.1 SECULAR EVOLUTION INCLUDING RADIATIVE DIFFUSION AND VISCOSITY

If a particular torus is dynamically stable, it must still evolve on either a thermal or a viscous time-scale. These two time-scales are, by assumption, separate from the dynamical time-scale, which is effectively the rotation period. In a radiative zone, photons will diffuse through the torus. The relativistic heat flux in the diffusion approximation is given by

$$q_\alpha = -\frac{1}{\bar{\varkappa}\varrho} P_\alpha^\beta (p_\beta + 4 p a_\beta). \tag{5.1}$$

Using the equation of hydrostatic (4.1), the poloidal part becomes

$$q_{\mathrm{a}} = \frac{a_a}{\bar{\varkappa}}. \tag{5.2}$$

To a fairly good approximation, the opacity $\bar{\varkappa}$ equals the constant Thomson opacity $\bar{\varkappa}_{\mathrm{T}}$. The $0, \varphi$ components are of smaller order and can be ignored.

The heat flux contributes an amount $q_\alpha u_\beta + u_\alpha q_\beta$ to the stress energy tensor $T_{\alpha\beta}$. It therefore contributes a term

$$u_\alpha [q^\alpha u^\beta + u^\alpha q^\beta]_{;\beta} = u^\alpha (u^\beta q_\beta)_{;\alpha} - a_\alpha q^\alpha - q^{\alpha;\alpha} \tag{5.3}$$

$$= -\frac{[\mathbf{a}^2 + \nabla \cdot \mathbf{a}]}{\bar{\varkappa}_T} \tag{5.4}$$

to the energy equation where we have assumed a Thomson opacity in equation (5.4). The first term in equation (5.4) is of purely relativistic origin and takes into account the gravitational redshift and Doppler shift of the escaping photons. The second term in equation (5.4) is the normal divergence of the non-relativistic heat flux and may be evaluated using a relation derivable from the Raychaudhuri equation.

$$\nabla \cdot \mathbf{a} = \frac{A^4 g^* \nabla\Omega \cdot \nabla\Omega + e^4 \nabla l \cdot \nabla l / g^*}{2} \tag{5.5}$$

A heat flux **q** will also carry off angular momentum. This contributes a term

$$\Delta T^\alpha_{i;\alpha} = q^\alpha u_{i,\alpha} + q^\alpha_{;\alpha} u_i \tag{5.6}$$

to the divergence of the stress energy tensor and a term

$$\mathbf{e}^2 (\mathbf{q} \cdot \nabla) l = \frac{e^2 (\mathbf{a} \cdot \nabla) l}{\bar{\varkappa}_{\mathrm{T}}} \tag{5.7}$$

to the angular momentum equation (3.5).

The appropriate shear viscosity in a radiative zone in the absence of significant magnetic stresses is

$$\eta = \frac{8p}{9 \bar{\varkappa}_{\mathrm{T}} \varrho}. \tag{5.8}$$

The associated viscous contribution to the stress energy tensor is

$$\tau_{\alpha\beta}=-2\eta\sigma_{\alpha\beta} \tag{5.9}$$

where $\sigma_{\alpha\beta}$ is the shear tensor. For the stationary, axisymmetric, toroidal flow of an equilibrium torus the only non-vanishing components of the shear tensor are

$$\sigma_{\mathrm{a}}^{0}=\tfrac{1}{2}leA^{2}\nabla\Omega$$

$$\sigma_{\mathrm{a}}^{\varphi}=\tfrac{1}{2}eA^{2}\nabla\Omega. \tag{5.10}$$

Viscous dissipation contributes a term

$$-\tau^{\alpha\beta}\sigma_{\alpha\beta}=\eta g^{*}A^{4}\nabla\Omega\cdot\nabla\Omega \tag{5.11}$$

to the energy equation (3.2). Equivalently, the entropy equation (3.3) augmented to include heat transport and viscosity becomes

