Mon. Not. R. astr. Soc. (1976) **176**, 465–481.

ACCRETION DISC ELECTRODYNAMICS— A MODEL FOR DOUBLE RADIO SOURCES

R. D. Blandford

Institute of Astronomy, The Observatories, Madingley Road, Cambridge CB3 0HA

(Received 1976 April 13; in original form 1976 February 12)

SUMMARY

A mechanism is proposed that may account for the production of two oppositely-directed beams of relativistic plasma such as have been inferred to be the energy supply of strong, double radio sources. High angular momentum material accreted by a compact object can settle into a stationary disc. If this material carries with it a vertical component of magnetic field (i.e. parallel to the rotation axis) then a magnetosphere can form above and below the disc similar in character to that found in an axisymmetric model of a pulsar. An exact analytic solution is given describing a force-free magnetosphere with this geometry. This solution has the property that energy and angular momentum are extracted in such a way that the disc can evolve in a stationary fashion, without invoking additional viscous torques. It may also provide some qualitative insight into the more difficult problem of the electromagnetic structure of the open field lines of a pulsar.

In the application to double radio sources such a machine could operate within the nucleus of an associated galaxy or quasar. In this case the two beams would be produced with a flow velocity which is already supersonic, thus evading some of the stability problems associated with models in which the flow is initially subsonic. This proposed explanation may also be appropriate for the double source positionally associated with the X-ray object Sco X-1.

1. INTRODUCTION

The primary observational fact that has emerged from studies of strong extragalactic radio sources is the occurrence of linear structure on length scales from $\sim$pc to Mpc. Correspondingly, the main requirement of a theoretical model is a satisfactory mechanism that will account for the establishment and maintenance of such a configuration. In this paper, we propose and analyse one such mechanism—the creation of a magnetosphere above and below a magnetized accretion disc in which electromagnetic momentum is focused continuously towards the rotation axis. A qualitatively similar model has been suggested independently by Lovelace (1976) (*cf.* also Ruffini & Wilson 1975): non-axisymmetric or non-stationary hydromagnetic versions have been proposed by Piddington (1970), Sturrock & Barnes (1972) and Ozernoi & Usov (1973).

2. NEWTONIAN MODEL

Before describing the detailed calculations, we present a simplified discussion of the general features of the model. An accretion disc (e.g. Lynden-Bell 1969; Pringle & Rees 1972; Shakura & Sunyaev 1973) containing a component of magnetic field parallel to the angular momentum is, in electrodynamical terms, similar to an axisymmetric model of a pulsar, as originally discussed by Goldreich & Julian (1969). Provided that the magnetic field, **B**, is not too large, the matter in the disc will rotate at the local Keplerian angular velocity dragging the magnetic field with it. It therefore acts as a unipolar inductor generating electric field, **E**, in the inertial frame. As charged particles of either sign are presumably free to leave the disc (in a pulsar this may not be strictly true; Ruderman & Sutherland 1975) a magnetosphere will rapidly be produced in which the total electromagnetic force, $\rho\mathbf{E}+\mathbf{j}\times\mathbf{B}$, is approximately zero and the divergence of the electric field is satisfied by a distribution of space charge ρ. A current distribution, $\mathbf{j}$, is also set up, thus modifying the magnetic field.

Each field line, assumed to be equipotential, will be characterized by a constant angular velocity, α, and therefore a given rotational velocity $\alpha\tilde{\omega}$ increasing with $\tilde{\omega}$. (We use cylindrical polar coordinates $\tilde{\omega}$, z, ϕ.) Near the speed-of-light surface, where $\tilde{\omega} = c/\alpha$, the material on the field lines can no longer co-rotate with the disc at the base of the field line and so must be swept backwards forcing the particles to travel outwards at speeds $\sim c$. There will therefore be an electromagnetic (and possibly also a particle) flux of energy and angular momentum crossing the light surface, which must be balanced by a loss of energy and angular momentum from the disc. This electromagnetic stress can provide the principal torque acting on the disc and, as we now show, it is possible to arrange the magnetic field so that the disc evolves in a stationary fashion. In this case, most of the gravitational energy will be liberated electromagnetically.

Consider an infinitely thin disc in the plane $z = 0$ containing a power-law distribution of toroidal surface current

$$J_\phi \propto \tilde{\omega}^{-n}. \tag{2.1}$$

If we assume that $B_{\tilde{\omega}} \sim B_z$ on the disc (d), then the general poloidal component of magnetic field satisfies

$$B_{\mathrm{pd}} \propto \tilde{\omega}^{-n} \tag{2.2}$$

and we can ignore contributions from the inner and outer edges of the disc provided $0<n<3$. As the rotational velocity equals $\tilde{\omega}\alpha$, the induced electric field near the disc is given by $E_\mathrm{d} \sim \tilde{\omega}\alpha B_{\mathrm{pd}} \propto \tilde{\omega}^{(1-n)}\alpha$ and the corresponding space charge is given by $\rho_\mathrm{d} = \epsilon_0\nabla.E \propto \tilde{\omega}^{-n}\alpha$.

Now in order to sweep back the field lines at the light surface (ls) through an angle $\sim 45°$, there must be a poloidal current given by the curl of the magnetic field, $j_{\mathrm{pls}} \sim (\alpha/c)(B_{\phi\mathrm{ls}}/\mu_0) \sim \alpha c\epsilon_0 B_{\mathrm{pls}}$. But flux and charge conservation require that $j_{\mathrm{pls}}/B_{\mathrm{pls}} \sim j_{\mathrm{pd}}/B_{\mathrm{pd}}$ and so we have

$$j_{\mathrm{pd}} \sim \rho_\mathrm{d} c \propto \tilde{\omega}^{-n}\alpha. \tag{2.3}$$

This current flows outwards along the field lines and must be supplied by a *radial* surface current lying in the disc, $J_{\tilde{\omega}} \sim j_\mathrm{p}\tilde{\omega} \propto \tilde{\omega}^{(1-n)}\alpha$. Therefore there will be a torque per unit area acting on the disc given by

$$G \sim \tilde{\omega} J_{\tilde{\omega}} B_{\mathrm{pd}} \propto \tilde{\omega}^{2(1-n)}\alpha \tag{2.4}$$

which must balance the change of angular momentum. Thus, if $v_{\tilde{\omega}}$ is the radial velocity in the disc and Σ the surface mass density, then

$$G \sim \Sigma v_{\tilde{\omega}} \frac{\partial}{\partial\tilde{\omega}}(\tilde{\omega}^2\alpha)$$

or

$$\Sigma\tilde{\omega}v_{\tilde{\omega}} \propto \tilde{\omega}^{2(1-n)}. \tag{2.5}$$

However, the left-hand side is proportional to the mass flow, which must be independent of $\tilde{\omega}$ in a steady state. Therefore we have shown that if the dominant torque is electromagnetic, then a power-law distribution of field and toroidal current in a stationary Newtonian disc will satisfy

$$B_{\mathrm{pd}} \propto J_\phi \propto \tilde{\omega}^{-1}. \tag{2.6}$$

Note that this is only a heuristic discussion designed to determine the necessary scaling with radius.

In the following Section, we present an exact formal magnetospheric solution for a Newtonian disc in which J_ϕ is approximately inversely proportional to $\tilde{\omega}$. In Section 4, we consider the braking of the disc and the validity of the approximations that are implicit in this Newtonian treatment. Finally in Section 5, the application to double radio sources and Sco X-1 is briefly described.

