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Compton scattering in a converging fluid flow – II. Radiation-dominated shock

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Summary. The problem of Compton scattering in an optically thick fluid flow in which bulk motion is the dominant source of photon heating is illustrated by analysing a radiation-dominated, plane-parallel shock of speed u with photon to electron ratio greatly exceeding $\sim (m_p/m_e)$. In traversing the shock (of thickness $\sim (c/u)$ Thomson optical depths), a typical photon experiences $(c/u)^2$ scatterings, each one giving a secular fractional energy increase $\sim (u/c)^2$ and a total average increase of order unity. In a converging fluid flow, an exponentially small number of photons are accelerated to an exponentially large energy. Thus, a power-law spectrum will be transmitted at high frequencies. For a shock of Mach number M, bulk acceleration produces a spectral index $\alpha = (M^2 - \frac{1}{2})(M^2 + 6)(M^2 - 1)^{-2}$, which tends to unity for a strong shock. The applicability of these results to quasars and the microwave background is briefly discussed.

1 Introduction

It is now well established that, in many astrophysical environments, radiation is emitted by hot plasmas that are far from thermodynamic equilibrium, but which may be quite optically thick to Thomson scattering by free electrons (i.e. active galactic nuclei, X-ray binaries). In the successive scatterings which they undergo prior to escaping from the plasma, photons are accelerated by the random motion of the scatterers and a power-law radiation spectrum can be produced (Katz 1976; Shapiro, Lightman & Eardley 1976; Payne 1980; Sunyaev & Titarchuk 1980). Such studies usually assume that the emitting plasma forms a static atmosphere, and ignore acceleration of the photons by bulk motion (self-consistently so, given the contexts of these problems).

These considerations mirror earlier developments in the theory of the acceleration of cosmic rays (Fermi 1949) in which the scatterers are usually clouds or hydromagnetic disturbances. In both instances, as the velocity of the scatterer is randomly directed, the acceleration is a *second-order process.* More recently (Axford, Leer & Skadron 1977; Bell 1978a,b; Blandford & Ostriker 1978; Eichler 1979) it has been pointed out that, in the vicinity of a shock front, a *first-order process* can operate between the converging flow on either side of the shock front and that, in this case also, a power-law cosmic ray spectrum (whose slope is fixed by the shock compression) can be produced. It is therefore of interest to ask whether this second mechanism has a photon analogue.

In the simplest treatment of the cosmic ray mechanism, test particles are scattered by 'cold' Alfvén waves and cross a collisionless shock (of speed u) in the background medium $\sim (c/u)$ times. Each crossing produces a systematic $\sim (u/c)$ gain in relative energy, leading to a net mean energy gain of order unity and a power-law spectrum at high energies. Replacing cosmic rays with photons and Alfvén waves with electrons, we readily see that the photons will be preferentially accelerated by the bulk motion if the Thomson optical depth of the preshock region is at least (c/u) and the electron thermal velocity is less than u. In a *gas-dominated* collisionless shock, the electrons and ions will, under such optically thick conditions, come quickly into equipartition so that the electron thermal velocity is $\sim (m_p/m_e)^{1/2} u$, rendering acceleration by the bulk motion unimportant.

However, in a *radiation-dominated* shock, most of the momentum flux will be converted into radiation pressure over a length-scale $\sim (c/u)$ Thomson optical depths (obtained by balancing convection of the radiation by the background medium with diffusion). In this case, the relative velocity across one optical depth is $\sim u^2/c$. Since an average photon undergoes $\sim (c/u)^2$ scatterings in crossing the shock, there is again a net gain in energy of order unity from the bulk acceleration. This is similar to a cosmic-ray mediated shock (Eichler 1979; Blandford 1980), with the important differences that the diffusion and emission coefficients are known confidently and the diffusion is energy-independent.

In most treatments of radiation hydrodynamics (e.g. Pai 1966; Zel'dovich & Raizer 1967, Weinberg 1972), it is assumed that the radiation field is a blackbody. However, when the gas is sufficiently tenuous, there will be insufficient photon production to establish thermodynamic equilibrium. If the medium is optically thick to Thomson scattering, then there may be time to establish, locally, a Bose–Einstein photon distribution in which the chemical potential is fixed by the photon production rate (e.g. Weaver 1976; Thorne 1981). However, at higher radiation density, bulk acceleration will be the dominant source of photon acceleration and a power-law photon distribution is produced.

In the preceding paper (Blandford & Payne 1981, hereafter referred to as Paper I), the equations describing Compton scattering in an optically thick, fluid flow were derived under the diffusion approximation. In addition, the relative importance of thermal Comptonization and bulk acceleration was discussed, and the conditions under which bulk acceleration dominates were derived. In Section 2 of this paper, we solve these equations for a radiation-dominated shock, in the limit of ignorable thermal effects and fresh photon production. The self-consistency of this analysis is discussed in Section 3. Two environments in which bulk acceleration may be important are quasars and the early Universe (see Section 4). The production of a power-law in a super-critical spherical accretion flow is described in a third paper.

2 Acceleration of photons in a radiation-dominated shock

As an example of the influence of bulk motion on the spectrum of radiation in a scattering medium, we consider a radiation-dominated, steady, plane-parallel, non-relativistic shock in which the gas density is so dilute relative to the photon density that production of fresh photons (e.g. through bremsstrahlung and line-emission) and Comptonization by thermal electrons are ignorable. We treat this case in detail because it can be solved analytically and because it illustrates the effects associated with bulk acceleration fairly clearly. Much of the physics of this relatively simple case is characteristic of the more general problem.

In the frame of the shock, the flow is in the positive x direction, and the asymptotic pre-shock velocity and radiation pressure are u_- and p_- respectively. (In this and the following sections, we use a subscript '–' to indicate the asymptotic pre-shock value of a variable, while a subscript '+' indicates the asymptotic post-shock value.) If the constant mass flux is denoted by J, then it is convenient to introduce dimensionless quantities

$$\mu = u(x)/u_- , \tag{1}$$

$$\pi = p(x)/Ju_- . \tag{2}$$

Conservation of momentum flux can then be expressed as

$$\mu + \pi = 1 + \pi_- , \tag{3}$$

where $\pi_- = p_-/Ju_-$. A convenient choice of independent variable is τ^*, defined by

$$d\tau^* = (u_-/c)\, n_e\, \sigma_T\, dx . \tag{4}$$

Henceforth we shall use a prime to denote differentiation with respect to τ^*.