$$4\varrho p^{1/2}\left[A\frac{\partial}{\partial t}\left(\frac{p^{3/4}}{\varrho}\right)+A\Omega\frac{\partial}{\partial\varphi}\left(\frac{p^{3/4}}{\varrho}\right)+\mathbf{u}\cdot\nabla\left(\frac{p^{3/4}}{\varrho}\right)\right]=\eta g^{*}A^{4}\nabla\Omega\cdot\nabla\Omega-\nabla\cdot\mathbf{q}-\mathbf{a}\cdot\mathbf{q}. \tag{5.12}$$

Similarly, a short calculation shows that viscous stress contributes a term

$$(u_{\varphi}P_{0\alpha}-u_{0}P_{\varphi\alpha})\tau^{\alpha\beta}_{;\beta}=\nabla\cdot(\eta g^{*}A^{2}\nabla\Omega)-\eta g^{*}A^{2}\mathbf{a}\cdot\nabla\Omega \tag{5.13}$$

to the angular momentum equation (3.5). The first term on the right-hand side of equation (5.13) appears in the non-relativistic equation whereas the second term is of relativistic origin. The full angular momentum equation including heat transport is

$$wAe^{2}\left[\frac{\partial l}{\partial t}+\Omega\frac{\partial l}{\partial\varphi}\right]+we^{2}\mathbf{u}\cdot\nabla l+le\frac{\partial p}{\partial t}+e\frac{\partial p}{\partial\varphi}=\nabla\cdot(\eta g^{*}A^{2}\Omega)-\eta g^{*}A^{2}\mathbf{a}\cdot\nabla\Omega-e^{2}(\mathbf{q}\cdot\nabla)l \tag{5.14}$$

with $\mathbf{q}$ given by equation (4.2).

5.2 INSTABILITY DRIVEN BY VISCOUS STRESS

In Section 4 we discussed the dynamical stability of radiation tori and gave the necessary and sufficient conditions for stability against local modes. We also gave the equations which must be solved in order to test a particular structure for global stability. However even if a torus is stable to dynamical modes, it may be unstable to short-wavelength modes driven by viscosity. This type of instability is the relativistic generalization of the Goldreich–Schubert–Fricke instability (Goldreich & Schubert 1967; Fricke 1968) and has also been discussed by Seguin (1975). If we slowly displace a slender ring of fluid radially, then it will remain in pressure equilibrium with its surroundings. This means that in so far as we neglect gas pressure there will be no heat exchange between the ring and its surroundings and we can ignore thermal conductivity in the perturbation equations. However the ring can exchange angular momentum with its surroundings through the action of viscous torques. (Seguin 1975) measures the relative importance of thermal conductivity and viscosity by the parameter (essentially the reciprocal of the Prandtl number).

$$L=\frac{Kw}{\eta\varrho C_{\mathrm{p}}}=\frac{9\varrho\beta}{32p} \tag{5.15}$$

where K is the thermal conductivity, β is the ratio of gas to radiation pressure and C_{p} is the specific heat per unit mass at constant pressure. L is generally much smaller than unity in radiation tori and so viscous effects dominate.

For the above reason, the continuity and entropy perturbation equations (4.11), (4.12) are unaffected. However, the angular momentum equation (4.13) must be augmented by a term $\eta k^2\delta l/w$ on the left-hand side (*cf.* equation 5.13). Similarly, the largest correction to the perturbed Euler equation (4.14) is a term $\eta k^2\delta\mathbf{u}$ on the left-hand side. The dispersion relation (4.16) now becomes

$$\bar{\sigma}\left(\bar{\sigma}-i\eta\frac{k^2}{w}\right)^2=\frac{\bar{\sigma}(\mathbf{k}\times\nabla l)\cdot(\mathbf{k}\times\gamma)}{k^2}-\frac{(\bar{\sigma}-i\eta k^2/w)\varrho(\mathbf{k}\times\nabla\ln S)\cdot(\mathbf{k}\times\nabla p)}{w^2k^2}. \tag{5.16}$$

Again, this is the simple generalization of the dispersion relation given by Seguin (1975) for axisymmetric perturbations to low-order non-axisymmetric disturbances of short wavelength.