3. SELF-CONSISTENT, FORCE-FREE MAGNETOSPHERE

(i) *Magnetospheric equations*

The equations governing the electromagnetic structure of a stationary, force-free, axisymmetric magnetosphere in which $\rho\mathbf{E}+\mathbf{j}\times\mathbf{B} = 0$ have been given by several authors. A convenient treatment, generalized to include differential rotation, is that of Okamoto (1974). A complete (but not independent) set of equations is

$$\nabla.\mathbf{E} = \rho/\epsilon_0 \tag{3.1}$$

$$\nabla.\mathbf{B} = 0 \tag{3.2}$$

$$\mathbf{E}+(\boldsymbol{\alpha}\times\mathbf{r})\times\mathbf{B} = 0 \tag{3.3}$$

$$\mathbf{j}/\epsilon_0 = \nabla\times\mathbf{B} \tag{3.4}$$

$$= \mu\mathbf{B}+\frac{\rho}{\epsilon_0}(\boldsymbol{\alpha}\times\mathbf{r}), \tag{3.5}$$

where

$$(\mathbf{B}.\nabla)\,\boldsymbol{\alpha} = (\mathbf{B}.\nabla)\,\mu = (\mathbf{B}.\nabla)(\tilde{\omega}B_\phi) = 0. \tag{3.6}$$

(For the whole of this Section we set $c = 1$.)
From equations (3.1)–(3.6) we obtain

$$\frac{\rho}{\epsilon_0} = \frac{\{\mu(\boldsymbol{\alpha}\times\mathbf{r}.\mathbf{B})-\mathbf{B}.\nabla\times(\boldsymbol{\alpha}\times\mathbf{r})\}}{\{1-(\boldsymbol{\alpha}\times\mathbf{r})^2\}} \tag{3.7}$$

which is non-singular at the light surface, where $|\boldsymbol{\alpha}\times\mathbf{r}| = 1$, as long as the fields are continuous.

One approach to the problem of finding solutions to these equations is to introduce a stream function and thus generate a second-order partial differential equation (e.g. Okamoto 1974, and references therein). We follow a much more restricted approach below and use a method that will only yield a self-consistent solution when a very stringent condition happens to be satisfied. Fortunately this is the case for the current distribution, equation (2.6).

Suppose there is an axisymmetric distribution of toroidal currents contained in a non-rotating conductor surrounded by vacuum. The magnetic field exterior to this conductor will be poloidal and curl-free and so it is in principle possible to set up an orthogonal, toroidal coordinate system (u, v, ϕ) such that v is constant along the field lines. If the line element can be expressed

$$ds^2 = h_u{}^2(u, v)\, du^2 + h_v{}^2(u, v)\, dv^2 + \tilde{\omega}^2(u, v)\, d\phi^2, \tag{3.8}$$

then the requirements that $\nabla . \mathbf{B} = 0$ and $\nabla \times \mathbf{B} = 0$ ensures that B_u be expressible in the forms

$$B_u = \frac{f(v)}{h_v \tilde{\omega}} = \frac{g(u)}{h_u}. \tag{3.9}$$

Now let the surface of the conductor have an arbitrary variation of angular velocity, α. As a direct consequence of the rotation, currents will flow in the magnetosphere modifying the field structure. We hypothesize that these currents only introduce a toroidal magnetic field and alter the magnitude of the poloidal component. They do not change the direction of the poloidal field. That is to say B_v remains zero and B_u becomes

$$B_u = \frac{f(v)\, a(v)}{h_v \tilde{\omega}}. \tag{3.10}$$

In addition, equations (3.6) show that α, μ and $(\tilde{\omega} B_\phi)$ are solely functions of v.

Taking the toroidal component of equation (3.5) and using equations (3.1), (3.3), (3.4) and (3.9) we obtain

$$\mu B_\phi \tilde{\omega} = f \left\{ \frac{a\alpha}{h_v{}^2} \frac{\partial}{\partial v} (\alpha \tilde{\omega}^2) - \frac{1}{h_v{}^2} (1 - \alpha^2 \tilde{\omega}^2) \frac{da}{dv} \right\}. \tag{3.11}$$

But from equation (3.6) we see that the left-hand side of equation (3.11) is only a function of v and so for self-consistency we must be able to choose functions $a(v)$, $b(v)$ such that

$$a\alpha \frac{\partial}{\partial v} (\alpha \tilde{\omega}^2) - (1 - \alpha^2 \tilde{\omega}^2) \frac{da}{dv} = h_v{}^2 b \tag{3.12}$$

for all u. In general this will not be possible and in particular it cannot be done for a dipolar field, as a direct calculation confirms. Nevertheless, we have presented this part of the calculation in some generality as it is possible to find alternative toroidal coordinate systems to that considered below for which equation (3.12) can be satisfied. (See the Appendix.)

(ii) *Poloidal field for a non-rotating magnetosphere*

We now return to the current distribution $J_\phi = C\epsilon_0 \tilde{\omega}^{-1}$ where C is a constant, and calculate the poloidal field. We assume that this current law holds for all $\tilde{\omega}$, although in a realistic disc this simply requires that we not be too close to the inner or outer edges.

The vector potential $\mathbf{A}$ has only a toroidal component which can be expressed

$$A_\phi = \frac{C}{2} \int_0^\infty da \int_0^\infty dk \exp(-k|z|)\, J_1(ka)\, J_1(k\tilde{\omega}) \tag{3.13}$$

(see e.g. Jackson 1962, p. 166). Taking the curl of this expression and using Bessel function identities and integrals, we obtain in cylindrical polar coordinates

$$B_{\tilde{\omega}} = \frac{C}{2\tilde{\omega}}\left\{1 - \frac{|z|}{(\tilde{\omega}^2+z^2)^{1/2}}\right\} \operatorname{Sign}(z),$$

$$B_z = \frac{C}{2}(\tilde{\omega}^2+z^2)^{-1/2}. \tag{3.14}$$

The discontinuity in $B_{\tilde{\omega}}$ at the disc is correctly given by J_ϕ/ϵ_0. (For a more general current power law, equation (2.1) with $0<n<3$, the field can be straightforwardly expressed in terms of hypergeometric functions. When the exponent $n = 2$, the field lines are purely radial.)

The magnetic field lines of equation (3.14) therefore satisfy

$$\frac{dz}{d\tilde{\omega}} = \frac{B_z}{B_{\tilde{\omega}}} \tag{3.15}$$

or

$$z = \frac{\tilde{\omega}^2 - \tilde{\omega}_0{}^2}{2\tilde{\omega}_0}, \quad z > 0, \tag{3.16}$$

i.e. parabolae with semi-latus rectum, $\tilde{\omega}_0$ (the radius at which the field line intersects the disc), and focus at the origin. Henceforth, we consider only the upper half of the disc, $z>0$. The corresponding expressions in the lower half follow immediately from symmetry considerations.

