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AMPLIFICATION OF RADIATION BY RELATIVISTIC PARTICLES IN A STRONG MAGNETIC FIELD

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SUMMARY

The possibility of maser action by relativistic charged particles confined to one-dimensional motion in a strong magnetic field is considered. It is shown that if the field lines are curved to form circular arcs, genuine maser action is impossible. Under more complex electrodynamical conditions when, for instance, points of inflection and torsion may be present in the field geometry or particle creation, electrostatic acceleration or plasma effects may be occurring, maser action is not impossible. Simple estimates suggest that this type of mechanism cannot be excluded from being responsible for the coherent radio emission from pulsars.

1. INTRODUCTION

There is now abundant evidence that pulsars can be identified with rotating magnetized neutron stars (see, e.g. Ruderman 1972 for a review), but while several relevant physical processes have been analysed, there is as yet no generally accepted mechanism for producing the coherent radio emission. In this paper, a further process—the possible amplification of radio frequency radiation by electrons constrained to one-dimensional motion in a strong magnetic field—is discussed, and some numerical estimates are given of the conditions under which such a mechanism could operate.

In Section 2 we introduce some relevant physical notions by calculating the angular absorption coefficient for 'curvature' radiation. The calculations are extended in Section 3 to apply to a more complex but highly specialized geometry and here we show that maser action is possible. These results are related to the problem of pulsar emission in Section 4. A similar process has been investigated by Cocke (1973).

2. THE ANGULAR ABSORPTION COEFFICIENT OF CURVATURE RADIATION

In a static magnetic field of strength B, an electron will decay into its ground state of gyrational motion in a time $\tau_D \sim \gamma c r_e^{-1} \omega_G^{-2}$, where ω_G is the non-relativistic electron gyro frequency, r_e is the classical electron radius and $\gamma = (1-\beta^2)^{-1/2}$ is the Lorentz factor corresponding to the electron velocity parallel to the field. The transverse motion of the electron in the guiding-centre frame will be non-relativistic (in the ground state) provided that $\omega_G \lesssim mc^2\hbar^{-1}$, and if in addition $\gamma \gg 1$, only the motion parallel to the field will be significant both dynamically and kinematically. In the absence of collisional frequencies $\gtrsim \tau_D^{-1}$ and a significant flux of radiation with frequency $\gtrsim\gamma^{-1}\omega_G$, an electron will remain in its lowest Landau orbital.

If the magnetic field exhibits weak inhomogeneity on length scales $\gg(\hbar/m\omega_G)^{1/2}$ or there is a gravitational or electric field present, the electron will drift perpendicular to the magnetic field direction (e.g. if the field lines have radius of curvature ρ, this drift velocity will be $\gamma c^2\omega_G^{-1}\rho^{-1}$ normal to the osculating plane (Ruderman 1972)). In the following calculations such drifts only introduce small corrections to the unperturbed trajectories and do not seriously alter the physics.

Thus, when the above conditions are satisfied, a relativistic electron behaves as if it were constrained to a wire and if its trajectory deviates from uniform rectilinear motion it can emit and absorb photons. If the magnetic field lines form an arc of a circle of radius ρ, the single particle emission (curvature radiation) is indistinguishable from synchrotron radiation. We now give a classical calculation of the corresponding angular absorption coefficient.

Consider a plane, monochromatic, vacuum, electromagnetic wave of angular frequency ω and linear polarization (with unit vector $\mathbf{e}$) interacting with a single ultra-relativistic electron. The wave is taken to propagate along a direction specified by a unit vector $\mathbf{n}$ at a small angle θ to the plane of the unperturbed trajectory. As is also the case in the calculation of the emission coefficient, the main interaction with the radiation field occurs when the electron moves almost parallel to $\mathbf{n}$. We take the point on the orbit where the curvature vector is normal to $\mathbf{n}$ as the space and time origin for the perturbation calculation. The electric field of the wave can thus be written as $E\mathbf{e}\cos(\chi+\phi)$, where $\chi = \omega(t-\mathbf{n}.\mathbf{r}/c)$.