Examining this dispersion relation, we see that it is always possible to find a growing mode unless $(\mathbf{k}\times\nabla\ln S)\cdot(\mathbf{k}\times\nabla p)$ is negative for all choices of $\mathbf{k}$. That is to say, radiation tori are locally unstable to modes driven unstable by radiative viscosity unless ∇S is antiparallel to ∇p. For sufficiently short wavelengths, angular momentum gradients are unable to stabilize a Schwarzschild unstable entropy gradient. These instabilities can only be suppressed if the fluid is barotropic, i.e. the entropy is a function of the pressure and so from equation (4.1), the angular velocity is constant on surfaces of constant angular momentum.

To calculate the fastest growing mode, it is convenient to introduce a unit poloidal vector $\mathbf{n}_\alpha=(0, n_\mathrm{a})$, in the direction of the displacement $\delta\mathbf{x}$ which satisfies

$$n_\alpha n^\alpha=n_\mathrm{a}n^\mathrm{a}=1,\quad n_\mathrm{a}k^\mathrm{a}=0. \tag{5.17}$$

For the unstable modes $\mathbf{n}$ lies between constant pressure and constant entropy surfaces; for a barotropic configuration $\mathbf{n}\cdot\nabla p=0$. For convenience introduce

$$x^2=(\mathbf{n}\cdot\gamma)(\mathbf{n}\cdot\nabla l) \tag{5.18}$$

$$y=\eta k^2/w \tag{5.19}$$

$$z^2=-(e/w^2)(\mathbf{n}\cdot\nabla\ln S)(\mathbf{n}\cdot\nabla p) \tag{5.20}$$

and write

$$\bar{\sigma}=v-i\omega \tag{5.21}$$

so that ω is the instability growth rate whose maximum value we are interested in. Then equation (5.16) can be written as

$$(i\bar{\sigma})^3+2y(i\bar{\sigma})^2+(x^2+y^2+z^2)(i\bar{\sigma})+yz^2=0. \tag{5.22}$$

Its imaginary part is

$$3\omega^2v-v^3+4y\omega v+(x^2+y^2+z^2)v=0 \tag{5.23}$$

which implies that either

$$v=0\quad\text{and}\quad\omega^3+2y\omega^2+(x^2+y^2+z^2)w+yz^2=0 \tag{5.24}$$

or

$$v^2=3\omega^2+4y\omega=x^2+y^2+z^2$$

and

$$\omega^3+2y\omega^2+\tfrac{1}{4}(x^2+5y^2+z^2)\omega+\tfrac{1}{8}(2x^2+2y^2+z^2)y=0. \tag{5.25}$$

If the configuration is dynamically stable, (4.16) implies that $x^2+z^2>0$, so there is only one unstable solution to (5.22) $v=0$ and $\omega>0$ for $y>0$. Solving (5.24) for y, we have the maximal instability rate

$$\omega_{\max}=-\frac{1}{2}\frac{z^2}{x^2}=\frac{1}{2}\frac{\varrho}{w^2}\frac{(\mathbf{n}\cdot\nabla\ln S)(\mathbf{n}\cdot\nabla\varrho)}{[(\mathbf{n}\cdot\gamma)(\mathbf{n}\cdot\nabla l)]^{1/2}} \tag{5.26}$$

which occurs for

$$y=x+z^2/2x. \tag{5.27}$$

$\lambda_v\sim(\eta r^2/wl)^{1/2}$ is the maximum radial wavelength of the perturbation for which viscous dissipation occurs on the time-scale of interest. For $\lambda\gg\lambda_v$ viscous effects are not important, for $\lambda\ll\lambda_v$ the perturbation is damped out. Instability driven by viscosity is most important for $\lambda\sim\lambda_v$.