Dropping terms that are, by assumption, ignorable, equation (22) of Paper I for the energy flux becomes

$$\frac{1}{2}\mu^2 + 4\pi\mu - \pi' = \frac{1}{2}\mu_-^2 + 4\pi_-\mu_- . \tag{5}$$

Equations (3) and (5) admit a shock solution,

$$\mu = \frac{4}{7}(1 + \pi_-) - \frac{1}{7}(3 - 4\pi_-)\tanh\,[(3 - 4\pi_-)\,\tau^*/2] \tag{6}$$

$$\pi = \frac{3}{7}(1 + \pi_-) + \frac{1}{7}(3 - 4\pi_-)\tanh\,[(3 - 4\pi_-)\,\tau^*/2], \tag{7}$$

generalizing Weaver (1976), Zel'dovich & Raizer (1967) and Pai (1966). μ and π can be expressed implicitly in terms of the spatial coordinate x measured from the midplane $\tau^* = 0$ by

$$\begin{aligned} x &= (c\bar{m}/J\sigma_T)\int u\, d\tau^* \\ &= (c\bar{m}/J\sigma_T)\left[\frac{4}{7}(1 + \pi_-)\tau^* - \frac{2}{7}\ln\,\{\cosh[(3 - 4\pi_-)\,\tau^*/2]\}\right], \end{aligned} \tag{8}$$

where $\bar{m}$ is the mean particle mass per scatterer.

Note that the velocity field is independent of the spectrum of the radiation and the number of photons, because the equation of state is frequency-independent. $\mu(x)$ and $\pi(x)$ are displayed in Fig. 1 for different values of the shock Mach number

$$M = (3/4\pi_-)^{1/2} . \tag{9}$$

There is an alternative solution to equations (3) and (5) which corresponds to a stationary, one-dimensional expansion of radiation-dominated gas from rest and stagnation pressure p_0, passing through a sonic point with speed $(4p_0/7J)$ (*cf.* Davidson & McCray 1980). Our formalism becomes invalid close to a photosphere where the pressure becomes very small

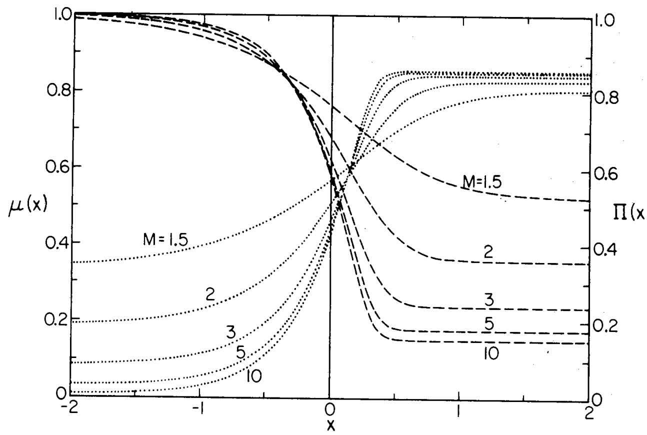

Figure 1. Fluid velocity in units of the pre-shock velocity, μ, (dashed curves) and radiation pressure in units of the pre-shock momentum flux, π, (dotted curves) for different values of the shock Mach number, M, which labels the curves. x, the scaled distance from the midplane is defined in equation (8) and increases in the direction of the flow.

and the velocity approaches p_0/J. For mass and energy fluxes J and F, respectively, we have

$$u = (4p_0/7J)\left[1 + \frac{3a}{7}\tan(3a\tau/2)\right] \tag{10}$$

$$p = \frac{3}{7}p_0\left[1 - \frac{4a}{7}\tan(3a\tau/2)\right], \tag{11}$$

with

$$-\frac{2}{3a}\tan^{-1}\left(\frac{7}{3a}\right) \leqslant \tau \leqslant \frac{2}{3a}\tan^{-1}\left(\frac{7}{4a}\right) \tag{12}$$

and

$$a = \frac{7}{3}[7JF/8p_0^2 - 1]^{1/2}. \tag{13}$$

The spectrum can be computed in this case as well, but is of little physical interest because in an expanding flow the power-law develops at the low-frequency end of the spectrum.

In order to compute the transmitted spectrum for a given incident photon distribution, we must solve equation (18) of Paper I in the limit of negligible photon production and thermal Comptonization, with the velocity field given by equation (6). As the problem is linear, it is sufficient to consider an incident δ-function distribution. The response to a more general distribution will be furnished by convolution. We can anticipate three features of the δ-function response. First, there will be a systematic (roughly adiabatic) increase in the energy of the photons associated with the converging flow. This will contribute a fractional energy-gain per photon of $\mu_+^{-1/3} = [(M^2+6)/7M^2]^{-1/3}$, which is smaller than the mean fractional energy-gain per photon computed from the shock jump conditions, $\pi_+\mu_+/\pi_- = (8M^2-1) \times (M^2+6)/49M^2$. Second, spatial diffusion into neighbouring fluid elements will contribute to a spreading of the photons in frequency. (Note that the probability of a photon emerging with zero frequency-shift is exponentially small and, within the approximations of this calculation, is identically zero.) Finally, a few photons will remain in the shock region to undergo $N \gg (c/u_-)^2$ scatterings. The probability of them doing so decreases exponentially with increasing N, but the energy gain increases exponentially with N. As in all Fermi processes, this combination produces a power-law spectrum at high energy. In summary, a sharp line is blue-shifted and broadened and develops a power-law wing on the blue side. Even after decompression back to the pre-shock density, the mean photon energy has been increased, as it must, since the shock can only increase the entropy of the radiation field.

