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KELVIN–HELMHOLTZ INSTABILITY OF RELATIVISTIC BEAMS

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SUMMARY

A discussion of the linearized Kelvin–Helmholtz instability in the vicinity of a plane infinite vortex sheet separating two fluids in relative motion is presented. The calculation generalizes existing treatments to include relativistic relative motion and relativistic internal sound speeds. The character of the unstable modes is outlined in the two limits (a) when the sound speed ratio is large, and (b) when it is equal to unity. The relevance of these results to beam models of extragalactic radio sources is briefly discussed.

1. INTRODUCTION

Recent observations of extended extragalactic radio sources have suggested that energy is supplied continuously to the extended components. In one class of theoretical models, the energy is essentially in the form of a light isotropic fluid which flows supersonically in antiparallel directions along cylindrical channels continuously feeding the radio-emitting regions. However, the stability of such a configuration is questionable and detailed calculations are necessary to decide upon the self-consistency of this picture.

We present here some calculations we believe to be relevant to this problem. In Section 2 we give the dispersion relation for the Kelvin–Helmholtz instability at a planar interface separating two fluids in relative, possibly relativistic, motion. In Section 3 an approximate analysis of the unstable modes is given for the limit when the ratio of the sound speeds, ϵ, is much less than the reciprocal of the Lorentz factor of the bulk velocity, γ. In Section 4, a corresponding analysis is given for the case when $\gamma^{-1} \ll \epsilon \ll 1$. In Section 5, the analysis given by Miles (1958) for instabilities involving identical fluids is extended into the relativistic domain. All three cases may be of relevance to models of extragalactic radio sources and this is discussed further in Section 6.

2. GENERAL DISPERSION RELATION

Consider two fluids, fluid 1 in the half space $x < 0$, at rest initially with sound speed s_1 and enthalpy w_1, and fluid 2 in the half space $x > 0$ moving with an unperturbed velocity u along the positive $\mathbf{z}$ direction with sound speed s_2 and enthalpy w_2. Both fluids are assumed to be isotropic and external forces (e.g. gravitation), dissipation, viscosity and heat flow are taken to be ignorable. The propagation of small disturbances $\propto \exp(i(\mathbf{k}.\mathbf{r} - \omega t))$ in the vicinity of the interface can be considered by linearizing the equation of motion. We find the dispersion relation

$$\phi^4\{\gamma^2(1-\eta^2)(\phi-M)^2+\eta^2\phi^2-1\} = \gamma^4\delta^2(\phi-M)^4\{\phi^2-\epsilon^2\}\,\epsilon^2 \qquad (1)$$

where we have introduced the notation (*cf.* Gerwin 1968),

$$\phi = \frac{\omega}{qs_2}, \qquad \epsilon = \frac{s_1}{s_2}, \qquad M = \frac{uk_z}{qs_2},$$

$$\eta = s_2, \qquad \delta = \frac{w_2 s_2^2}{w_1 s_1^2}, \qquad \mathbf{q} = (0, k_y, k_z),$$

$$\gamma = (1-u^2)^{-1/2}, \tag{2}$$

and we set c = 1 throughout. The derivation is analogous to non-relativistic treatments and is given by Turland & Scheuer (1976).

For two non-relativistic fluids, w is simply the density of rest mass and δ is the ratio of ratios of specific heats, Γ_2/Γ_1. If both sound speeds are subrelativistic, we may take η to be zero and the dispersion relation for relativistic motion can be transformed exactly to the more usual non-relativistic dispersion relation (Gerwin 1968), by the substitutions $\phi \to \phi/\gamma$, $\epsilon \to \epsilon/\gamma$ and $M \to M/\gamma$. When either or both of the fluids are extremely relativistic, we have again $\delta = \Gamma_2/\Gamma_1$.

To determine the conditions for instability, we must search for complex solutions for ϕ in equation (1) with $\mathrm{Im}(\phi) > 0$. All other quantities in the equation are real. However, not all such solutions are physically acceptable. The perturbations must either be localized to the region around $x = 0$ or satisfy the Sommerfeld radiation condition in the rest frame of each fluid (Miles 1958). This requires $\mathrm{Im}(q) = 0$ in the frame in which we are working and

$$\mathrm{Im}(k_{1x}) < 0 \quad \textit{or} \quad \mathrm{Im}(k_{1x}) = 0 \text{ and } \mathrm{Re}(\omega/k_{1x}) < 0,$$

$$\mathrm{Im}(k_{2x}) > 0 \quad \textit{or} \quad \mathrm{Im}(k_{2x}) = 0 \text{ and } \mathrm{Re}[(\omega - uk_z)/k_{2x}] > 0, \tag{3}$$

where

$$\frac{k_{1x}^2}{q^2} = \frac{\{\phi^2 - \epsilon^2\}}{\epsilon^2},$$

$$\frac{k_{2x}^2}{q^2} = \{\gamma^2(1-\eta^2)(\phi - M)^2 + \eta^2\phi^2 - 1\}. \tag{4}$$

In deriving equation (1) we have introduced spurious solutions and so we further stipulate that the choice of signs for k_{1x}, k_{2x} satisfies

$$\arg\left(\frac{k_{1x}}{k_{2x}}\right) = \arg\left[\left(\frac{\phi}{\phi - M}\right)^2\right]. \tag{5}$$