6 Evolution of radiation tori

Even if a portion of a torus is created in a dynamically stable configuration, then it may evolve to become dynamically unstable as discussed in Section 4. If the fluid is sufficiently optically thick, then slow convective overturn should be able to maintain the entropy and angular momentum distributions close to marginal stability just as in a deep stellar convection zone. However, one additional relationship must be specified to fix conditions within the convection zone. One prescription, suggested by Bardeen (1973) and used by Paczyński & Abramowicz (1982) is to assume that the medium is gyrotropic, i.e. there exists a unique function $S(l)$. This is appropriate if the entropy and angular momentum are both efficiently transported together by subsonic motions. This prescription which implies marginal stability according to the axisymmetric condition (4.18) and also enables us to write the equation of hydrostatic support in the form

$$\nabla[\ln(we/\varrho)]=F\nabla l$$

where

$$F=\frac{\Omega}{1-\Omega l}+\frac{p}{w}\frac{d\ln S}{dl}. \tag{6.1}$$

This in turn implies that the Bernoulli constant, $B=we/\varrho$ and the quantity F are, in addition to the entropy S, constant on surfaces of constant angular momentum. So, if S, l, B, F are specified on the surface of the convection zone, then we can solve for them everywhere else within the convection zone. In fact the solution is implicit. One procedure is to solve $B(l)=w[\varrho,S(l)]e(l,r,\theta)/\varrho$ for $S(l,r,\theta)$ and similarly to solve $F(l)=F\{\Omega(l,r,\theta),l,p[\varrho,S(l)],w[\varrho,S(l)]\}$ for $\varrho(l,r,\theta)$. Then we eliminate ϱ to obtain the isogyres [$l(r,\theta)$=constant surfaces].

We use mixing length theory to quantify the efficiency of heat and angular momentum transport. The convective velocity is roughly

$$v_{\mathrm{con}}\sim[(\gamma\cdot\mathbf{H})(\nabla l\cdot\mathbf{H})+(\mathbf{a}\cdot\mathbf{H})(\nabla\ln S\cdot\mathbf{H})]^{1/2} \tag{6.2}$$

where $\mathbf{H}$ is a vector of length equal to the mixing length (typically a pressure scale height) along the direction of heat and angular momentum transport. The convective heat flux is

$$q_{\mathrm{con}}\lesssim-4pv_{\mathrm{con}}(\mathbf{H}\cdot\nabla\ln S) \tag{6.3}$$

and the angular momentum flux (or equivalently the torque per unit area) is

$$G\lesssim-wV_{\mathrm{con}}(\mathbf{H}\cdot\nabla l). \tag{6.4}$$

The gyrotropic approximation should be valid when $|\mathbf{H}\cdot\nabla\ln S|$, $|\mathbf{H}\cdot\nabla\ln l|\ll1$. We estimate the convective heat flux by the radiative heat flux, $q_{\mathrm{con}}\sim|\mathbf{a}|/\bar{\varkappa}_{\mathrm{T}}$ (the ratio of the two heat fluxes is expected to be O(1) in the deep interior) and the angular momentum flux by $\sim5R\Omega/\bar{\varkappa}_{\mathrm{T}}$, where we have assumed that the mass inflow is sufficient to maintain an Eddington luminosity and the efficiency is 20 per cent. On the basis of these estimates, we derive an expression for the ratios of the gradients of $\ln S$ and $\ln l$,

$$\frac{|\mathbf{H}\cdot\nabla\ln S|}{|\mathbf{H}\cdot\nabla\ln l|}\sim 10^{-1}. \tag{6.5}$$

It seems that to within a factor ≈ 5, heat and angular momentum are transported with comparable efficiency by convective motions. This provides further support for the gyrotropic approximation.