We now introduce paraboloidal coordinates (u, v, ϕ) where

$$\tilde{\omega} = uv,$$

$$z = \tfrac{1}{2}(u^2-v^2),$$

$$h_u = h_v = (u^2+v^2)^{1/2}. \tag{3.17}$$

In terms of these coordinates the only non-zero component of the magnetic field is

$$B_u = \frac{C}{u(u^2+v^2)^{1/2}},$$

i.e.

$$f(v) = Cv. \tag{3.18}$$

(iii) *Rotating magnetosphere*

Let us impose an arbitrary angular velocity variation $\alpha(v)$ on the disc and search for a field solution of the form of equation (3.10) in paraboloidal coordinates. Substituting equations (3.17), (3.18) in equation (3.12) we find that there exists an acceptable solution with

$$a = (1+\alpha^2 v^4)^{-1/2},$$

$$b = \alpha(1+\alpha^2 v^4)^{-3/2}\frac{d}{dv}(\alpha v^2). \tag{3.19}$$

The fields, currents and charges in the magnetosphere are then given by

$$B_u = \frac{C}{u(u^2+v^2)^{1/2}(1+\alpha^2 v^4)^{1/2}}, \tag{3.20}$$

$$B_\phi = \frac{C}{\mu u(1+\alpha^2 v^4)^{3/2}}\frac{d}{dv}(\alpha v^2), \tag{3.21}$$

$$E_v = \frac{-Cv\alpha}{(u^2+v^2)^{1/2}(1+\alpha^2v^4)^{1/2}}, \tag{3.22}$$

$$\frac{\rho}{\epsilon_0} = \frac{-C}{(u^2+v^2)\,v(1+\alpha^2v^4)^{3/2}}\frac{d}{dv}(\alpha v^2), \tag{3.23}$$

$$\frac{j_u}{\epsilon_0} = \frac{C\mu}{u(u^2+v^2)^{1/2}(1+\alpha^2v^4)^{1/2}}, \tag{3.24}$$

$$\frac{j_\phi}{\epsilon_0} = \frac{C\alpha v^2}{u(u^2+v^2)(1+\alpha^2v^2)^{3/2}}\frac{d}{dv}(\alpha v^2). \tag{3.25}$$

Note that when $\alpha \to 0$ we recover the field structure for the non-rotating magnetosphere.

In deriving these expressions, we have used all the Maxwell equations save one, the component of equation (3.4) resolved parallel to the poloidal field. Using equation (3.24), we obtain

$$\frac{\mu v}{(1+\alpha^2v^4)^{1/2}} = \frac{d}{dv}\left\{\frac{\alpha v}{\mu(1+\alpha^2v^4)^{3/2}}\frac{d}{dv}(\alpha v^2)\right\}, \tag{3.26}$$

which has the solution

$$\mu = \frac{-\alpha v}{(1+\alpha^2v^4)\{A(1+\alpha^2v^4)-1\}^{1/2}}\frac{d}{dv}(\alpha v^2) \tag{3.27}$$

where A is an arbitrary constant and the negative sign corresponds to a trailing toroidal field.

Even after specifying the radial variation of the angular velocity on the disc, we have not yet found a unique solution and must specify an additional boundary condition. The most relevant choice would seem to be the behaviour of the toroidal magnetic field and the radial surface current at large radii where the rotational velocity presumably tends to zero. On the disc, where $u = v = \tilde{\omega}^{1/2}$,

$$\frac{J_{\tilde{\omega}}}{\epsilon_0} = -2B_\phi = \frac{2C\{A(1+\alpha^2\tilde{\omega}^2)-1\}^{1/2}}{\tilde{\omega}(1+\alpha^2\tilde{\omega}^2)^{1/2}}. \tag{3.28}$$

As $\alpha\tilde{\omega} \to 0$, the total radial current $2\pi\tilde{\omega}J_{\tilde{\omega}} \to 4\pi C(A-1)^{1/2}$. If, as would seem natural in a disc, $J_{\tilde{\omega}}$, $B_\phi \to 0$ as $\alpha\tilde{\omega} \to 0$ then $A = 1$, and we shall assume this condition henceforth. However, we emphasize that solutions with $A > 1$ *could* be physical, depending on the way that the configuration is set up.

We can now transform back to cylindrical polar coordinates. We label the radius at which a particular field line intersects the disc by $\tilde{\omega}_0$, α is then a function of $\tilde{\omega}_0$. We obtain

$$\mathbf{B} = \frac{C\tilde{\omega}_0}{\tilde{\omega}(\tilde{\omega}^2+\tilde{\omega}_0^2)(1+\alpha^2\tilde{\omega}_0^2)^{1/2}}\{\tilde{\omega}_0,\ \tilde{\omega},\ -\alpha(\tilde{\omega}^2+\tilde{\omega}_0^2)\}, \tag{3.29}$$

$$\mathbf{E} = \frac{C\tilde{\omega}_0\alpha}{(\tilde{\omega}^2+\tilde{\omega}_0^2)(1+\alpha^2\tilde{\omega}_0^2)^{1/2}}\{-\tilde{\omega},\ \tilde{\omega}_0,\ 0\}, \tag{3.30}$$

$$\frac{\rho}{\epsilon_0} = \frac{-2C\tilde{\omega}_0}{(\tilde{\omega}^2+\tilde{\omega}_0^2)(1+\alpha^2\tilde{\omega}_0^2)^{3/2}}\frac{d}{d\tilde{\omega}_0}(\alpha\tilde{\omega}_0), \tag{3.31}$$

$$\frac{\mathbf{j}}{\epsilon_0} = \frac{2C\tilde{\omega}_0}{\tilde{\omega}(\tilde{\omega}^2+\tilde{\omega}_0^2)(1+\alpha^2\tilde{\omega}_0^2)^{3/2}}\frac{d}{d\tilde{\omega}_0}(\alpha\tilde{\omega}_0)\{-\tilde{\omega}_0,\ -\tilde{\omega},\ \alpha\tilde{\omega}_0^2\}, \tag{3.32}$$

where $\tilde{\omega}_0$ is related to z by equation (3.16). The corresponding stream function, P, in the notation of Okamoto (1974) is readily shown to be

$$P = \frac{C}{2}\int^{\tilde{\omega}_0} \frac{d\tilde{\omega}_0'}{(1+\alpha^2(\tilde{\omega}_0')\,\tilde{\omega}_0'^2)^{1/2}}. \tag{3.33}$$

To complete the electromagnetic specification we give the surface charges and currents in the disc

$$\frac{\sigma}{\epsilon_0} = 2E_z = \frac{C}{(1+\alpha^2\tilde{\omega}_0^2)^{1/2}}, \tag{3.34}$$

$$\frac{J_{\tilde{\omega}}}{\epsilon_0} = -2B_\phi = \frac{2C\alpha}{(1+\alpha^2\tilde{\omega}_0^2)^{1/2}}, \tag{3.35}$$

$$\frac{J_\phi}{\epsilon_0} = 2B = \frac{C}{\tilde{\omega}_0(1+\alpha^2\tilde{\omega}_0^2)^{1/2}}. \tag{3.36}$$

Note that the current conservation equation,

$$\frac{1}{\tilde{\omega}_0}\frac{\partial}{\partial\tilde{\omega}_0}(\tilde{\omega}_0 J_{\tilde{\omega}}) = -2j_z, \tag{3.37}$$

is automatically satisfied and that the current law has been modified slightly from $J_\phi \propto \tilde{\omega}^{-1}$. The fields, charges and currents are displayed in Fig. 1.