The partial differential equation satisfied by the photon occupation number is

$$\frac{\partial^2 n}{\partial \tau^{*2}} - 3\mu \frac{\partial n}{\partial \tau^*} + \nu \frac{\partial n}{\partial \nu} \frac{\partial \mu}{\partial \tau^*} = 0 , \tag{14}$$

with μ given by equation (6). We seek separable solutions by writing

$$n(\nu,\chi) = f(\chi)\, \nu^{-\lambda} , \tag{15}$$

where λ is an eigenvalue of the ordinary differential equation

$$\chi(1-\chi) \frac{d^2 f}{d\chi^2} - \left[(M^2-1)^{-1} + \frac{8}{7}\chi\right] \frac{df}{d\chi} + \frac{2}{7}\lambda f = 0 , \tag{16}$$

which is in standard hypergeometric form, with

$$\chi = \frac{1}{2} \{1 + \tanh[3(1 - M^{-2})\tau^*/2]\} . \tag{17}$$

Making a power-series expansion about $\chi = 0$,

$$f(\chi) = \sum_{n=0}^{\infty} a_n \chi^{n+m} , \tag{18}$$

we find that the indicial equation has roots

$$m = 0, (1 - M^{-2})^{-1} . \tag{19}$$

Only the second solution, with $m = (1 - M^{-2})^{-1}$, satisfies the boundary condition at $\chi = 0$. Furthermore, for series (18) not to diverge at $\chi = 1$, it must terminate. This generates a set of discrete eigenvalues,

$$\lambda_n = \frac{7}{2} \left[(n+m)\left(n+m+\frac{1}{7}\right)\right] ; \quad n = 0,1,2, \ldots . \tag{20}$$

The solution for $f(\chi)$ is then given in terms of orthogonal Jacobi polynominals $J_n(p,q;\chi)$. We use the normalization of Morse & Feshbach (1953).

A monochromatic spectrum of intensity $I_0 \delta(\nu - \nu_0)$ will initially be compressed adiabatically, as diffusion is ignorable in the limit $x \to -\infty$. In the limit $\nu \to \nu_0$ we have then

$$\begin{aligned} n(\nu_0,\chi) &= \frac{I_0}{2\nu_0^3} \delta(\nu - \nu_0 \mu^{1/3}) \\ &= \frac{7 I_0 m}{4\nu_0^4} \delta(\chi) . \end{aligned} \tag{21}$$

The solution to equation (14), subject to our boundary conditions, is

$$\begin{aligned} n(\nu,\chi) = {} & \frac{7 I_0 \chi^m}{4\nu_0^4 \Gamma(m) \Gamma(1+m)} \sum_{n=0}^{\infty} \frac{(^1\!/_7 + 2n + 2m)\,\Gamma(^1\!/_7 + n + 2m)}{\Gamma(1+n)\,\Gamma(^1\!/_7 + n + m)} \\ & \times J_n(^1\!/_7 + 2m, 1 + m; \chi)\, (\nu/\nu_0)^{-^7\!/_2 (n+m)(n+m+^1\!/_7)} , \end{aligned} \tag{22}$$

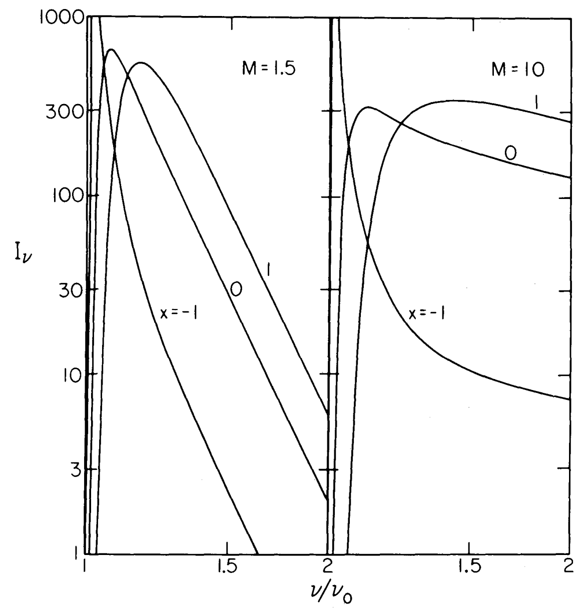

Figure 2. Intermediate intensities I_ν (in arbitrary units) for incident photons of frequency ν_0, plotted logarithmically. The curves are labelled by their scaled distance from the midplane x (equation 8). Results for two values of the Mach number (1.5 and 10) are displayed. Note that as the spectrum develops, the peak shifts to high frequencies and broadens, and the power-law tail develops.

for $\nu \geqslant \nu_0$. In particular, the transmitted spectrum ($\chi = 1$) for an incident δ-function is

$$n(\nu,1) = \frac{7I_0}{4\nu_0^4 \Gamma(m)\,\Gamma(m + 1/7)} \sum_{n=0}^{\infty} (-1)^n\,(2n + 2m + 1/7)\,\frac{\Gamma(n + 2m + 1/7)}{\Gamma(n+1)} \times \left(\frac{\nu}{\nu_0}\right)^{-7/2\,(n+m)(n+m+1/7)} . \tag{23}$$

In Fig. 2 we show how the spectrum builds up through the shock for different values of the Mach number M. In Fig. 3 we exhibit the transmitted spectrum for a δ-function incident spectrum, again for different values of the Mach number.

As a check on the analysis, we can compute the transmitted photon flux and pressure, respectively,

$$\lim_{\nu \to \nu_0} \left[u_+ \int_\nu^\infty 8\pi n(\nu', 1)\,\nu'^2 d\nu' \right] = \left(\frac{4\pi I_0}{h\nu_0}\right) u_- , \tag{24}$$

and

$$\lim_{\nu \to \nu_0} \left[\int_\nu^\infty \frac{8\pi}{3} n(\nu', 1)\,\nu'^3 d\nu' \right] = \left(\frac{m + 1/7}{m - 1}\right) \frac{4\pi}{3} I_0 . \tag{25}$$

To establish the results of equations (24) and (25), we have integrated the series in (23) term by term (the order of integrating and summing can be interchanged since the series is uniformly convergent) and made use of the result in the Appendix. Equations (24) and (25) confirm that the downstream flux and pressure satisfy the shock jump conditions.