It appears to be harder to transport entropy than angular momentum in the outer parts of the torus and we can use the estimate of the convective heat flux to determine the optical depth at which the superadiabatic entropy gradient becomes substantial and where consequently the gyrotropic approximation breaks down

$$|\mathbf{H}\cdot\nabla\ln S|\sim\tau_{\mathrm{T}}^{-2/3}(\mathbf{a}\cdot\mathbf{H})^{-1/3}. \tag{6.6}$$

Now $(\mathbf{a}\cdot\mathbf{H})\lesssim(\Omega r)^2$ and so convection will certainly be inefficient when $\tau_{\mathrm{T}}\lesssim(\Omega r)^{-1}\approx 30$. The gyrotropic approximation is probably inappropriate when $|\mathbf{H}\cdot\nabla\ln S|$ as given by equation (6.6) with H estimated by the pressure scale height is less than 0.1.

This result viz that convection (as usual) is fairly inefficient near the photosphere, has a particularly striking consequence in the case of a radiation torus. If a significant fraction of the heat flux is carried by convection through the outer envelope, then the atmospheric layers cannot be in hydrostatic equilibrium and will presumably be blown off by radiation pressure in the form of a dense wind. Only if the atmospheric scale height is comparable with the radius can a compensatory change in the centrifugal force allow there to be both radiative and hydrostatic equilibrium.

In Section 5, we considered the influence of radiative viscosity. It is also of interest to enquire whether or not a torus can evolve radiatively under the action of radiative viscosity alone in the absence of other more powerful instabilities. It turns out that radiative viscosity is roughly ten times too small to transport angular momentum efficiently in the innermost part of the torus and as a result a torus evolving under radiative viscosity alone would simply cool and deflate to form a thin ring. [This can be confirmed by estimating $p\approx 0.1\varrho$ and comparing the viscous heating and radiative cooling rates according to equations (5.4) and (5.11).]

As was also discussed in Section 5, radiative viscosity can cause a double diffusive instability which drives the fluid towards barotropicity. The issue now is can this instability maintain barotropicity in the face of radiative diffusion which will tend to destroy it? The answer appears to be yes as long as the optical depth of the torus is sufficiently large. To demonstrate this we follow Durney & Spruit (1979) and treat the convection as a small-scale turbulence and introduce a specific form of the (anisotropic) viscous stress tensor. This turbulent viscosity, which is computed in the Appendix behaves much like a molecular viscosity with the viscous stress tensor being proportional to the shear tensor; angular momentum is transported against the angular velocity gradient lowering the mechanical energy of the system (*cf*. Bardeen 1973).

We introduce the scalar quantity η_1 which measures the value of viscosity (equation A.11). Using equation (A.8), the heat dissipation rate becomes

$$-\tau_{\alpha\beta}\sigma^{\alpha\beta}=\eta_1 g^*A^4(\mathbf{n}\cdot\nabla\Omega)(\mathbf{n}\cdot\nabla\Omega) \tag{6.7}$$

which can be compared with equation (5.11). Similarly, the contribution of viscous torques to the angular momentum equation becomes:

$$(u_\varphi P_{0\alpha}-u_0P_{\varphi\alpha})\tau^{\alpha\beta}_{;\beta}=\nabla[A\eta_1 g^*A^2\mathbf{n}(\mathbf{n}\cdot\nabla\Omega)]. \tag{6.8}$$

There is only one term on the RHS of the above equation (*cf.* equation 5.13), because the angular momentum flux is perpendicular to the acceleration **a**.