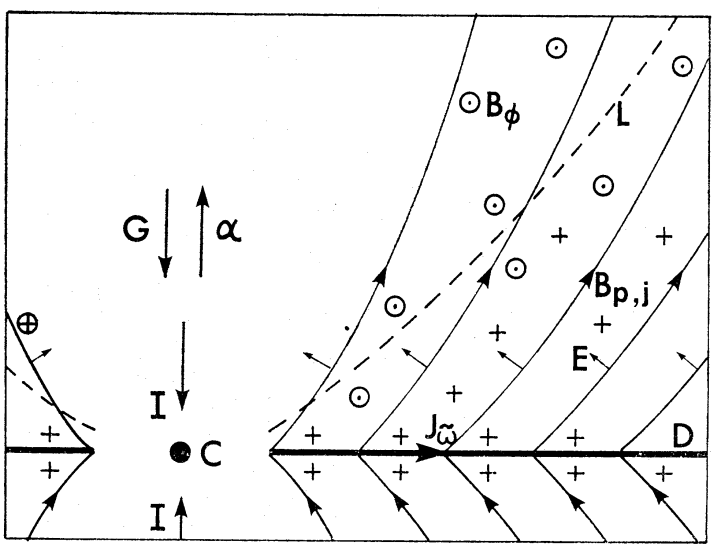

FIG. 1. *Schematic representation of the magnetosphere above and below a magnetized accretion disc, D, surrounding a compact object, C. For the Newtonian solution (with* $\boldsymbol{\alpha}.\mathbf{B}>0$*) described by equations* (3.29)–(3.36), *the poloidal field,* B_p*, lies on paraboloidal surfaces with the toroidal component,* B_ϕ*, becoming increasingly important as the light surface, L, is reached. The light surface drawn is appropriate for a Keplerian disc,* $\alpha^2 = R_s c^2/(2\tilde{\omega}_0^3)$ *where* R_s *is the Schwarzschild radius of the compact object. The current, j, and charge density,* ρ*, are related by* $j \sim 2^{1/2}\,\rho c$ *close to the disc.*

We have therefore found an electromagnetic solution of the force-free field equations that appears to be compatible with physical boundary conditions imposed on an infinite, rotating current disc and, as we show below, has an acceptable behaviour well beyond the light surface. (The field structure is not of course force-free within the disc, which is a massive source of angular momentum in an exactly analogous fashion to a pulsar.) We are free to specify the rotational velocity $\alpha\tilde{\omega}_0$ as a function of $\tilde{\omega}_0$, with the sole restriction that it not exceed the speed of light anywhere. If, for instance, $\alpha \to 0$ at large and small radii, $J_{\tilde{\omega}}$ also goes to zero and both signs of current leave the disc along the field lines so as to preserve the surface charge. In this case the disc behaves similarly to the polar cap of a pulsar. Alternatively, if α increases monotonically with decreasing radius until some inner boundary, then a current must flow outwards from within this inner boundary carrying away charges of sign opposite to those lost by the disc. This is probably the case if the material in the disc is in Keplerian orbit as we discuss in Section 4. In this connection, we remark at this stage that we have explicitly solved the problem in a flat space. In a gravitational field, the Einstein–Maxwell equations should be solved in the relevant spacetime, and this will introduce corrections $O(\alpha\tilde{\omega})^2$ to the solution. For the problem of a Keplerian disc therefore, the solution, equations (3.29)–(3.32), is really only valid to lowest order in $(\alpha\tilde{\omega})^2$. (The generalization of these ideas to a Kerr metric is discussed in Blandford & Znajek, 1976, in preparation.) In the following two Sections we only use the lowest order terms.

(iv) *Electromagnetic wind*

We now consider the properties of the solution we have just described beyond the light surface and check that they are physically acceptable. From equations (3.23), (3.24), (3.27) we have that

$$j_u = \frac{\alpha v^2(u^2+v^2)^{1/2}}{u\{A(1+\alpha^2 v^4)-1\}^{1/2}}\,\rho, \tag{3.38}$$

$$j_\phi = \frac{-\alpha v^3}{u}\,\rho. \tag{3.39}$$

At large distances from the disc $u \gg v$ and, for the solution with $A = 1$, $j_u \to \rho$, $j_\phi \to 0$, i.e. both signs of charge stream out poloidally with an asymptotic speed c. (If $A > 1$ the asymptotic speed is subrelativistic, confirming that such solutions are acceptable beyond the light surface.)

At large distances (for $A = 1$) the poloidal and toroidal fields satisfy

$$\frac{B_\phi}{B_u} = -\alpha(\tilde{\omega}^2+\tilde{\omega}_0{}^2)^{1/2} \to -\alpha\tilde{\omega}, \tag{3.40}$$

$$\frac{E_v}{B_\phi} = \tilde{\omega}(\tilde{\omega}^2+\tilde{\omega}_0{}^2)^{-1/2} \to 1. \tag{3.41}$$

As in the model by Goldreich & Julian (1969), the magnetic field is predominantly toroidal at large distances. A non-zero poloidal component is, however, necessary to ensure that angular momentum as well as energy is carried off by the electromagnetic fields. The poloidal component of the Poynting flux lies on the paraboloidal surfaces v = constant and so the energy is focused towards the rotation axis as the distance from the disc is increased.

(v) *Electromagnetic torque*

The rate of transport of electromagnetic angular momentum can be calculated from the Maxwell stress tensor, or equivalently from the electromagnetic force acting on the disc. The torque per unit area, tending to decelerate the disc, is given by

$$\mathbf{G} = \mathbf{r}\times(\mathbf{J}\times\langle\mathbf{B}\rangle) = -\frac{C^2\boldsymbol{\alpha}\epsilon_0}{(1+\alpha^2\tilde{\omega}_0{}^2)}, \tag{3.42}$$

where $\langle\mathbf{B}\rangle$ is the mean value of the magnetic field through the disc. The corresponding rate of loss of energy per unit area is given by

$$W = -\mathbf{G}.\boldsymbol{\alpha} = \frac{C^2\alpha^2\epsilon_0}{(1+\alpha^2\tilde{\omega}_0{}^2)}. \tag{3.43}$$

There are no irreversible energy losses heating the disc with this type of torque, unlike the case of a disc evolving under viscous stresses (e.g. Pringle & Rees 1972). In addition to the torque given by equation (3.42), there are electromagnetic stresses tending to compress the disc, resulting from the radial component of the magnetic field.

(vi) *Inertial effects*

From equations (3.31), (3.32) we have that, for $\alpha\tilde{\omega}_0 \ll 1$,

$$\frac{j}{\rho} \simeq \left(1+\frac{\tilde{\omega}_0{}^2}{\tilde{\omega}^2}\right)^{1/2} \tag{3.44}$$

which is $\sim 2^{1/2}$ close to the disc, decreasing to ~ 1 near the light surface and beyond. The magnetosphere therefore cannot be completely charge-separated and charges of both signs must flow away from the disc with different velocities. However, the magnitudes of the particle fluxes cannot be determined from the purely electromagnetic equations we have used so far. Neither can they be fixed in this case by the equation proposed by Scharlemann & Wagoner (1973) who assume that there is a small component of electric field resolved parallel to the magnetic field, and equate the resulting changes in the momenta of the two types of particle present. This is because, if the asymptotic particle speed is c, particles with both signs of charge must be accelerated.

Nevertheless, it is presumably inertial effects that dictate the particle flux along the field lines, as the force-free electromagnetic equations are unaffected by the addition of pairs of oppositely-charged particles. If the particle density becomes too large, inertial effects near the light surface will introduce additional bending of the field lines.