We may restrict our attention to positive values of M without loss of generality. The reason for this is that if $(\omega, k_{1,2})$ is a particular physical solution with $M > 0$ then $(-\omega^*, -k_{1,2}{}^*)$ is a solution of (1) with M replaced by $-M$, which also satisfies (3), (5), and is the corresponding solution in the sense that it displays similar behaviour in both space and time. We also restrict attention to values of $\epsilon \leq 1$. These results are applicable when the moving fluid has the lower sound speed only in so far as they refer to instabilities with $\mathbf{q}$ real in its frame. As a Lorentz transformation with complex ω introduces an imaginary part of $\mathbf{q}$, the conditions for instability with $\mathbf{q}$ real in the higher sound speed fluid frame must be treated separately. We do not consider this case which can be covered by a modification of the method of Section 3.

3. INSTABILITIES IN THE LIMIT: $\gamma\epsilon \ll 1$

Consider the two fluids to be such that one fluid, (1), (stationary) has a sound speed much less than the other (moving) fluid (2): that is $\epsilon \ll 1$. Our procedure therefore is to search for approximate roots of equation (1) with $\mathrm{Im}(\phi) > 0$ for different values of the effective Mach number, M, a real quantity. We then test these roots using (3) and (5) to see if they are physical.

Consideration of (1) shows that the roots are particularly easy to evaluate when the quantity $|1 - M^2\gamma^2(1-\eta^2)|$ is either much smaller or much larger than $(\gamma\epsilon)^{2/3}$. By considering the solutions in these limits, we find that we can also learn about the behaviour of the instabilities for intermediate values of $1 - M^2\gamma^2(1-\eta^2)$.

(i) $1 - M^2\gamma^2(1-\eta^2) \gg (\gamma\epsilon)^{2/3}$

We discuss this case in detail in order to illustrate a technique for determining approximation solutions of the dispersion relation. We expand (1) as a sextic in ϕ, keeping only the leading terms in the coefficients, under the assumption that $\gamma\epsilon \ll 1$. The equation becomes

$$\gamma^{-4}[\gamma^2(1-\eta^2)+\eta^2]\,\phi^6 - 2M(1-\eta^2)\,\gamma^{-2}\phi^5 - \gamma^{-4}[1-(1-\eta^2)\,\gamma^2M^2]\,\phi^4$$
$$+4M\delta^2\epsilon^2(M^2-\epsilon^2)\,\phi^3 + \delta^2\epsilon^2M^2(6\epsilon^2 - M^2)\,\phi^2 - 4\delta^2M^3\epsilon^4\phi + \delta^2M^4\epsilon^4 = 0. \tag{6}$$

Treating ϵ as a small parameter, we write $\phi = O(\epsilon^n)$ and then search for values of the exponent, n, which cause two or more terms to dominate the dispersion relation. At other values of n, only one term dominates the dispersion relation and a solution is therefore not possible. Assume for the moment that $M \gg \epsilon$ (and remember that $\gamma^{-1}(1-\eta^2)^{-1/2} > M$). Then, when $n = 0$, the first three terms dominate and the resultant equation furnishes the two roots

$$\phi = \frac{M\gamma^2(1-\eta^2) \pm \{\gamma^2(1-\eta^2)(1-M^2\eta^2)+\eta^2\}^{1/2}}{\gamma^2(1-\eta^2)+\eta^2} \tag{7a}$$

which must be real. When $n = 1$, the third, fifth and seventh terms dominate, yielding the remaining four roots

$$\left(\frac{\phi}{\epsilon}\right)^2 = \frac{-\delta^2M^4\gamma^4}{2[1-(1-\eta^2)\,\gamma^2M^2]}\left[1 \pm \left\{1 + \frac{4[1-(1-\eta^2)\,\gamma^2M^2]}{\gamma^4M^4\delta^2}\right\}^{1/2}\right]. \tag{7b}$$

Examination of the remaining terms in (6) shows that these equations are also valid for $M \lesssim \epsilon$. It is clear that one of the roots of (7b) has $\mathrm{Im}(\phi) > 0$ and is potentially growing. Two limiting forms of interest of this solution are

$$\phi \simeq \delta^{1/2}\epsilon M\gamma \exp[i\pi/2]; \quad M \ll \gamma^{-1},$$

$$\phi \simeq \frac{\delta\epsilon \exp[i\pi/2]}{(1-\eta^2)[1-(1-\eta^2)\,\gamma^2M^2]^{1/2}}; \quad (\gamma\epsilon)^{2/3} \ll [1-(1-\eta^2)\,\gamma^2M^2] \ll 1.$$

In the former limit,

$$\mathrm{Im}(\omega) = \omega_i \simeq q\delta^{1/2}M\gamma s_1,$$
$$k_{1x} \simeq q\exp(3\pi i/2),$$
$$k_{2x} \simeq q\exp(\pi i/2), \tag{8}$$

and

$$\arg(k_{1x}/k_{2x}) \simeq \arg[\{\phi/(\phi - M)\}^2] \simeq \pi,$$

consistent with equations (3), (5). This therefore represents a physical solution.