Next we calculate expressions for the mean and mean square energy transfers in a single revolution on the assumption that ϕ is uniformly distributed in [0, 2π]. The perturbation to γ that is linear in the field strength is

$$\gamma_1 = \frac{eE}{mc}\int^t dt'(\mathbf{e}.\boldsymbol{\beta}')\cos(\chi'+\phi). \tag{1}$$

In (1), the variables are considered functions of time (to which the primes refer) and we ignore the contributions from the beginning and end of the interaction which can be shown not to be significant provided that, $(\omega\rho/c)$, γ, $\theta^{-1}\gg 1$. The first-order perturbation to the electron velocity is

$$\boldsymbol{\beta}_1 = \frac{eE}{mc\gamma^3}\boldsymbol{\beta}\int^t dt'(\mathbf{e}.\boldsymbol{\beta}')\cos(\chi'+\phi), \tag{2}$$

and the perturbation in position satisfies

$$\mathbf{r}_1 = \frac{eE}{m\gamma^3}\int^t\int^{t'} dt'\,dt''\boldsymbol{\beta}'(\mathbf{e}.\boldsymbol{\beta}'')\cos(\chi''+\phi).$$

Now $\mathbf{n}.\boldsymbol{\beta} = 1-\omega^{-1}(\partial\chi/\partial t)$, and so

$$\mathbf{n}.\mathbf{r}_1 = \frac{eE}{m\omega\gamma}\int^t dt''(\mathbf{e}.\boldsymbol{\beta}'')\{\omega(t-t'')-(\chi-\chi'')\}\cos(\chi''+\phi) \tag{3}$$

after changing the order of integration. To sufficient accuracy, we can approximate χ by

$$\frac{\omega}{2}\left\{(\gamma^{-2}+\theta^2)\,t+\frac{c^2}{3\rho^2}\,t^3\right\},$$

(Jackson 1962). Now when $|t''|$ is sufficiently large, $|\chi''| \gg 1$ and the integrand in (3) oscillates very rapidly with time. As is also the case with the calculation of the emission coefficient, the dominant contribution to the integrand comes from times, t'', such that $|\chi''| \lesssim 1$—the region of stationary phase. However, in order that the integral be significant for finite frequencies, it is also necessary that the second term in the expression for χ ($\propto t^3$) should be at least comparable with the first term ($\propto t$). This imposes the usual condition, $\omega\rho/c \lesssim (\gamma^{-2}+\theta^2)^{-3/2}$. Thus the integrand contributes effectively for times $|t''| \sim (\omega\rho/c)^{2/3}\,\omega^{-1}$ and so in (3) the term containing $(\chi-\chi'')$ can be dropped in comparison with the term containing $\omega(t-t'')$. (Evaluating these smaller terms in the final integrals shows that they do not contribute to $\langle\Delta\gamma\rangle$ as long as $\omega\rho/c \gg (\gamma^{-2}+\theta^2)^{-3/4}$).

For the second order change in γ, we have

$$\Delta\gamma_2 = \frac{eE}{mc}\int dt\left\{(\mathbf{e}\,.\,\boldsymbol{\beta}_1)\cos(\chi+\phi)+\frac{\omega}{c}(\mathbf{e}\,.\,\boldsymbol{\beta})(\mathbf{n}\,.\,\mathbf{r}_1)\sin(\chi+\phi)\right\}$$

$$= \frac{e^2E^2}{m^2c^2\gamma^3}\int\int^t dt\,dt''(\mathbf{e}\,.\,\boldsymbol{\beta})(\mathbf{e}\,.\,\boldsymbol{\beta}'')\{\cos(\chi+\phi)\cos(\chi''+\phi) + \omega(t-t'')\sin(\chi+\phi)\cos(\chi''+\phi)\},$$

substituting (2) and (3). Again the term involving $\omega(t-t'')$ dominates. We can therefore average over phase ϕ, to obtain