So, for an incident δ-function spectrum, the transmitted flux will have a power-law character at high frequency. If we define the spectral index as

$$\alpha \equiv -\frac{d(\log I_\nu)}{d(\log \nu)} , \tag{26}$$

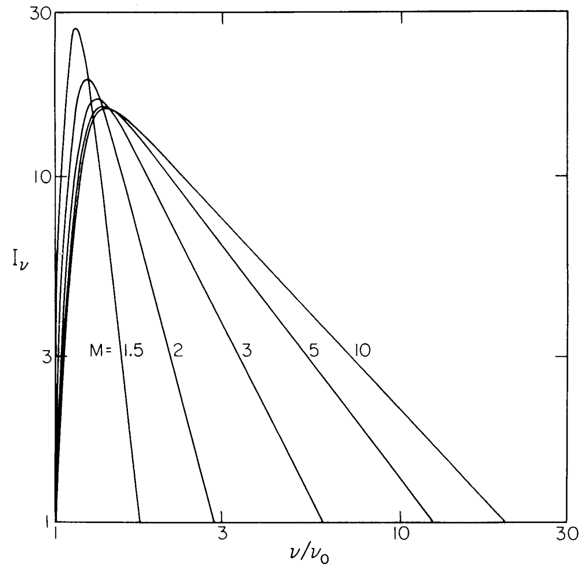

Figure 3. Transmitted intensities I_ν (in arbitrary units) for incident photons of frequency ν_0, plotted logarithmically. The curves are labelled by their Mach numbers. Note that the power-law portions of the spectra contain large fractions of the total intensity.

then from the leading eigenvalue we obtain

$$\lim_{\nu \to \infty} \alpha = (m+1)(7/2\, m - 3) = \frac{(M^2 - 1/2)(M^2+6)}{(M^2-1)^2}. \tag{27}$$

For a strong shock, $\alpha \to 1$. From Fig. 3 we see that the fundamental mode (i.e. $n = 0$) dominates the spectrum above the peak frequency, and that the transmitted spectra have a single power-law shape to quite high accuracy.

It is of interest to compute the transmitted spectrum for an incident blackbody (or dilute blackbody) radiation field by convolving the δ-function response (equation 23) with a Planck function. We replace I_0 by $2h\nu_0^3$ $[\exp(h\nu_0/kT) - 1]^{-1} d\nu_0$ and integrate over $d\nu_0$ to obtain

$$n(\nu, 1) = \frac{7}{2\Gamma(m)\,\Gamma(m + 1/7)} \sum_{n=0}^{\infty} (-1)^n\,(2n + 2m + 1/7)\,\frac{\Gamma(n + 2m + 1/7)}{\Gamma(n+1)}\,\mathscr{F}\!\left(\frac{h\nu}{kT}, \lambda_n\right), \tag{28}$$

where

$$\mathscr{F}(y, k) = \int_0^1 dx\, \frac{x^{k-1}}{\exp(xy) - 1}. \tag{29}$$

The corresponding intensity for different values of the Mach number (prior to decompression) is shown in Fig. 4. Again, we find that the high frequency spectrum is very straight.

The post-shock density and pressure are above that in the ambient medium, so the post-shock parameters must be decompressed. The mean photon energy as well as its density and pressure will decrease due to adiabatic expansion as the plasma returns to some average initial ambient state. There are two limiting prescriptions for decompression:

(i) Each volume element returns, on average, from the post-shock density $n_{e_+} = rn_{e_-}$ $(r \equiv \mu_+^{-1} = 7M^2/[M^2 + 6])$ to its pre-shock density. Then the final mean photon energy and pressure are given by

$$\frac{\bar{\nu}_+}{\bar{\nu}_{bb}} = \frac{p_+}{p_-} = \left(\frac{8M^2 - 1}{7}\right)\left(\frac{M^2+6}{7M^2}\right)^{4/3}. \tag{30}$$

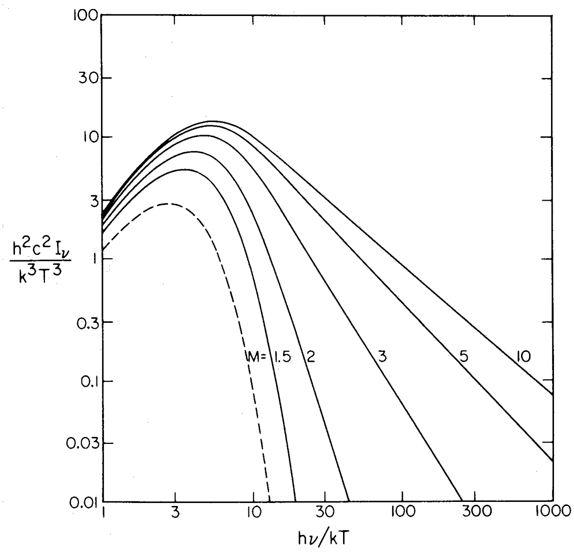

Figure 4. Transmitted intensity I_ν for incident blackbody spectrum (dashed curve) for different values of the shock Mach number, plotted logarithmically. Note that the high-frequency spectra are dominated by the power-law response to a δ-function. The total energy-density increases by the pressure ratio $(8M^2 - 1)/7$ and the photon energy-density by the compression ratio $r = 7M^2/(M^2 + 6)$.

(ii) Each volume element returns, on average, from the post-shock pressure $p_+ = [(8M^2 - 1)/7]\, p_-$ to its pre-shock pressure p_-. Then the final mean photon energy and density (n_ϕ) are, respectively,

$$\bar{\nu}_+ = \left(\frac{8M^2-1}{7}\right)^{3/4} \left(\frac{M^2+6}{7M^2}\right) \bar{\nu}_{\mathrm{bb}}\,, \tag{31}$$

$$n_{\phi+} = \left(\frac{8M^2-1}{7}\right)^{-3/4} \left(\frac{M^2+6}{7M^2}\right)^{-1} n_{\phi-}\,. \tag{32}$$

Even after decompression back to the average initial ambient state, there is still an increase in the mean photon energy, since the shock increases the entropy of the radiation field.