In the second limit, for which the growth rate is larger,

$$\begin{aligned} \omega_i &\simeq q\delta s_1[1-(1-\eta^2)\,\gamma^2M^2]^{-1/2}(1-\eta^2)^{-1}, \\ k_{1x} &\simeq q\delta(1-\eta^2)^{-1}[1-(1-\eta^2)\,\gamma^2M^2]^{-1/2}\exp(3\pi i/2), \\ k_{2x} &\simeq q[1-(1-\eta^2)\,\gamma^2M^2]^{1/2}\exp(\pi i/2). \end{aligned} \tag{9}$$

(ii) $|1-M^2\gamma^2(1-\eta^2)| \ll (\gamma\epsilon)^{2/3}$

In this limit the ϕ^4 term in (6) may be ignored. The six roots are approximately

$$\phi \simeq \frac{2\gamma(1-\eta^2)^{1/2}}{\gamma^2(1-\eta^2)+\eta^2},$$

$$\phi \simeq \pm\epsilon,$$

and the three roots of

$$\phi^3+\tfrac{1}{2}M^3\delta^2\epsilon^2\gamma^2(1-\eta^2)^{-1} = 0. \tag{10}$$

One root of (10) corresponds to a growing mode which can be seen to be physical. The growth rate is given by

$$\omega_i \simeq 2^{-4/3}3^{1/2}q\delta^{2/3}(\gamma\epsilon)^{-1/3}(1-\eta^2)^{-5/6}\,s_1. \tag{11}$$

We also find

$$\begin{aligned} k_{1x} &\simeq q(2\gamma\delta\epsilon)^{-1/3}\,\delta(1-\eta^2)^{-5/6}\exp(4\pi i/3), \\ k_{2x} &\simeq q(2\gamma\delta\epsilon)^{1/3}(1-\eta^2)^{-1/6}\exp(2\pi i/3). \end{aligned}$$

(iii) $M^2\gamma^2(1-\eta^2)-1 \gg (\gamma\epsilon)^{2/3}$

In this limit, equation (7) is again appropriate. The only possibility for a growing mode is if the argument of the square root in (7b) becomes negative. For this to be possible, we require that

$$\delta < 1-\eta^2. \tag{12}$$

There is then a root with $Im(\phi) > 0$ when M^2 satisfies

$$1-\left\{1-\frac{\delta^2}{(1-\eta^2)^2}\right\}^{1/2} < \frac{\delta^2\gamma^2M^2}{2(1-\eta^2)} < 1+\left\{1-\frac{\delta^2}{(1-\eta^2)^2}\right\}^{1/2}. \tag{13}$$

The most quickly growing mode for M in this range is

$$\begin{aligned} \phi_{\max} &\simeq \frac{\delta\epsilon}{(1-\eta^2)}\left[\left\{\left(\frac{1-\eta^2}{\delta}\right)+1\right\}^{1/2}+i\left\{\left(\frac{1-\eta^2}{\delta}\right)-1\right\}^{1/2}\right]; \quad 1<\left(\frac{1-\eta^2}{\delta}\right)\leq 2 \\ &\simeq \frac{\epsilon}{2}(3^{1/2}+i); \quad \left(\frac{1-\eta^2}{\delta}\right)\geq 2. \end{aligned}$$

Inequality (12) is unlikely to be satisfied if one fluid is ultrarelativistic ($\eta = 3^{-1/2}$). If both fluids are subrelativistic ($\eta = 0$) and the hotter moving fluid has the lower specific heat ratio, there will be unstable effective Mach numbers in the range given by equation (13). In this case, however, δ is likely to exceed $\frac{1}{2}$ and so the maximum growth rate is given by

$$\omega_i \simeq q\delta s_1(\delta^{-1}-1)^{1/2}, \tag{14}$$

and a sufficient condition for stability is

$$M \geqslant M_c \simeq 2^{1/2}\gamma^{-1}\delta^{-1}\{1+(1-\delta^2)^{1/2}\}^{1/2}. \tag{15}$$

When $\delta \geqslant (1-\eta^2)$ the maximum unstable value of M is given by

$$M_c \simeq \frac{1}{\gamma(1-\eta^2)^{1/2}}\left[1+\frac{3}{2}\left\{\frac{\delta^2}{(1-\eta^2)}\right\}^{1/3}(\gamma\epsilon)^{2/3}+O(\gamma\epsilon)^{4/3}\right]. \tag{16}$$

For the special case $\delta = 1$, $\eta = 0$ but general ϵ and γ, we can adapt the analysis of Miles (1958) to obtain the exact result

$$M_c = \gamma^{-1}\{1+(\gamma\epsilon)^{2/3}\}^{3/2}. \tag{17}$$

We stress that equations (16) and (17) do not necessarily provide a good estimate of the maximum unstable effective Mach number when $\delta \leqslant (1-\eta^2)$ as has been assumed by some authors (e.g. Blake 1972).