Using the estimate of η_1 given in equation (A.11) and assuming that the mixing length is of the same order as the entropy or pressure scale heights we check again whether the structure can be quasi-stationary. For topological reasons the configuration must be locally barotropic near the equatorial plane and the following arguments are both invalid and unnecessary there. At higher latitudes, equating the heating rate to the cooling rate gives an estimate for the angle, 2ε, between the isentropes (S=constant surfaces) and the isobar (p=constant surfaces)

$$\varepsilon^2 \approx \frac{\varkappa}{\bar{\varkappa}}\left(\frac{w}{\varrho p}\right) \tag{6.9}$$

where 2ε is the angle between the isentropes (S=constant surfaces) and isobars (p=constant surfaces) and $\varkappa$ is the epicyclic frequency $\sim\Omega$. Solving for ε

$$\varepsilon \approx \left(\frac{w\Omega r}{p\tau_{\rm T}}\right)^{1/2} \tag{6.10}$$

we see again that $\varepsilon \ll 1$ and the barotropic approximation should be valid in a radiative zone.

Our conclusion then is that provided $\tau \gg \varrho\Omega rc/p$ the angle ε is small and any radiative zone will remain approximately barytropic. It should be possible to devise an implicit numerical procedure in which the heat always flows normal to the isobars according to equation (5.2) and angular momentum is redistributed along the isobars in just such a way as to keep the radiative zone barytropic and in hydrostatic equilibrium.

7 Conclusion

In this paper, we have outlined some procedures for investigating further the properties of relativistic radiation tori. The most pressing issue is the non-axisymmetric global stability which can be studied by numerical integration of equations (4.4)–(4.10). If essentially toroidal configurations are unstable it will be necessary to perform fully only numerical calculations along the lines of Hawley *et al.* (1984) to determine if the unstable modes are limited to small amplitudes by non-linear effects. If the large amplitudes are always present then the assumption of hydrostatic equilibrium will be violated and the remainder of this paper, and much earlier work on this topic, is irrelevant. To date the non-axisymmetric stability investigations have been limited to non-relativistic equilibria. The inclusion of the relativistic terms given above will probably only exacerbate instabilities but the fully relativistic problem should strictly be addressed before final conclusions are drawn.

If, alternatively, there exist dynamically stable configurations, then the prescription outlined in Section 6 (namely, that the fluid is gyrotropic in a convection zone and barytropic in a radiave zone) can be followed in modelling the evolution of a radiation torus. Of course, the evolution path is dictated by the rate of supply of mass which is quite probably intermittent. Nevertheless, it may turn out that the tori generally evolve towards a simple structure. This would be a considerable simplification in confronting observational data on active galactic nuclei with physical models of radiation tori.

Acknowledgments

We thank Martin Rees and Jim Pringle for comments. RB acknowledges support by the National Science Foundation under grant AST82-13001, and the Alfred P. Sloan Foundation, RB also acknowledges hospitality at the Institute of Astronomy, Cambridge, during 1983 July where some of these calculations were carried out. MJ acknowledges an AST grant at the California Institute of Technology where this work was initiated, and the partial support of the Polish Academy of Sciences. SK would like to thank Trinity College, Cambridge, for a research studentship.

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Appendix: Turbulent viscosity

We estimate the viscosity in a radiative zone using the mixing length approximation. The exponential growth of perturbations with the rate ω implies the following relation between the velocity v and the travelled distance d;

$$v \sim \omega d. \tag{A1}$$

Averaging in space over one mixing length H one obtains for the typical eddy velocity

$$v \sim \tfrac{1}{2}\omega H. \tag{A.2}$$

The linear perturbation equations (4.11–15) augmented by the linearized viscous terms from equation (5.14) give the following conditions

$$\delta l = -\frac{\delta \mathbf{u} \cdot \nabla l}{\omega + y}. \tag{A.3}$$

Introducing the toroidal unit vector m_α

$$m_\alpha m^\alpha = 1, \qquad m_\alpha u^\alpha = 0 \tag{A.4}$$

one can decompose the eddy velocity v_α into toroidal and poloidal parts

$$v_\alpha = \left(\frac{e^2}{g^{*1/2}}\right) \delta l m_\alpha + \delta \mathbf{u}. \tag{A.5}$$