As a check on the numerical computations, we see from Table 1 that the increase in the radiation energy density (decompressed back to the pre-shock gas density) is consistent with the shock jump conditions (*cf.* equation 30).

Table 1. Characteristics of the transmitted spectrum for an incident blackbody spectrum with temperature T. For different values of the Mach number M, we give the compression ratio $r = \mu(+\infty)^{-1}$, the high-frequency spectral index α, the increase in $\bar{\nu}$ (or equivalently, the radiation energy-density) after decompression back to the pre-shock gas density $\bar{\nu}/\nu_{\mathrm{bb}} = [(8M^2 - 1)/7]\,[(M^2 + 6)/7M^2]^{4/3}$, and the post-shock equilibrium electron temperature for different values of the upper cut-off frequency $\nu_{\max}$.

M	r	α	$\bar{\nu}/\bar{\nu}_{\mathrm{bb}}$	T_e/T		
			$h\nu_{\max}/kT$	100	1000	∞
1	1	–	1	–	–	0.95
1.5	1.9	9.2	1.03	1.24	1.24	1.24
2	2.8	3.9	1.12	1.72	1.73	1.73
3	4.2	2.0	1.50	3.5	5.9	–
5	5.6	1.3	2.83	5.9	21	–
10	6.6	1.1	9.21	7.1	34	–

It is of interest to consider equation (23) in the weak shock limit. Averaging the mean and mean squared energy gains by means of the identity proved in the Appendix, we find that we can compute Fokker–Planck coefficients $\langle \Delta\nu \rangle$, $\langle \Delta\nu^2 \rangle$ for a weak shock as follows

$$\langle \Delta\nu^2 \rangle_{\mathrm{d,p}} = \begin{Bmatrix} \tfrac{1}{2}\nu \langle \Delta\nu \rangle_{\mathrm{d}} \\ \tfrac{2}{3}\nu \langle \Delta\nu \rangle_{\mathrm{p}} \end{Bmatrix} = \frac{32}{147}(M-1)^3 \tag{33}$$

per shock. The subscripts d, p denote expansion back to the pre-shock density and pressure, respectively. Note that the entropy jump is third order in the shock strength as is generally true for weak shocks, because the photons are carrying all the pressure. (For test particles accelerated at a shock discontinuity in the background medium, the entropy jump in the test particles is second order in the shock strength (Blandford & Ostriker 1980).) For a photon gas weakly shocked at a rate R (and allowed to re-expand) the Fokker–Planck equation for n is

$$\frac{\partial n}{\partial t} = \frac{R}{\nu^2}\frac{\partial}{\partial \nu}\left\{ -n\nu^2 \langle \Delta\nu \rangle + \frac{1}{2}\frac{\partial}{\partial \nu} n\nu^2 \langle \Delta\nu^2 \rangle \right\}. \tag{34}$$

Solving for $n(\nu, t)$ given the intensity at $t = 0$, we obtain

$$n(\nu, t)_{\mathrm{d}} = \int \frac{I(\nu', 0)}{4\nu'^4 \, (\pi\tau)^{1/2}} \exp\left[-\{\ln(\nu/\nu') + 3\tau\}^2/4\tau\right] d\nu', \tag{35a}$$

$$n(\nu, t)_{\mathrm{p}} = \int \frac{I(\nu', 0)}{4\nu'^3 \nu \, (\pi\tau)^{1/2}} \exp\left[-\{\ln(\nu/\nu') + 2\tau\}^2/4\tau\right] d\nu', \tag{35b}$$

where

$$\tau = \frac{16R}{147} \langle (M-1)^3 \rangle \, t.$$

Additional terms corresponding to escape and adiabatic expansion can be added to the right-hand side of equation (34), and it is straightforward to generate power-law solutions by direct analogy with conventional treatments of Fermi acceleration.

3 Thermal effects and photon production

3.1 THERMAL EFFECTS

We are now in a position to determine under what conditions this shock model is physically self-consistent. The photon acceleration rate by bulk motion, which is similar to the reciprocal of the time for the gas to flow through the shock, t_{u}, satisfies

$$t_{\mathrm{u}}^{-1} \sim \tfrac{1}{3} |\nabla \cdot \mathbf{u}| \sim 0.2\, n_{\mathrm{e}\_}\, \sigma_{\mathrm{T}}\, u_{-}^2/c, \tag{36}$$

(see equation 30 of Paper I; the numerical coefficients in this and the following estimates are consistent with the detailed results).

In a given radiation field, the equilibrium electron temperature is determined by the radiation field,

$$T_{\mathrm{e}} = \frac{h\langle \nu \rangle}{4k}, \tag{37}$$

(where $\langle \nu \rangle$ is given by equation (26) of Paper I). The time-scale for this is

$$t_{\mathrm{r}}^{-1} \sim n_\phi \sigma_{\mathrm{T}} h\bar{\nu}/m_{\mathrm{e}} c$$

$$\sim 3n_{\mathrm{e}-} \sigma_{\mathrm{T}} \left(\frac{m_{\mathrm{p}}}{m_{\mathrm{e}}}\right)\left(\frac{u_-^2}{c}\right). \tag{38}$$

In a radiation-dominated shock,

$^1/_3 n_\phi\, n_\phi h\bar{\nu} \sim n_{\mathrm{e}-} m_{\mathrm{p}} u^2$ and hence $t_{\mathrm{r}}/t_{\mathrm{u}} \sim m_{\mathrm{e}}/15\, m_{\mathrm{p}} \ll 1$.

However, in a radiation-dominated fluid it takes much longer for the photons to equilibrate. Let us assume that, for most of the photons, bulk acceleration is dominant and a power-law can develop. The upper cut-off in the spectrum, $\nu_{\max}$, is determined by balancing the bulk acceleration rate t_{u}^{-1} with the rate of loss of energy by Compton recoil, i.e.

$$\nu_{\max} \sim 0.2\, m_{\mathrm{e}} u_-^2/h. \tag{39}$$

(Since we are interested in non-relativistic flows, we see from equation (39) that electron–positron pair production will never be important.) For $\langle \nu \rangle \lesssim \nu \lesssim \nu_{\max}$, the rate of Compton cooling of photons ($\propto \nu$) will exceed the Compton heating rate (independent of frequency), and so, for $\nu < \nu_{\max}$, Compton effects are less important than bulk acceleration.