To summarize this section, we have shown that in the limit $\gamma\epsilon \ll 1$, there are essentially three types of growing mode. For small values of $M(<\gamma^{-1}(1-\eta^2)^{-1/2})$ there is an unstable wave, with wave vector $\mathbf{q}$ parallel to the interface, that is evanescent in both fluids. A perturbation with length scale $\sim 2\pi/q$ grows in a time $\sim q^{-1}s_1^{-1}$. For a Mach number $M \sim \gamma^{-1}(1-\eta^2)^{-1/2}$, there is a mode which grows faster in a time $\sim (\gamma\epsilon)^{1/3}\, q^{-1}s_1^{-1}$. In this mode, the wave appears to propagate from fluid 2 to fluid 1, being refracted towards the normal. In the rest frame of fluid 2, however, it appears to propagate away from the interface. For large $|x|$, the wave decays exponentially. The third type of growing mode exists only if $\delta < 1-\eta^2$. This too appears to propagate from fluid 2 to fluid 1. It grows on a time scale $\sim q^{-1}s_1^{-1}$. For large $|x|$, $(\sim (q\epsilon)^{-1}$ in fluid 2) it decays exponentially. Therefore all values of the relative velocity, u, can display instability simply by a choice of k_z/q small enough to make M small enough to give one of these three types of mode.

4. INSTABILITIES IN THE LIMIT: $\gamma\epsilon \gg 1$; $\epsilon \ll 1$

A second limit of some physical interest arises when the sound speed of the moving fluid is somewhat more than that of the stationary fluid and when the relative velocity is extremely relativistic. In this limit, $\gamma\epsilon \gg 1$, and the terms which dominate the dispersion relation differ from those considered in Section 3. Careful examination in this limit shows that we may expect four roots in the vicinity of $\phi = M$ and two near $\phi = \pm\epsilon$. We consider two limits (i) $M \gg \epsilon$, (ii) $M \lesssim \epsilon$. Note that $M = \epsilon$ corresponds to modes having an effective Mach number of unity with respect to the stationary fluid.

(i) $M \gg \epsilon$

For the roots with $\phi \simeq M$, we write $\phi = M+m$ with $|m| \ll M$. We then obtain the quadratic

$$M^2\{\gamma^2(1-\eta^2)\, m^2 + M^2\eta^2 - 1\} = \delta^2\gamma^4 m^4\epsilon^2,$$

from which it can be seen that m^2 must be real and positive. ϕ is therefore real and none of these four roots grow. The roots near $\phi = \pm\epsilon$ are similarly shown to be real.

(ii) $M \lesssim \epsilon$

As $\phi \lesssim \epsilon \ll 1$ we see that the term $\eta^2\phi^2$ can be dropped from the bracket on the left-hand side of the complete dispersion relation, equation (1). With this approximation, the dispersion relation is invariant under the following transformation:

$$\phi \to \phi - M, \quad M \to -M,$$

$$\frac{\delta}{(1-\eta^2)} \leftrightarrow \frac{(1-\eta^2)}{\delta}, \quad \epsilon \leftrightarrow \gamma^{-1}(1-\eta^2)^{-1/2}.$$

The inequality $\gamma\epsilon \gg 1$ transforms into $\gamma\epsilon \ll 1$ (assuming $(1-\eta^2) = O(1)$). We can therefore write down expressions for the growth rates and corresponding **k**-vectors directly from the results of the previous section.

We find approximate expressions for the physical unstable modes as follows. For $M \ll \epsilon \ll 1$,

$$\begin{aligned}(\phi - M) &\simeq \delta^{-1/2}\gamma^{-1}\epsilon^{-1}M \exp(\pi i/2),\\ \omega_i &\simeq q\delta^{-1/2}\gamma^{-1}\epsilon^{-1}Ms_2,\\ k_{1x} &\simeq -k_{2x} \simeq q \exp(3\pi i/2).\end{aligned} \tag{19}$$

For $\epsilon^{1/3}\gamma^{-2/3} \ll \epsilon - M \ll \epsilon \ll 1$,

$$\begin{aligned}(\phi - M) &\simeq \frac{(1-\eta^2)^{1/2}\,\epsilon}{\delta\gamma(\epsilon^2 - M^2)^{1/2}} \exp(\pi i/2),\\ \omega_i &\simeq \frac{q(1-\eta^2)^{1/2}\, s_1}{\delta\gamma(\epsilon^2 - M^2)^{1/2}},\\ k_{1x} &\simeq q\epsilon^{-1}(\epsilon^2 - M^2)^{1/2}\exp(3\pi i/2),\\ k_{2x} &\simeq q\delta^{-1}(1-\eta^2)\,\epsilon(\epsilon^2 - M^2)^{-1/2}\exp(\pi i/2).\end{aligned} \tag{20}$$

For $|\epsilon - M| \ll \epsilon^{1/3}\gamma^{-2/3}$,

$$\begin{aligned}(\phi - M) &\simeq 2^{-1/3}(1-\eta^2)^{1/3}\,\epsilon(\delta\gamma\epsilon)^{-2/3}\exp(2\pi i/3),\\ \omega_i &\simeq 2^{-4/3}3^{1/2}q(\delta\gamma\epsilon)^{-2/3}(1-\eta^2)^{1/3}\, s_1,\\ k_{1x} &\simeq 2^{1/3}q(\delta\gamma\epsilon)^{-1/3}(1-\eta^2)^{-1/6}\exp(4\pi i/3),\\ k_{2x} &\simeq 2^{-1/3}q(\gamma\epsilon)^{1/3}(1-\eta^2)^{5/6}\,\delta^{-2/3}\exp(2\pi i/3).\end{aligned} \tag{21}$$