For a magnetosphere in which the electron density n_e has the minimum value, $n_e \sim \rho/e$, the plasma frequency ω_p and electron gyro frequency Ω_e are connected by the usual relation $\tilde{\omega}_p{}^2 \sim \alpha\Omega_e$ along the field lines. The condition that inertial terms not be important near the light cylinder is then that $\epsilon_0 B_{ls}{}^2 \gg n_e m_p$, where m_p is the proton mass. Equivalently we require that the ion gyro-frequency at the light cylinder greatly exceed the angular frequency α. In terms of the ion gyro-frequency at the disc, Ω_{id}, this condition becomes

$$\tilde{\omega}_0{}^2\alpha\Omega_{id} \gg 1 \tag{3.45}$$

using the field structure of equation (3.29). This condition also guarantees that gravitational forces can be ignored. However, for a denser magnetosphere, a more stringent condition must be satisfied. Undoubtedly a realistic treatment would be much more complex, involving streaming instabilities, pair creation and radiative losses which, coupled with inertial effects, would be able to sustain potential differences along the field lines that are eventually comparable with those across the disc, giving a consequent conversion of electromagnetic into particle energy flux. We do not consider this further in this paper.*

In this Section, we have presented an idealized electromagnetic model of a machine that seems in principle capable of extracting gravitational energy from a disc of matter and converting it continuously into a coupled electromagnetic/relativistic particle fluid efficiently collimated into directions parallel and anti-parallel to the rotation axis. Although superficially dissimilar, the solution described above and that outlined in the Appendix may provide some insight into the difficult problem of the structure of the open field lines of a pulsar. We now turn to the constraints this solution imposes on the disc and the central source of gravitation.

4. ELECTROMAGNETIC BRAKING OF KEPLERIAN ACCRETION DISCS

In equation (3.42) of the previous Section, we have given an expression for the electromagnetic torque acting on the disc, that is steadily extracting both angular momentum and orbital energy. If the disc is stationary, and the velocity in the disc is $\mathbf{v}$, with $v_{\tilde{\omega}} \ll v_\phi$; $v_z = 0$, then the Navier–Stokes equation gives

$$v_{\tilde{\omega}} \frac{\partial}{\partial \tilde{\omega}} (\tilde{\omega} v_\phi) = \frac{G}{\Sigma}. \tag{4.1}$$

We now assume that the tangential velocity is Keplerian and ignore all but the lowest order terms in $(\alpha\tilde{\omega})^2$. Introducing the mass flux, $F = -2\pi\Sigma\tilde{\omega}v_{\tilde{\omega}}$, which is constant for a stationary disc, we obtain the self-consistent relation

$$F = 8\pi\epsilon_0 c B_\mathrm{d}^2 \tilde{\omega}_0^2, \tag{4.2}$$

where B_d is the magnitude of the poloidal field strength at the disc (*cf.* equation (2.5)).

If we further stipulate that magnetic flux is conserved then, as $B_z\tilde{\omega}v_{\tilde{\omega}}$ is constant on the disc, so is $v_{\tilde{\omega}}$, and $\Sigma \propto \tilde{\omega}^{-1}$. However, it is clear that in a *strictly* stationary situation we cannot convect the flux radially inwards. If, near the central object where the solution will in any case be invalid, there is magnetic flux of opposite sign to that convected inwards, it is in principle possible to reconnect the flux within the central region so as not to disturb the outer solution very seriously (*cf.* also Sturrock & Barnes 1972). This reconnection can continue until all the central flux has been removed by the reconnection process. Equivalently, the sense of the magnetic field accreted can alternate on time-scales long compared with the transit time through the disc.

Alternatively, we can rely on a finite conductivity within the disc allowing the matter to cross the field lines. In practice, what will probably happen is that the field lines which are predominantly poloidal in the magnetically-dominated region

* It is an interesting speculation that a similar situation prevails on at least some of the open field lines of a non-axisymmetric pulsar—that the force-free electromagnetic solution requires the current to exceed ρc. In this case, parallel electric fields would be set up trying to force the plasma to counter-stream. If, as seems plausible, the plasma is sufficiently dense to allow streaming instabilities to grow within a pulse period, coherent bunches of charged particles might well be able to form. These could then emit the observed radio-frequency curvature radiation of high brightness temperature. Note that this instability would be driven by an electromagnetically maintained potential difference rather than a separate distribution of high-energy charged particles originating in a spark as in the model of Ruderman & Sutherland (1975) (*cf.* also the Appendix).

above the disc will be sheared by the differential rotation and become mainly toroidal within the disc. They will then be able to reverse on a length scale no larger than the height of the disc, and quite plausibly much smaller. Reconnection of the field lines then occurs at a rate balancing the amplification due to shearing. Such a process has been analysed by Lynden-Bell (1969) and Eardley & Lightman (1975), and it is obviously capable of introducing an additional torque into the system larger than the electromagnetic braking torque described above. The detailed conditions for self-consistency of the electromagnetic picture depend upon, amongst other factors, a knowledge of the vertical structure of the disc and we do not consider them here. Even if the dominant torques arise from magnetic viscosity (or from turbulent viscosity), electromagnetic extraction of energy can still occur, although the fraction of the gravitational energy liberated in such a form will be much less than unity and the field law, $B_{\rm d} \propto \tilde{\omega}^{-1}$, will no longer be required.

It is possible to justify the neglect of the work done against the radial magnetic stresses in the energy equation (3.43). The shear stress acting in the radial direction is given by $2B_{\tilde{\omega}}B_z/\mu_0$ and so the ratio of the work done (per unit area) by this stress to the work done by the tangential stress, $2B_zB_\phi/\mu_0$, is $v_{\tilde{\omega}}c/2v_\phi^2$. But this is also twice the ratio of the radial stress to the gravitational force (per unit area) acting on the disc and so, if we are assuming that the disc is Keplerian (i.e. the gravitational force dominates the radial equation), we can ignore the work done by the disc pulling inwards against the poloidal field. (In fact an even stronger condition must probably be satisfied for self-consistency. If the disc itself is not to be unstable, its gas pressure must exceed the magnetic pressure.) Taking the height of the disc as h and the gravitational potential difference across half the disc as $\sim \alpha^2 h^2$, we obtain the inequality

$$\frac{v_{\tilde{\omega}}c}{v_\phi^2} \lesssim \frac{h}{\tilde{\omega}} \ll 1. \tag{4.3}$$

We next consider the inner and outer edges of the disc. The outer edge can be determined by one of several different criteria—the radius at which the velocity in the disc first becomes Keplerian; the field line for which the electromagnetic pressure evaluated at the light cylinder becomes comparable with the external pressure; or the field line on which inertial effects (as possibly measured by inequality (3.45)) first become important. However, as long as the electromagnetic solution is valid over several decades of radius, conditions at the outer edge will have little influence on the energy liberation rate from the inner disc. Of much more importance is the electrodynamical configuration at the inner edge, as this is where most of the gravitational energy is released. In the case of accretion on to a self-magnetized central object (e.g. a neutron star), the inner edge of the disc is probably given by the Alfvén radius, $R_{\rm A}$, where the magnetic stresses balance the tangential component of the momentum flux in the disc. If the radius of the central object is R_*, then at most a fraction $R_*/R_{\rm A}$ of the power radiated as a result of infall can be radiated electromagnetically. The angular velocity is unlikely to reach zero at $R_{\rm A}$ and so there will be a radial current

$$2I \sim 2\pi J_{\tilde{\omega}} R_{\rm A} \sim \frac{4\pi}{\mu_0 c} C\alpha(R_{\rm A})\, R_{\rm A}, \tag{4.4}$$

and the circuit will presumably be completed by the escape of charged particles (of opposite sign to those lost from the disc) along the rotation axes in the form of two currents of magnitude $\frac{1}{2}I$. These particles may also be accelerated to relativistic speeds but the power associated with their escape cannot be estimated without making detailed assumptions about the electromagnetic nature of the central object. There will be a toroidal field associated with the axial current but this will not seriously alter the solution that we have been using. To see this, it is simplest to observe that if we preserve the poloidal field structure but let the angular velocity decrease to zero at some finite radius, then the solution is exact and from equation (3.35) there need be no axial current. Changing the distribution of currents within radius R will therefore only introduce fractional errors $\sim R/\tilde{\omega}$ in the calculated toroidal field, and at sufficiently large values of $\tilde{\omega}$ these can be ignored.