From Fig. 4, we see that if the incident radiation spectrum is roughly thermal, with temperature T, then the lower cut-off in the transmitted spectrum is roughly

$$\nu_{\min} \sim 5kT/h. \tag{40}$$

Combining equations (39) and (40), the condition that bulk acceleration dominates Comptonization for $\nu_{\min} \lesssim \nu \lesssim \nu_{\max}$ is

$$u \gtrsim 5\left(\frac{kT}{m_{\mathrm{e}}}\right)^{1/2} \left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/2}$$

$$\sim (2\times 10^9\,\mathrm{cm\,s^{-1}})\left(\frac{T}{10^6\,\mathrm{K}}\right)^{1/2}\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{1/2}. \tag{41}$$

So thermal Comptonization is ignorable in the shock region for fast ($u \gtrsim 10\,000\,\mathrm{km\,s^{-1}}$), cold ($T \lesssim 10^6\,\mathrm{K}$) flows (see Fig. 5).

In order to determine the electron temperature and relate the transmitted spectrum to the photon-to-electron ratio ahead of the shock, we must distinguish three spectral cases:

(i) *Weak Shocks* ($\alpha \gtrsim 2, M \lesssim 3$)

$$T_{\mathrm{e}} \sim 1.2\left(\frac{\alpha-1}{\alpha-2}\right) T, \tag{42}$$

$$\left(\frac{n_\phi}{n_{\mathrm{e}}}\right)_- \sim \left(\frac{15 m_{\mathrm{p}}}{r m_{\mathrm{e}}}\right)\left(\frac{\alpha-1}{\alpha}\right)\left(\frac{\nu_{\max}}{\nu_{\min}}\right). \tag{43}$$

For example, with $\alpha = 2.5$ ($M = 2.5$) and $\nu_{\max} = 10\,\nu_{\min}$, $T_{\mathrm{e}} = 3.7\,T$ and $(n_\phi/n_{\mathrm{e}})_- \simeq 4.5\times 10^4$.

(ii) *Intermediate Shocks* ($1.2 \lesssim \alpha \lesssim 2, 3 \lesssim M \lesssim 8$)

$$T_{\mathrm{e}} \sim 1.2\left(\frac{\alpha-1}{2-\alpha}\right)\left(\frac{\nu_{\max}}{\nu_{\min}}\right)^{2-\alpha} T, \tag{44}$$

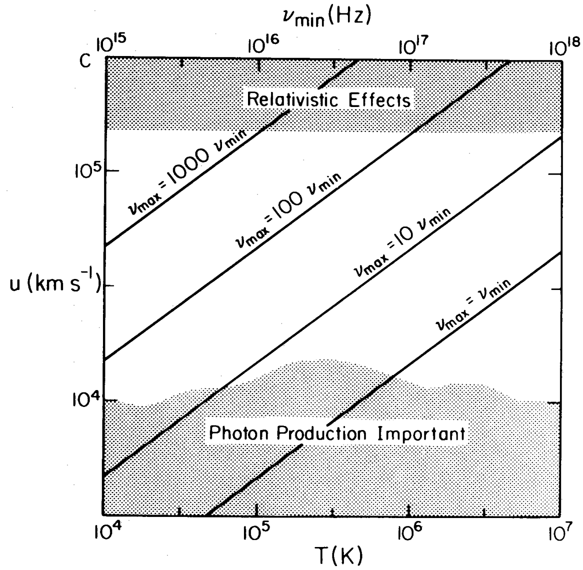

Figure 5. Regime of (u, T)-parameter space for which the radiation-dominated, bulk-heating shock model is self-consistent (u being the shock speed and T the temperature of the input radiation field). The solid lines indicate the extent of the power law in frequency space, with ν_{min} given on the upper abscissa. In the shaded regions our calculations are no longer fully self-consistent; however, we expect the qualitative features of our spectral calculations to carry over.

while $(n_\phi/n_e)_-$ is again given by equation (43). For example, with $\alpha = 1.5$ $(M = 4.1)$ and $\nu_{max} = 10\nu_{min}$, $T_e \simeq 4.0\,T$ and $(n_\phi/n_e)_- \simeq 1.8 \times 10^4$.

(iii) *Strong Shocks* $(1 \lesssim \alpha \lesssim 1.2, M \gtrsim 8)$

$$T_e \sim 1.2 \frac{(\nu_{max}/\nu_{min})}{\ln(\nu_{max}/\nu_{min})} T, \tag{45}$$

$$\left(\frac{n_\phi}{n_e}\right)_- \sim \left(\frac{15 m_p}{r m_e}\right) \frac{(\nu_{max}/\nu_{min})}{\ln(\nu_{max}/\nu_{min})} . \tag{46}$$

For example, with $\nu_{max} = 10\nu_{min}$, $T_e \sim 5.4\,T$ and $(n_\phi/n_e)_- \sim 1.7 \times 10^4$.

For the original thermal spectrum, $T_e \simeq 0.95\,T$ (ignoring stimulated scattering). The electron temperature typically increases by a factor ~ 4 through the shock and the ratio of gas-to-radiation pressure in the pre-shock gas is $\sim 10^{-2}\,(\nu_{min}/\nu_{max})_+$.

We can use the fact that the transmitted spectrum is a power-law to estimate the spectral index α. The mean photon frequency $\bar{\nu}$ is,

$$\bar{\nu} \simeq \left(\frac{\alpha}{\alpha - 1}\right) \nu_{min} \left[\frac{1 - (\nu_{min}/\nu_{max})^{\alpha - 1}}{1 - (\nu_{min}/\nu_{max})^{\alpha}}\right], \qquad \alpha \neq 0, 1. \tag{47}$$

However, this quantity can also be computed through the shock from the fluid solution, equation (7). In particular, in the post-shock region,

$$\bar{\nu} \simeq \frac{(8M^2 - 1)(M^2 + 6)}{49 M^2} \nu_0 \tag{48}$$

for incident photons of frequency ν_0. Estimating a value for $\nu_{min} \sim 2\nu_0$, we can equate equations (47) and (48) to solve for $\alpha(M)$. The values thus obtained are in good agreement with equation (27), obtained from the leading eigenvalue, when $\nu_{\max} \gg \nu_{\min}$. In this way, we see how the existence of upper and lower cut-offs in the spectrum, coupled with the jump conditions, effectively fixes the spectral slope. Note that when thermal Comptonization is marginally important, $(\nu_{\max}/\nu_{\min})$ is not very large and, if the shock is strong, we expect that the spectrum will be flattened and α can be reduced below unity.