When $\delta > (1-\eta^2)$, there are modes directly analogous to those discussed in Section 3(iii). For values of M satisfying

$$1 - \left\{1 - \frac{(1-\eta^2)^2}{\delta^2}\right\}^{1/2} < \frac{(1-\eta^2)^2 M^2}{2\delta^2\epsilon^2} < 1 + \left\{1 - \frac{(1-\eta^2)^2}{\delta^2}\right\}^{1/2}, \tag{22}$$

the solutions with the maximum growth rates are

$$\begin{aligned}\phi &\simeq M + \frac{(1-\eta^2)^{1/2}}{\delta\gamma}\left[\left\{\frac{\delta}{(1-\eta^2)} + 1\right\}^{1/2} + i\left\{\frac{\delta}{(1-\eta^2)} - 1\right\}^{1/2}\right]; \quad 1 < \frac{\delta}{(1-\eta^2)} \leqslant 2\\ &\simeq M + \frac{1}{2(1-\eta^2)^{1/2}}(3^{1/2} + i); \quad \frac{\delta}{(1-\eta^2)} \geqslant 2.\end{aligned}$$

When $1 < \delta/(1-\eta^2) \lesssim 2$, the corresponding growth rate is

$$\omega_i \simeq q \frac{(1-\eta^2)^{1/2}}{\delta\gamma} \left\{ \frac{\delta}{(1-\eta^2)} - 1 \right\}^{1/2} s_2, \tag{23}$$

and a sufficient condition for stability is

$$M \geqslant M_c \simeq \frac{2^{1/2}\delta\epsilon}{(1-\eta^2)} \left[1 + \left\{ 1 - \frac{(1-\eta^2)^2}{\delta^2} \right\}^{1/2} \right]^{1/2}. \tag{24}$$

When $\delta < (1-\eta^2)$, the maximum unstable value of M is given by

$$M_c \simeq \epsilon \left[1 + \frac{3}{2} \left(\frac{1-\eta^2}{\delta^2} \right)^{1/3} (\gamma\epsilon)^{-2/3} + O(\gamma\epsilon)^{-4/3} \right]. \tag{25}$$

To summarize this section, we have shown for the case $\gamma\epsilon \gg 1$, $\epsilon \gg 1$, that there are unstable modes for all $M < \epsilon$ and again all values of the relative velocity are unstable. The most rapidly growing mode occurs when $|M-\epsilon| \ll \epsilon(\gamma\epsilon)^{-2/3}$ and a perturbation of length scale $2\pi/q$ grows in a time $\sim (\gamma\epsilon)^{2/3} q^{-1} s_1^{-1}$. Modes with effective Mach number $M > M_c$ given by equations (24), (25) are stable. We show schematically in Fig. 1 the location and character of the growing modes when $\epsilon \ll 1$ in the (γ, M) plane.

5. INSTABILITIES FOR IDENTICAL FLUIDS IN RELATIVE MOTION

We now consider the case when the two fluids are characterized by the same sound speed and specific heat ratio; that is $\delta = \epsilon = 1$. In this case the dispersion relation (1) can be factorized to give

$$[\gamma^2(\phi-M)^2 - \phi^2][\phi^2(1-\eta^2\phi^2) - \gamma^2(\phi-M)^2(\phi^2-1)] = 0. \tag{26}$$

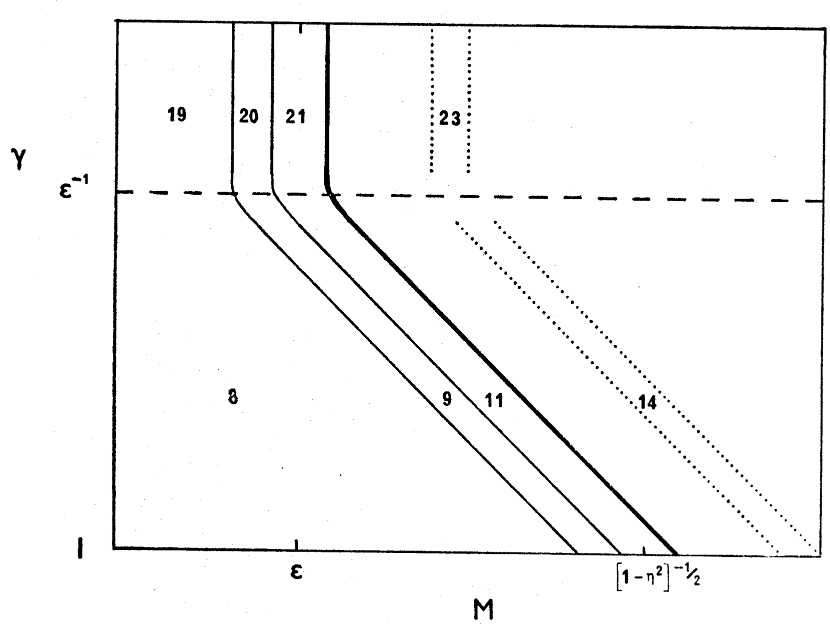

FIG. 1. *Schematic representation of unstable effective Mach number, M, for different values of the Lorentz factor, γ, plotted logarithmically. M, ϵ, η are defined in equation (2). Different regions of instability are labelled by the equation number in the text giving the corresponding growth rate. All regions to the left of the bold line are unstable. Regions enclosed by the dotted lines are only unstable for a restricted range of values of δ, η (equations (13), (22)).*