The eddy motions are practically two dimensional (if we are near marginal stability) and so the viscosity must be described by a tensor quantity,

$$\tau_{\alpha\beta} = -\eta_{\alpha\beta\gamma\delta} \omega^{\gamma\delta} \tag{A.6}$$

where $\eta_{\alpha\beta\gamma\delta} = \eta_{\alpha\beta\delta\gamma} = \eta_{\gamma\delta\alpha\beta}$ is the viscosity tensor (Lifshitz & Pitaevskii 1981). The viscosity tensor also satisfies the orthogonality condition $u^\alpha \eta_{\alpha\beta\gamma\delta} = 0$. We assume that the turbulent eddies can be described in a manner similar to that used for molecules of an ideal gas. In the spirit of Lifshitz & Pitaevskii, we postulate that the turbulent viscosity tensor has the form

$$\frac{1}{15} \eta_{\alpha\beta\gamma\delta} = wvL \left\langle \frac{v_\alpha v_\beta (v_\gamma v_\delta - \tfrac{1}{3} v^2 P_{\gamma\delta})}{v^4} \right\rangle \tag{A.7}$$

where the angular brackets mean the averaging over the 'eddy distribution'. The product wvL is the usual mixing length theory estimate of the viscosity. The projection tensor $P_{\gamma\delta}$ replaces the unit tensor in three dimensions. The numerical factor is necessary to reduce to the usual relation $\tau_{\alpha\beta} = -2wvL\sigma_{\alpha\beta}$ in the case when v_α is isotropic.

Using the equations (A5, 6, 7) and (5.9) one obtains for the viscous stress

$$\tau_{\alpha\beta} = -15wvL\left(\left\langle \frac{v_n^3 v_m}{v^4} \right\rangle n_\alpha n_\beta + \left\langle \frac{v_n^2 v_m^2}{v^4} \right\rangle (n_\alpha m_\beta + m_\alpha n_\beta) + \left\langle \frac{v_n v_m^3}{v^4} \right\rangle m_\alpha m_\beta \right) A^2 g^{*1/2} \mathbf{n} \cdot \nabla \Omega. \tag{A.8}$$

The second group of terms describes the angular momentum transport. We define

$$\eta_L = 15wvL \left\langle \frac{v_n^2 v_m^2}{v^4} \right\rangle = \frac{15}{2} w\omega L^2 \left\langle \frac{v_n^2 v_m^2}{v^4} \right\rangle. \tag{A.9}$$

For the most rapidly growing perturbations one can approximate $15/2 \langle v_n^2 v_m^2 / v^4 \rangle \approx 1$. Let 2ε be the small angle between the $S = \text{const}$ and $p = \text{const}$ surfaces. Let H_S and H_p be the entropy and pressure scale heights respectively. If we are not too close to the equatorial plane the growth rate ω can be estimated as

$$\omega \approx \frac{1}{2} \frac{\varrho}{w^2} \frac{\varepsilon}{H_S} \frac{\varepsilon p}{H_p} \frac{1}{\varkappa} \tag{A.10}$$

where we have employed equations (A1, 3, 9). $\varkappa$ is the epicyclic frequency given in (4.24). For the viscosity we get

$$\eta_1 = \frac{1}{2} \varepsilon^2 \frac{p}{w} \varrho \frac{L^2}{H_S H_p} \frac{1}{\varkappa}. \tag{A.11}$$

Note added in proof

Drs Goodman and Narayan have pointed out to us that the boundary conditions used in deriving equation (4.29) (i.e. $\delta h \to 0$ or $\nabla \delta h \to 0$) are probably not appropriate in practice. They should be replaced by the condition that the Lagrangian pressure perturbation vanish at the surface of the torus (i.e. $\bar{\sigma}^2 \delta h - \mathbf{a} \cdot \nabla \delta h = 0$). If this boundary condition is used, surface terms are introduced into equation (4.29) and the association of instability with compressibility does not necessarily follow.