Even when a power-law spectrum is transmitted, the photons can still be Comptonized in the post-shock flow. After a time t_+, the upper cut-off will have fallen to $\nu_{\max}(t_u/t_+)$ through successive Compton recoil losses, while the lowest energy photons will have been Fermi-accelerated by the electrons to $\sim \nu_{\min} \exp[(\langle\nu\rangle/\nu_{\max})(t_+/t_u)]$, thus further reducing the range of the power law. A power-law will only be observed if the post-shock fluid quickly expands to become optically thin to Thomson scattering. This will occur most readily for a high-speed flow, but even here, it is unlikely that this re-expansion or time-dependence can ever be self-consistently omitted from the problem. For this reason, we regard the calculations in Section 2 as no more than an idealized demonstration of the production of a power-law by a converging fluid-flow.

When the photon-to-electron ratio $\lesssim 10^4$, then the upper and lower cut-offs are separated by less than a decade in frequency and we expect that thermal Comptonization will start to dominate the photon acceleration. For $1 \ll n_\phi/n_e \ll 10^4$, a Wien spectrum can be established locally and the effective photon approximation of Weaver (1976) is appropriate.

3.2 PHOTON PRODUCTION

We now investigate our assumption that photon production can be ignored.

The emissivity of the gas must be calculated with the electron temperature and ionization equilibrium dictated by the radiation field. If the gas is fully ionized, we can determine when free–free emission is unimportant from the condition

$$j_{\mathrm{ff}+} \lesssim n_\phi \, h\bar{\nu} \, t_u^{-1} \; , \tag{49}$$

where $j_{\mathrm{ff}+}$ is the standard free–free emissivity evaluated ahead of the shock. If thermal Comptonization is not effective, then we need consider only those photons emitted with $h\nu \sim kT$. For temperatures $T_e \lesssim 10^7$K, line emission will provide a greater source of fresh photons. The enhancement over free–free emission will not be as great as under standard optically thin conditions (e.g. Raymond, Cox & Smith 1976) (by up to $\sim 10^3$) because the ionic species normally responsible for collisional cooling will be strongly photo-ionized. We have estimated the emissivity by collisional cooling, as a function of u and T, and corrected equation (49) accordingly. The emissivity is dominated by forbidden-line cooling from highly ionized metals. Recombination radiation will never be important under the conditions of interest here, while cyclotron emission will be unimportant within the shock if the magnetic pressure is dynamically unimportant.

From equation (49) we obtain the inequality to be satisfied for neglect of photon production to be self-consistent,

$$u \gtrsim (3 \times 10^3 \,\mathrm{km\,s^{-1}})\,(T/10^6\,\mathrm{K})^{1/8} A_{\mathrm{ff}}{}^{1/4} r^{1/2} \; , \tag{50}$$

where A_{ff} is the amplification factor for the emissivity over the free–free emissivity and r is again the compression ratio.

When this inequality is not satisfied, our calculations are no longer directly applicable. However, when photon production is important, we expect the qualitative features of our results to carry over. That is, for frequencies greater than the effective frequency at which photons are introduced, we expect that the power-law shape will be maintained.

When the shock speed u becomes large ($u \gtrsim c/2$), photons undergo fewer than 5 scatterings within the shock and the diffusion approximation breaks down. Furthermore, radiation viscosity terms cannot be omitted from the momentum equation (*cf.* Weinberg 1972), while our Fokker–Planck treatment of energy-momentum transfer and the use of the Thomson cross-section become invalid. Again, when relativistic effects are important, we expect the qualitative features of our spectral calculations to remain.

The conditions discussed in this section under which the bulk-heating shock model is valid, are displayed schematically in Fig. 5.

4 Applications

In the previous sections, we have shown that fast, cold, radiation-dominated shocks can produce power-law spectra. Clearly, the size of any astrophysical source exhibiting these conditions must exceed the shock thickness $\sim (c/u)$ Thomson optical depths. If the power law is to extend over more than a decade in frequency, then the source must have $n_\phi/n_e \gtrsim 3 \times 10^4$ also. These conditions may be obtained in the gas cloud surrounding a quasar or active galactic nucleus.

If the gas within 10–100 Schwarzschild radii of a black hole is highly radiation-dominated, optically thick (to electron scattering), and supported by random transonic motions, then it may be possible to account for the production of the power-law continua associated with Seyferts, quasars and Lacertids. If the incident spectrum is thermal, the infrared–optical spectrum cannot be explained by Fermi acceleration of photons (either first- or second-order). With a temperature $T \sim 5000$ K and a radius $R \sim 10$ Schwarzschild radii (for a black hole of mass $M \sim 10^9 M_\odot$) a gas cloud has much too small a surface area to provide the requisite input flux. (Alternatively, if the infrared radiation were thermal, then with a luminosity $L \sim 10^{46}\,\mathrm{erg\,s^{-1}}$ and a temperature $T \sim 5000$ K, it would have to be produced at a distance $R \sim 0.05$ pc.) However, any copious flux of low-frequency, non-thermal photons could provide the required source. This scenario would then resemble the models of Katz (1976) and Colgate & Petschek (1978) which invoke synchrotron radiation or coherent plasma processes to produce the required low-frequency photons.