The six roots are

$$\phi = \frac{\gamma M}{\gamma \pm 1},$$

which are both real, and the four roots of

$$\gamma^2(\phi - M)^2 = \frac{\phi^2(1-\eta^2\phi^2)}{(\phi^2-1)}. \tag{27}$$

(a) $\gamma \gg 1$

Equation (27) can be solved approximately in the limit $\gamma \gg 1$.

(i) $M \ll 1$. In this case it is clear that (27) has two real roots for $\phi \simeq \pm 1$ and two complex roots for $\phi \simeq M$. One of these represents a growing mode with

$$\begin{aligned} \omega_i &\simeq qM\gamma^{-1}\eta\{(1-\eta^2M^2)/(1-M^2)\}^{1/2}, \\ k_{1x} &\simeq q(1-M^2)^{1/2} \exp(3\pi i/2), \\ k_{2x} &\simeq q(1-\eta^2M^2)(1-M^2)^{-1/2} \exp(\pi i/2). \end{aligned} \tag{28}$$

(ii) $M \gg 1$. For $M > M_c \sim 1$, the two roots near $\phi \simeq \pm 1$ are again real. Writing $\phi = 1+f$, $M = 1+m$ with f, $m \ll 1$ (27) becomes

$$f^3 - 2f^2m + fm^2 - \frac{(1-\eta)^2}{2\gamma^2} = 0.$$

This has complex roots if and only if

$$m < \frac{3}{2}\left(\frac{1-\eta^2}{\gamma^2}\right)^{1/3}.$$

Therefore the critical Mach number, M_c, is

$$M_c \simeq 1 + \frac{3}{2}\left(\frac{1-\eta^2}{\gamma^2}\right)^{1/3}. \tag{29}$$

For $M > M_c$, the modes are stable. Unlike the cases when $\epsilon \ll 1$, there are no further unstable modes with $M > M_c$.

For $M < M_c$, the maximum growth rate occurs when $m \ll f$. Thus for $|M-1| \lesssim \gamma^{-2/3}$,

$$\begin{aligned} (\phi - M) &\simeq \left(\frac{1-\eta^2}{2\gamma^2}\right)^{1/3} \exp(2\pi i/3), \\ \omega_i &\simeq 2^{-4/3}3^{1/2}q\eta(1-\eta^2)^{1/3}\gamma^{-2/3}, \\ k_{1x} &\simeq 2^{1/3}q(1-\eta^2)^{1/6}\gamma^{-1/3} \exp(4\pi i/3), \\ k_{2x} &\simeq 2^{-1/3}q(1-\eta^2)^{5/6}\gamma^{1/3} \exp(2\pi i/3). \end{aligned} \tag{30}$$

(b) $\gamma \sim 1$

For mildly-relativistic relative motion, the quartic equation (27) can be solved analytically but the general expression so obtained is very lengthy. Miles (1958) has shown that the case $\gamma = 1$, $\eta = 0$ can be solved fairly simply and he finds the maximum unstable effective Mach number, M_c, to be $8^{1/2}$. The existence of a maximum unstable effective Mach number can be investigated in general without too much difficulty. If such a Mach number exists, then the corresponding value of ϕ, ϕ_c, must be a real double root of the dispersion relation (27) and therefore also a root of its derivative with respect to ϕ. Simultaneous solution of the two equations leads to a parametric relation for $M_c(\gamma^2)$ in terms of ϕ_c:

$$M_c = \frac{(1-\eta^2)\,\phi_c^3}{\eta^2(\phi_c^2-1)^2+(1-\eta^2)},$$

and

$$\gamma^2 = \frac{[\eta^2(\phi_c^2-1)^2+(1-\eta^2)]^2}{(\phi_c^2-1)^3(1-\eta^2\phi_c^2)},$$

where ϕ_c must satisfy

$$1<\phi_c^2<M_c^2\leqslant\eta^{-2}(1-\gamma^{-2}).$$

These relationships are displayed in Fig. 2 for the case of two ultrarelativistic fluids, $\eta = 3^{-1/2}$. We note that there is in general a critical Lorentz factor γ_c such that, for all $\gamma<\gamma_c$, modes with all possible values of M are unstable. When $\eta = 3^{-1/2}$, we find $\gamma_c = 3$ and the corresponding values $M_c(\gamma_c) = 3^{-1/2}8^{1/2}$, $\phi_c = 2^{-1/2}3^{1/2}$. For large γ, equation (29) is recovered. The (unphysical) cusp at $\gamma_c = 1.69$, $M_c(\gamma_c) = 1.90$ ensures that for $\gamma\gg 1$, $M_c \sim 1$ is the only relevant critical Mach number.