If the central object is a black hole, then the inner boundary conditions depend on electromagnetic and geometrical considerations that are completely absent from this treatment. In the case of a Schwarzschild geometry a very rough estimate of the total fraction of rest-mass energy that can be released is ~ 5 per cent, the binding energy of the last stable orbit (*cf.* Lynden-Bell 1969). Without a self consistent discussion of the dynamics of the disc within this orbit, it does not seem possible to improve upon the statement that in principle an appreciable fraction of the rest-mass energy of the infalling material can be released in the form of electromagnetic fields and relativistict particles by this mechanism.

5. ASTROPHYSICAL APPLICATIONS

The electrodynamical model outlined in Section 3 is no more than an idealization of a general mechanism that could operate in several distinct astrophysical environments. We consider briefly two obvious examples:

(i) *Extragalactic double radio sources*

Recent observations of powerful, double radio sources have indicated that energy must be supplied continuously throughout the lifetime of the source. In models discussed by Rees (1971), Longair, Ryle & Scheuer (1973), Scheuer (1974) and Blandford & Rees (1974), it has been proposed that the energy flows from a compact galactic nucleus along two channels out towards the extended radio components. In fact the occurrence of linear radio structure appears to be very general and has been observed on much smaller length scales than that associated with extended sources (e.g. Kellermann *et al.* 1975; Schilizzi *et al.* 1975; Fomalont & Miley 1975; Pauliny-Toth *et al.* 1976). In Blandford & Rees (1974) it was postulated that the energy is transferred as a very light fluid comprising relativistic particles and electromagnetic fields, through two de Laval nozzles where the flow becomes supersonic. The most serious theoretical criticism of this idea involves the stability of these channels. Recent studies of the relativistic Kelvin–Helmholtz instability (Turland & Scheuer 1976; Blandford & Pringle 1976), whilst inconclusive, do indicate that the problem is much more severe for subsonic and trans-sonic than supersonic flows. There are therefore attractions in a model in which the fluid is created already in a supersonic form.

In the context of these models it is reasonable to associate the radio-source axis with the rotation axis of the nuclear region but the nature of the ultimate energy source is less obvious. Either clusters of stellar-mass objects (e.g. Rees 1971; Arons, Kulsrud & Ostriker 1975) or single coherent massive objects (e.g. Lynden-Bell 1969; Morrison & Cavaliere 1971) have been postulated. Electromagnetic extraction of gravitational energy from a differentially-rotating accretion disc could occur in either category, e.g. for a dense cluster of stellar-mass black holes within a nucleus, it would not be surprising if the majority had angular momenta approximately parallel to the nucleus' spin axis. For the reasons discussed in Section 4 the efficiency of the energy extraction is most likely to be large if the central object is a black hole and we assume this for the purposes of making simple numerical estimates.

In Pringle, Rees & Pacholczyk (1973) some physical processes likely to be associated with accretion by massive black holes are discussed. In particular, it is argued that the inner regions of these discs are liable to be unstable and release loops of magnetic field that become coiled around the rotation axis. In this case, the flux problem is circumvented by violating axisymmetry and, if the magnetosphere is electromagnetic rather than hydromagnetic, two collimated beams can be produced by a mechanism similar to that analysed above.

For a large black hole of mass $10^9\, M_9 M_\odot$, radiating $10^{39}\, L_{39}$ W relativistically with an efficiency of $0 \cdot 1\, \epsilon_{-1}$, we need a mass accretion rate of

$$F \sim 1 \cdot 5\, L_{39} \epsilon_{-1}^{-1}\, M_\odot\, \mathrm{yr}^{-1}. \tag{5.1}$$

The Schwarzschild radius is given by

$$R_\mathrm{s} \sim 3 \times 10^{12}\, M_9\, \mathrm{m}, \tag{5.2}$$

and if we use equation (4.2) the corresponding magnetic field strength on the disc is

$$B \sim 0.3 \left(\frac{R_\mathrm{s}}{\tilde{\omega}}\right) L_{39}^{1/2} \epsilon_{-1}^{-1/2} M_9^{-1}\, \mathrm{T}. \tag{5.3}$$

If, as an example, we set the height of the disc to be $h \sim 0 \cdot 1\, R_\mathrm{s}$ at $\tilde{\omega}_0 \sim 3R_\mathrm{s}$ and the infall velocity $u_{\tilde{\omega}} \sim -10^{-2}\, c$ (thus satisfying equation (4.3)), then

$$\Sigma \sim 500\, L_{39} \epsilon^{-1} M_9^{-1}\, \mathrm{kg\, m}^{-2} \tag{5.4}$$

and the corresponding electron density, n_e, is

$$n_\mathrm{e} \sim 10^{18}\, L_{39} \epsilon^{-1} M_9^{-2}\, \mathrm{m}^{-3}. \tag{5.5}$$

If, as discussed in Section 3, the particle density in the magnetosphere has the value $\sim \rho/e$, then the mass flowing out through the magnetosphere predominantly from the inner part of the disc $\sim 10^{-14}\, L_{39}^{1/2} \epsilon_{-1}^{-1/2}\, M_\odot\, \mathrm{yr}^{-1}$, which is a negligible fraction of the rate at which mass is swallowed by the black hole. However, without a prescription for determining the equilibrium field strength and a better understanding of inertial effects, these estimates can only confirm that it is not impossible for the model to be self-consistent.

We assume that somewhere, probably in the vicinity of the light surface, the directed flux of electromagnetic momentum is at least partially converted into a flux of relativistic particles, shorting out some of the potential difference across the field lines. The asymptotic state is probably that there is no electric field in a frame moving with a single wind velocity. Thereafter the magnetized plasma behaves as a coupled fluid which will expand transversely until the total transverse pressure balances the confining pressure of the surrounding material. The flow will subsequently develop in accord with the considerations of Longair, Ryle & Scheuer

(1973) and Blandford & Rees (1974). The dominant stress within this fluid may, on the other hand, be due to a transverse magnetic field, and this could seriously change the stability requirements of the beam from that of an isotropic fluid.

As with a pulsar magnetosphere, there are several possible electromagnetic processes involving the outflowing plasma that we have ignored, e.g. synchrotron cooling of the electrons in the strong magnetic field, and inverse Compton scattering of radiation originating from the inner parts of the disc. Such processes could undoubtedly account for much of the non-thermal variability associated with galactic nuclei, but our present ignorance of the relevant physics makes it profitless to attempt any detailed comparison. (It is an interesting aspect of this model that it provides a means of circumventing the Eddington limit, because the energy liberated by the disc can be predominantly non-radiative and can escape with high anisotropy.)