If the incident radiation is thermal, then to produce a power-law extending over more than a decade in frequency from radiation with a temperature T, the size of the emitting region must satisfy

$$R \gtrsim (10^{17}\,\mathrm{cm})\, T_4^{-3} u_9^{-1}\ , \tag{51}$$

(where $T_4 \equiv T/[10^4\,\mathrm{K}]$ and $u_9 \equiv u/[10^9\,\mathrm{cm\,s^{-1}}]$). If we associate u with the virial velocity a distance R away from a black hole of mass M, then inequality (51) is satisfied for

$$R \lesssim (3 \times 10^{16}\,\mathrm{cm}) M_9^{1/2} L_{46}^{3/4}\ , \tag{52}$$

(with $M_9 \equiv M/[10^9 M_\odot]$ and $L_{46} \equiv L/[10^{46}\,\mathrm{erg\,s^{-1}}]$). As a numerical example, for $R \sim 3 \times 10^{16}$ cm, $T \sim 3 \times 10^4$ K, $u \sim 30\,000\,\mathrm{km\,s^{-1}}$ and $n_e \sim 10^{10}\,\mathrm{cm^{-3}}$. A density of $\sim 10^{12}\,\mathrm{cm^{-3}}$ closer to the black-hole horizon suffices to produce a luminosity of $\sim 10^{46}\,\mathrm{erg\,s^{-1}}$ through bound–bound, bound–free, and free–free emission. For u as high as $(c/3)$, the spectrum extends from ~ 50 eV to ~ 15 keV (*cf.* Holt 1979). Variability on time-scales perhaps as short as a day is expected. Note that this mechanism is unlikely to account for the power-law γ-ray fluxes reported for NGC 4151 and 3C 273.

Although the chaotic conditions we have described involve far more complex flows than the simple shock model we have analysed, we would argue that the Comptonization mechanism whereby bulk kinetic energy is converted into suprathermal photons is very general and, as described in Section 3, a spectral index $\alpha \lesssim 1$ is to be expected on energetic grounds. This mechanism is not likely to be relevant for black holes of stellar mass, where radiation pressure is unlikely to dominate gas pressure by so large a factor, and where the straightforward thermal Comptonization described by Katz (1976) and Shapiro *et al.* (1976) seems more appropriate.

A second example of an extended, radiation-dominated fluid occurs in the early Universe. In the conventional hot big-bang cosmology, the sound speed just prior to recombination ($z_r \sim 1500$) is $c_s \sim (40\,000\,\mathrm{km\,s^{-1}})\ \Omega^{-1/2}$, for a Hubble constant of $100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. Thermalization of the radiation spectrum on an expansion time-scale is ineffective for redshifts $z \lesssim z_{th} \sim 4 \times 10^5\ \Omega^{-3/2}$. Large-amplitude, adiabatic perturbations that come within the horizon for $z_r \lesssim z \lesssim z_{th}$ will fall below the Jeans' mass and turn into non-linear sound waves which can break to form shocks that are effectively damped. In this way, perturbations on mass scales well above the usual damping mass (e.g. Jones 1976) can be attenuated and may lead to a substantial energy input into the radiation field and a consequently non-Planckian spectrum such as may have been observed (Woody & Richards 1979). In order to bring about the large ($\gtrsim 5$ per cent) changes in the brightness temperature of the microwave background reported by these authors, the wave energy damped must be a similarly large fraction of the radiation energy density prior to recombination. However, large-amplitude adiabatic fluctuations on large baryon mass-scales ($\gtrsim 10^{13} M_\odot$) will survive recombination and lead to unobserved density fluctuations currently (e.g. Press & Vishniac 1980). In addition, large-amplitude waves at small mass-scales ($\lesssim 100 M_\odot$) will alter the standard view of cosmological nucleosynthesis. The calculations presented in this paper are therefore only applicable if large-amplitude waves were produced in a narrow range of mass scales in the matter-dominated era just prior to recombination. This requires the selection of a specific mass-scale in the initial perturbation spectrum. Any residual small-scale anisotropy in the microwave background (e.g. Partridge 1980) associated with these waves may be erased if re-ionization occurs at a redshift ~ 100–300 so that the Universe at this epoch is optically thick to Thomson scattering (e.g. Rees 1977). In view of uncertainties in both the observational position and the nonlinear evolution of large-amplitude adiabatic fluctuations, we do not pursue this application further.

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Appendix

In Section 2, we computed the downstream photon flux and pressure from the distribution function (equations 24 and 25). To do so, we required the following identity:

$$\sum_{n=0}^{\infty} (-1)^n \frac{(2n+a+b)}{(n+a)(n+b)} \frac{\Gamma(n+a+b)}{\Gamma(n+1)} z^{(n+a)(n+b)}$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\Gamma(n+1)} \int_0^{\infty} du\, u^{n+a+b-1} \exp(-u) \left(\frac{1}{n+a} + \frac{1}{n+b}\right) z^{(n+a)(n+b)}$$

$$= \int_0^{\infty} du \int_0^{u} dv \exp(-u)\,(u^{a-1} v^{b-1} + u^{b-1} v^{a-1}) \sum_{n=0}^{\infty} (-1)^n \frac{v^n}{\Gamma(n+1)} z^{(n+a)(n+b)}$$

$$= \frac{z^{ab}}{\sqrt{2\pi}} \int_0^{\infty} du \int_0^{u} dv \int_{-\infty}^{+\infty} dw\,(u^{a-1} v^{b-1} + u^{b-1} v^{a-1}) \exp(-u) \exp(-w^2/2)$$

$$\times \sum_{n=0}^{\infty} (-1)^n \frac{v^n}{\Gamma(n+1)} z^{n(a+b)} \exp(nw\sqrt{2\ln z})$$

$$= \frac{z^{ab}}{\sqrt{2\pi}} \int_0^{\infty} du \int_0^{u} dv \int_{-\infty}^{+\infty} dw\,(u^{a-1} v^{b-1} + u^{b-1} v^{a-1}) \exp\left[-\left\{u + \frac{w^2}{2} + zv \exp(w\sqrt{2\ln z})\right\}\right].$$

We can now take the limit $z \to 1$ to establish the result,

$$\lim_{z \to 1} \sum_{n=0}^{\infty} (-1)^n \frac{(2n+a+b)}{(n+a)(n+b)} \frac{\Gamma(n+a+b)}{\Gamma(n+1)} z^{(n+a)(n+b)} = \Gamma(a)\,\Gamma(b)\,. \tag{A1}$$