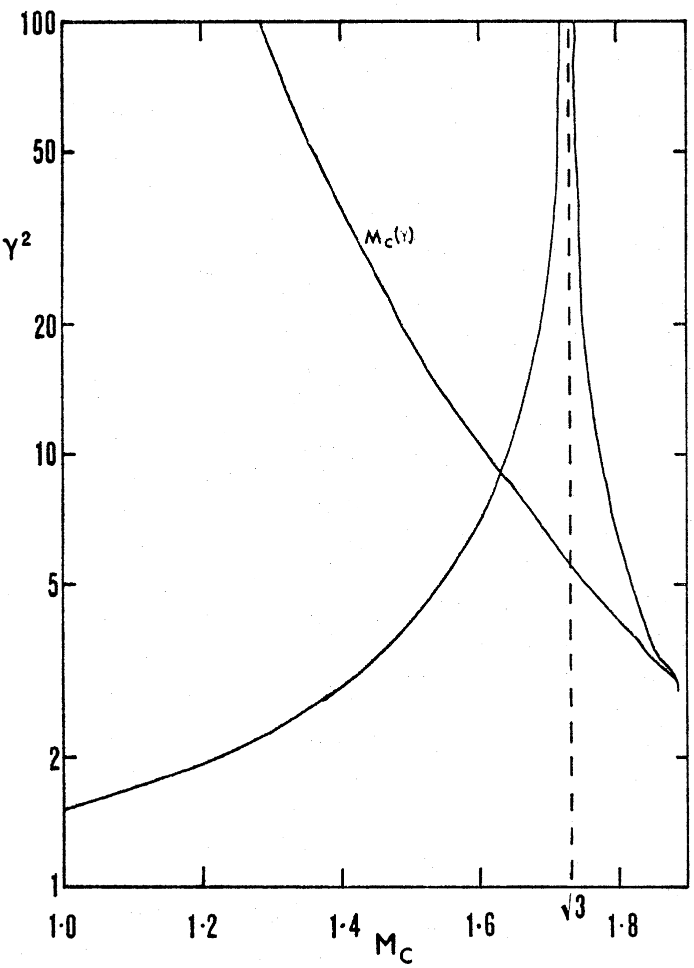

FIG. 2. *Critical effective Mach number $M_c(\gamma)$ when both sound speeds equal $3^{-1/2}c$. The curve $M^2\eta^2 = (1-\gamma^{-2})$ is also drawn and physical modes lie to the left of this line. Physical modes with $M>M_c(\gamma)$ are stable.*

6. APPLICATION TO EXTRAGALACTIC RADIO SOURCE MODELS

In the previous three sections we have discussed semi-quantitatively the nature of the growing modes that exist at a planar tangential discontinuity. We now consider the difficult problem of the fluid dynamical stability of beam models of extragalactic radio sources in the light of these results.

In Rees (1971), Longair, Ryle & Scheuer (1974), Scheuer (1974) and Blandford & Rees (1974), specific models postulating the existence of a continuous flow of energy from an active nucleus to an extended extragalactic radio source are outlined. In Rees (1971), it was assumed that the energy took the form of a spectrum of low frequency electromagnetic waves, possibly generated by a cluster of pulsars. However, it is now believed that such waves are unlikely to be generated in the absence of plasma and static field and in later papers a more general viewpoint is adopted—that the electromagnetic fields and particles are sufficiently well coupled to form an isotropic fluid.

In Blandford & Rees (1974), a particular mechanism for channelling the energy in antiparallel directions, the formation of a pair of de Laval nozzles, is proposed and some discussion of the fluid dynamics of relativistic flow is presented. However, neither the nozzle mechanism nor relativistic velocities are essential. Collimated supersonic flow could be produced directly in the nucleus (e.g. using electromagnetic effects near the event horizon of a rotating black hole) and, as extended radio components do not appear to be expanding relativistically, we require only that the fluid velocity exceed this expansion velocity (typically $\lesssim 0{\cdot}1$ c). A viable beam will be supersonic for most of its length, and its sound speed will probably be much greater than that of the surrounding intergalactic medium with which it must be in pressure equilibrium. Typically $\epsilon \sim 10^{-1}$–10^{-3}. The calculations of Sections 3 and 4 are relevant to this case.

In a source like Cygnus A (Hargrave & Ryle 1974) most of the radio power is radiated by extended regions of low surface brightness rather than the compact hot spots located at the head of the source. This fits in naturally with a beam model as there is inevitably a surplus of energy if sufficient momentum be supplied to push against the ram pressure of the intergalactic medium. Scheuer (1974) has discussed in some detail the possibility that this 'waste' energy is deposited in a cocoon surrounding the channel and separating it from the shocked intergalactic medium. Scheuer makes the assumption that both the beam and the surrounding material are ultrarelativistic. The discussion of Section 5 is relevant to this case.