It is regrettably difficult to think of observational tests by which this mechanism can be discriminated from the alternatives that have been proposed. However, if VLB studies of compact radio structure within galactic nuclei and quasars show that the linear structure associated with the extended components is also generally found on the scale of $\lesssim 1$ pc, *without* substantial variability in position, then the type of model suggested here would seem preferable to, for instance, the 'twin-exhaust' model in which the collimation is achieved on a length scale estimated to be ~ 10–100 pc and in which the compact structure must probably be attributed to independent events in a disc sharing a common rotation axis with the nucleus. A further indication that radio outbursts occur in a fairly permanent magnetic field structure might be if the observed polarization has a preferred orientation probably parallel to the axis, when the source is in a quiescent state.

Finally, in the context of radio-source models, this mechanism could actually be operating within the individual extended components if they contain large collapsed objects as proposed by Rees & Saslaw (1975, and references therein).

(ii) *Sco X-1*

Sco X-1 is the strongest X-ray source in the sky and it has recently been claimed to have a binary orbital period of 0·78 day (Gottlieb, Wright & Liller 1975). In theoretical models (e.g. Pringle & Rees 1972), it has been proposed that the transferred stellar material forms an accretion disc around a compact object, probably a neutron star. However, Sco X-1 has also been positionally identified with a triple radio source that appears to be a scale model of the typical extragalactic configuration (Braes & Miley 1971). In particular, only the central component appears to be variable. This radio emission can also be attributed to the creation of two beams of relativistic plasma by the magnetized accretion disc. In this case, since the integrated radio luminosity is only $\sim 10^{-10}$ of the X-ray luminosity, the electromagnetic extraction of energy need not be very efficient, and the disc can terminate at the Alfvén radius where the binding energy per unit mass $\ll c^2$. An observable test of this idea may arise if X-ray polarization were detected, resulting from scattering in the disc (*cf.* Rees 1975). This would then indicate the orientation of the rotation axis in the sky, which ought to agree with the orientation of the radio source axis. High-sensitivity radio maps, in the vicinity of other galactic X-ray sources (in particular those associated with globular clusters) may reveal further examples of double sources and confirm that the source associated with Sco X-1 is not a chance coincidence.

ACKNOWLEDGMENTS

I thank H. Ardavan, M. J. Rees and R. Znajek for many invaluable discussions. I also acknowledge helpful comments by D. Lynden-Bell, F. D. Kahn, M. McEllin, L. Mestel and J. E. Pringle.

REFERENCES

Arons, J., Kulsrud, R. M. & Ostriker, J. P., 1975. *Astrophys. J.*, **198**, 687.
Blandford, R. D. & Rees, M. J., 1974. *Mon. Not. R. astr. Soc.*, **169**, 395.
Blandford, R. D. & Pringle, J. E., 1976. *Mon. Not. R. astr. Soc.*, **176**, 443.
Braes, L. L. E. & Miley, G. K., 1971. *Astr. Astrophys.*, **14**, 160.
Eardley, D. M. & Lightman, A. P., 1975. *Astrophys. J.*, **200**, 187.
Fomalont, E. B. & Miley, G. K., 1975. *Nature*, **257**, 99.
Goldreich, P. & Julian, W. H., 1969. *Astrophys. J.*, **157**, 869.
Gottlieb, E. W., Wright, E. L. & Liller, W., 1975. *Astrophys. J. Lett.*, **195**, L33.
Jackson, J. D., 1962. *Classical Electrodynamics*, John Wiley & Sons, New York.
Kellermann, K. I., Clark, B. G., Niell, A. E. & Shaffer, D. B., 1975. *Astrophys. J. Lett.*, **197**, L113.
Longair, M. S., Ryle, M. & Scheuer, P. A. G., 1973. *Mon. Not. R. astr. Soc.*, **164**, 243.
Lovelace, R. V., 1976. (preprint)
Lynden-Bell, D., 1969. *Nature*, **223**, 690.
Michel, F. C., 1973. *Astrophys. J. Lett.*, **180**, L133.
Morrison, P. & Cavaliere, A., 1971. *Nuclei of galaxies*, ed. O'Connell, North Holland, Amsterdam.
Okamoto, I., 1974. *Mon. Not R. astr. Soc.*, **167**, 457.
Ozernoi, L. M. & Usov, V. V., 1973. *Astrophys. Space Sci.*, **25**, 149.
Pauliny-Toth, I. I. K., Preuss, E., Witzel, A., Kellermann, K. I., Shaffer, D. B., Purcell, G. H., Grove, G. W., Jones, D. L., Cohen, M. H., Moffett, A. T., Romney, J., Schilizzi, R. T. & Rinehart, R., 1976. *Nature*, **259**, 17.
Piddington, J. H., 1970. *Mon. Not. R. astr. Soc.*, **148**, 131.
Pringle, J. E. & Rees, M. J.. 1972. *Astr. Astrophys.*, **21**, 1.
Pringle, J. E., Rees, M. J. & Pacholczyk, A. G., 1973. *Astr. Astrophys.*, **29**, 179.
Rees, M. J., 1971. *Nature*, **229**, 312, errata p. 510.
Rees, M. J. & Saslaw, W. C., 1975. *Mon. Not. R. astr. Soc.*, **171**, 53.
Ruderman, M. & Sutherland, P. G., 1975. *Astrophys. J.*, **196**, 51.
Ruffini, R., & Wilson, J. R. 1975. *Phys. Rev. D*, **12**, 2959.
Scharlemann, E. T. & Wagoner, R. V., 1973. *Astrophys. J.*, **182**, 951.
Scheuer, P. A. G., 1974. *Mon. Not. R. astr. Soc.*, **166**, 513.
Schilizzi, R. T., Cohen, M. H., Romney, J. D., Shaffer, D. B., Kellermann, K. I., Swenson, G. W., Yen, J. L. & Rinehart, R., 1975. *Astrophys. J.*, **201**, 263.
Shakura, N. I. & Sunyaev, R. A., 1973. *Astr. Astrophys.*, **24**, 337.
Sturrock, P. A. & Barnes, C., 1972. *Astrophys. J.*, **176**, 31.
Turland, B. D. & Scheuer, P. A. G., 1976. *Mon. Not. R. astr. Soc.*, **176**, 421.

APPENDIX

FORCE-FREE MAGNETOSPHERE ABOVE A ROTATING MAGNETIZED DISC

In this Appendix we present a force-free electromagnetic solution, alternative to that given in equations (3.29)–(3.32), which may be of more relevance to the problem of the structure of a pulsar magnetosphere.

Instead of the paraboloidal coordinates used in Section 3, we introduce oblate spheroidal coordinates (u, v, ϕ) for which

$$\begin{aligned} \tilde{\omega} &= Ruv, \\ z &= R(u^2-1)^{1/2}(1-v^2)^{1/2}, \\ h_u &= R(u^2-v^2)^{1/2}(u^2-1)^{-1/2}, \\ h_v &= R(u^2-v^2)^{1/2}(1-v^2)^{-1/2}. \end{aligned} \qquad \text{(A1)}$$

In the upper half space, $1 \lesssim u \lesssim \infty$, $0 \lesssim v \lesssim 1$, and the rotation axis is again taken to be along the z direction. The magnetic field lines, on which v is constant, lie on hyperboloidal surfaces and emerge normally from a finite thin disc of radius R in the x–y plane, becoming radial at distances much greater than R.