It has long been recognized that a potentially serious defect of these models is the possible existence of disruptive instabilities at the channel walls which might terminate the beam long before the radio source be reached. It is therefore very important to try to elucidate the conditions (if any) under which a beam of this type might be stable, at least in its time-averaged properties. Unfortunately, any reliable investigation will have to include consideration of thermal and radiative dissipation, magnetic fields, sound and turbulence generation and the development of a boundary layer. There are no quick answers. The calculations presented above apply to the idealized problem of the linear development of the Kelvin–Helmholtz instability at an infinite planar surface and must be applied with extreme caution.

The boundary conditions we have imposed are that the disturbance must either be localized close to the vortex sheet, or at least propagate away from it. In addition, $\mathbf{q}$, the $\mathbf{k}$-vector resolved in the plane of the vortex sheet has been taken to be real.

A realistic beam could be approximated by a finite circular cylinder, radius r. In this case the azimuthal component of $\mathbf{k}$ (corresponding to k_y in our presentation) would have to be an integral multiple of r^{-1}. The normal modes are then naturally expressible in terms of Bessel functions (*cf.* Gill 1965). This raises no difficulties of principle although the resulting dispersion relation is inevitably fairly complex. The choice of boundary conditions at the end of the beam is somewhat more problematical. If the local pressure scale height (along the beam) is L, the dispersion relation, equation (2.15), is invalid for modes with $k_z \lesssim L^{-1}$. One might naïvely expect such modes to be excluded, but this depends more strictly on the manner in which solutions are joined between adjacent pressure scale heights.

When the relative flow is subsonic or trans-sonic, and when $\epsilon \ll 1$ (for instance in the nuclear region of the twin-nozzle model), the bulk velocity will be no more than mildly relativistic. From Section 3 we see that unstable growing modes exist for $M \lesssim 1$, that is effectively for all values of $\cos\theta = k_z/q$. In particular, unless non-linear effects lead to stabilization, modes with $k_y \sim r^{-1}$, which would seem capable of completely destroying the beam, can grow on time scales $\sim \epsilon^{1/3}(r/s_1)$. We note, however, that this instability propagates into the beam with a transverse group velocity of only $\epsilon^{1/3}s_2$ which is subsonic. If non-linearities and mixing could keep the amplitude of the disturbance limited at the effective interface (boundary layer) between the two fluids, a small transverse velocity $\epsilon^{1/3}s_2$ within the channel towards the boundary might be able to prevent the instability from significantly affecting most of the beam. In other words, if $\epsilon^{1/3}s_2/u \ll r/L$, there is a possibility that only a small fraction of the beam is stripped away by instabilities as it flows through each scale height.

When the beam is highly supersonic ($u \gg s_2$, and possibly relativistic) with $\epsilon \ll 1$, but such that $\gamma\epsilon \lesssim 1$, there are again always growing modes (Section 3), but in this case they may not be so serious. The growing modes are confined to values of $M \lesssim \gamma^{-1}$, that is to values of θ very close to $\pi/2$. To be more precise, the growing modes require

$$\frac{k_y}{k_z} \sim \left|\frac{\pi}{2} - \theta\right| \sim \frac{s_2}{u\gamma(1-\eta^2)^{1/2}} \ll 1.$$

If, however, k_z is effectively limited to be $\lesssim L^{-1}$ (and this depends on the precise boundary conditions), and if $s_2/(u\gamma) \lesssim r/L$, the only growing modes have $k_y \gg r^{-1}$ and therefore need not lead to direct destruction of the channel. This condition seems plausible since it simply requires that the time taken for a sound wave to cross the beam be less than the time taken by the beam to flow one pressure scale height. The characteristics of these short-wavelength unstable modes could perhaps be used to derive estimates of the amount of mixing and beam stripping that would be expected in a highly supersonic beam.

If the beam moves so fast that $\gamma\epsilon \gtrsim 1$, $\epsilon \ll 1$, then (Section 4) the unstable modes require $M < \epsilon$, that is $|\pi/2 - \theta| \lesssim s_1/u \ll 1$. In particular, there are unstable modes with $|\pi/2 - \theta| > s_2 u^{-1}\gamma^{-1}(1-\eta^2)^{-1/2}$ for which the sound crossing-time of the channel can be less than the time the beam takes to traverse a pressure scale height. In this case, the instabilities are more likely to be capable of disrupting the beam entirely.

Finally, at the end of the beam where both fluids may have similar sound speeds, the fastest growing modes (Section 5) have $M \ll 1$. If $s_1 \sim s_2 \sim 3^{-1/3}c$, $|\pi/2 - \theta|$ need not be small and instabilities on all relevant length scales can grow freely at speeds $\sim c$. This is not necessarily an undesirable result, since the observations indicate that continuous particle acceleration is taking place within this region. Energy discharged in the beam can be liberated just as easily by surface instabilities as by a transverse shock.

In summary, the results of these calculations, interpreted in a straightforward though possibly naïve manner, indicate that highly supersonic beams with $\gamma\epsilon \lesssim 1$ and $\epsilon \ll 1$ could be reasonably stable. Subsonic and trans-sonic flows ($\gamma\epsilon \ll 1$), such as might be found in the vicinity of a nozzle, are unstable but there is a possibility that the instability might just strip the outer layers of the beam, leaving the main flow comparatively unaffected. We emphasize, however, that a much more detailed study of the physical processes involved is necessary before such conclusions can be accepted with any confidence.

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