The calculation of the electromagnetic structure proceeds analogously to that presented in Section 3 and we obtain

$$
\begin{aligned}
B_u &= \frac{B_0}{u(u^2-v^2)^{1/2}(1-\alpha^2 v^4 R^2)^{1/2}}, \\
B_\phi &= \frac{B_0(1-v^2)^{1/2}\,\alpha}{\mu u(1-\alpha^2 v^4 R^2)^{3/2}} \frac{d}{dv}(\alpha v^2), \\
E_v &= -\frac{B_0 \alpha v R}{(u^2-v^2)^{1/2}(1-\alpha^2 v^4 R^2)^{1/2}}, \\
\frac{\rho}{\epsilon_0} &= -\frac{B_0(1-v^2)^{1/2}}{v(u^2-v^2)(1-\alpha^2 v^4 R^2)^{3/2}} \frac{d}{dv}(\alpha v^2), \\
\frac{j_u}{\epsilon_0} &= \frac{B_0 \mu}{u(u^2-v^2)^{1/2}(1-\alpha^2 v^4 R^2)^{1/2}}, \\
\frac{j_\phi}{\epsilon_0} &= -\frac{B_0 \alpha v^2 R(1-v^2)^{1/2}}{u(u^2-v^2)^{1/2}(1-\alpha^2 v^4 R^2)^{3/2}} \frac{d}{dv}(\alpha v^2),
\end{aligned} \tag{A2}
$$

where B_0 is the field strength at the centre of the disc. The solution for μ corresponding to equation (3.27) is

$$
\mu = \frac{-\alpha v R(1-v^2)^{1/2}}{(1-\alpha^2 v^4 R^2)\{1-A(1-\alpha^2 v^4 R^2)\}^{1/2}} \frac{d}{dv}(\alpha v^2). \tag{A3}
$$

If we stipulate that the radial surface current in the disc (and the toroidal field component) vanish at the origin, then $A = 1$. Transforming back to cylindrical polar coordinates, we obtain

$$
\begin{aligned}
\mathbf{B} &= \frac{B_0 \tilde{\omega}_0^2 R}{\tilde{\omega}(\tilde{\omega}^2 R^2 - \tilde{\omega}_0^4)(R^2-\alpha^2\tilde{\omega}_0^4)^{1/2}} \\
&\quad \times \{\tilde{\omega}_0 R(\tilde{\omega}^2-\tilde{\omega}_0^2)^{1/2},\ \tilde{\omega} R(R^2-\tilde{\omega}_0^2)^{1/2},\ -\alpha(\tilde{\omega}^2 R^2-\tilde{\omega}_0^4)\}, \\
\mathbf{E} &= \frac{B_0 \alpha R^2 \tilde{\omega}_0^2}{(\tilde{\omega}^2 R^2-\tilde{\omega}_0^4)(R^2-\alpha^2\tilde{\omega}_0^4)^{1/2}} \{-\tilde{\omega}(R^2-\tilde{\omega}_0^2)^{1/2},\ \tilde{\omega}_0(\tilde{\omega}^2-\tilde{\omega}_0^2)^{1/2},\ 0\}, \\
\frac{\rho}{\epsilon_0} &= -\frac{B_0 \tilde{\omega}_0 R^4 (R^2-\tilde{\omega}_0^2)^{1/2}}{(\tilde{\omega}^2 R^2-\tilde{\omega}_0^4)(R^2-\alpha^2\tilde{\omega}_0^4)^{3/2}} \frac{d}{d\tilde{\omega}_0}(\alpha\tilde{\omega}_0^2), \\
\frac{\mathbf{j}}{\epsilon_0} &= -\frac{B_0 R^2 \tilde{\omega}_0 (R^2-\tilde{\omega}_0^2)^{1/2}}{\tilde{\omega}(\tilde{\omega}^2 R^2-\tilde{\omega}_0^4)(R^2-\alpha^2\tilde{\omega}_0^4)^{3/2}} \frac{d}{d\tilde{\omega}_0}(\alpha\tilde{\omega}_0^2) \\
&\quad \times \{R\tilde{\omega}_0(\tilde{\omega}^2-\tilde{\omega}_0^2)^{1/2},\ R\tilde{\omega}(R^2-\tilde{\omega}_0^2)^{1/2},\ \alpha\tilde{\omega}_0^4\},
\end{aligned} \tag{A4}
$$

where the equation of the field lines is

$$
(R^2-\tilde{\omega}_0^2)\left(\frac{\tilde{\omega}^2}{\tilde{\omega}_0^2}-1\right) = z^2. \tag{A5}
$$

In the far field,

$$\frac{B_\phi}{B_u} = -\frac{\alpha(\tilde{\omega}^2R^2-\tilde{\omega}_0{}^4)^{1/2}}{R} \rightarrow -\alpha\tilde{\omega},$$

$$\frac{E_v}{B_\phi} = \frac{\tilde{\omega}R}{(\tilde{\omega}^2R^2-\tilde{\omega}_0{}^4)^{1/2}} \rightarrow 1 \tag{A6}$$

as in Section 3. Note that, at large distances, the field lines are asymptotically radial in this solution rather than parallel to the rotation axis as in the infinite disc solution. We have therefore found a second electromagnetic magnetospheric solution. A third example is Michel's (1973) monopole solution.

The ratio of the poloidal current to the charge density is given by

$$\frac{j_u}{\rho} = \frac{(\tilde{\omega}^2R^2-\tilde{\omega}_0{}^4)^{1/2}}{\tilde{\omega}R}, \tag{A7}$$

which tends to unity when $\tilde{\omega} \gg R$. The electromagnetic structure of the far field is therefore precisely as deduced on general grounds by Goldreich & Julian (1969). Closer to the disc, we see that $j_u < \rho$ and so the magnetosphere can be completely charge-separated, and counter-streaming plasma is not required.

Finally, the electromagnetic torque acting on a uniformly-rotating disc, in the limit $\alpha R \ll 1$, is given by

$$\int_0^R 2\pi\tilde{\omega}_0{}^2 J_{\tilde{\omega}} B_z \, d\tilde{\omega}_0 = \frac{8\pi}{3}\epsilon_0\alpha B_0{}^2R^4, \tag{A8}$$

where B_0 is the field strength at the axis. However, for an axisymmetric pulsar of radius R, this torque is estimated to be $\sim 4\pi\epsilon_0\alpha^3B_0{}^2R^6$. The reason for the discrepancy is that in this solution all the magnetic flux crosses the light cylinder, whereas for a pulsar only a fraction $\sim \alpha R$ of the total flux threading the star does not return to the opposite hemisphere within the light cylinder.

To what extent is this solution applicable to pulsars? The most obvious difference is in the field behaviour close to the equatorial plane. In the solution, the toroidal and radial fields reverse sign discontinuously when $z = 0$ and therefore must be supported by a radial current. In a realistic pulsar model, on the other hand, these field lines are expected to be dipolar and, for the low-latitude field lines in particular, the sign of $\boldsymbol{\alpha}.\mathbf{B}$ should change, thus leading to a change in the sign of the charge density. The solution therefore offers no insight into the difficult problems posed by this property, although it may provide a convenient approximate form for the structure of the high-latitude open-field lines in an *axisymmetric* pulsar model, simply by restricting the solution to values of v less than some intermediate value $\sim \frac{1}{2}$. In particular it may be relevant to some models of the pulsar emission mechanism. However, we would emphasize that the result, $j < \rho$, will not necessarily carry over to a non-axisymmetric magnetosphere.