$$\langle\Delta\gamma\rangle = \frac{e^2E^2}{2m^2c^2\gamma^3}\int\int^t dt\,dt''(\mathbf{e}\,.\,\boldsymbol{\beta})(\mathbf{e}\,.\,\boldsymbol{\beta}'')\,\omega(t-t'')\sin(\chi-\chi'')$$

$$= \frac{e^2E^2}{4m^2c^2\gamma^3}\left[\int\int^t dt\,dt''(\mathbf{e}\,.\,\boldsymbol{\beta})(\mathbf{e}\,.\,\boldsymbol{\beta}'')\,\omega(t-t'')\sin(\chi-\chi'') + \int\int_t dt\,dt''(\mathbf{e}\,.\,\boldsymbol{\beta})(\mathbf{e}\,.\,\boldsymbol{\beta}'')\,\omega(t-t'')\sin(\chi-\chi'')\right]$$

$$= \frac{-ie^2E^2}{4m^2c^2\gamma^3}\int\int dt\,dt''(\mathbf{e}\,.\,\boldsymbol{\beta})(\mathbf{e}\,.\,\boldsymbol{\beta}'')\,\omega(t-t'')\,e^{i\chi}\,e^{-i\chi''}.$$

Similarly from (1), we obtain

$$\langle\Delta\gamma^2\rangle = \frac{e^2E^2}{2m^2c^2}\int\int dt\,dt''(\mathbf{e}\,.\,\boldsymbol{\beta})(\mathbf{e}\,.\,\boldsymbol{\beta}'')\,e^{i\chi}\,e^{-i\chi''}.$$

Now to lowest order, $(\mathbf{e}\,.\,\boldsymbol{\beta})$ and ρ are independent of γ and so

$$\frac{\omega t}{\gamma^3} = -\frac{\partial\chi}{\partial\gamma}.$$

We can now substitute the standard form for the energy radiated per unit frequency per unit solid angle in linear polarization, i, $I_{\omega\Omega}{}^i$ (e.g. Jackson 1962; eq. 14.67) to obtain

$$\langle\Delta\gamma\rangle = \frac{\pi^2E^2}{m^2c\omega^2}\frac{\partial}{\partial\gamma}I_{\omega\Omega}{}^i.$$
$$\langle\Delta\gamma^2\rangle = \frac{2\pi^2E^2}{m^2c\omega^2}I_{\omega\Omega}{}^i. \qquad (4)$$

These relations can just as well be derived quantum mechanically, most simply by adapting the method, of for example McCray (1969), for synchrotron radiation in which the principle of detailed balance is applied to three electron states of energy $\gamma mc^2 \pm \hbar\omega$, γmc^2. Three important modifications must however be incorporated into the calculation.

(i) It is only necessary to consider transitions involving radiation in a fixed polarization state propagating within an infinitesimal element of solid angle about a particular direction, rather than integrate over all angles.

(ii) As the periodic time is (to lowest order) independent of γ, probabilities rather than probabilities per unit time can be used.

(iii) The density of available states in energy space is constant (for $\gamma \gg 1$) rather than proportional to γ^2, as the motion is one- rather than three-dimensional.

Equation (4) can then be straightforwardly recovered.

Before substituting the analytic form for $I_{\omega\Omega}{}^i$ we first dispose of a point of some subtlety concerning the treatment of polarization. Relativistic electrons moving along a single fixed circular trajectory radiate one specific elliptical polarization for given θ, ω, γ. From the point of view of both classical and quantum mechanical formalisms, an electron can only absorb radiation in the same polarization state as it spontaneously emits. (Strictly, in quantum mechanical terms, this polarization is emitted in a transition between two electron eigenstates, and absorption must be regarded as the difference between stimulated absorption and stimulated emission.) Thus any incident radiation will be resolved into the spontaneously emitted polarization and the orthogonal polarization of which only the former will be absorbed.

The generalization of (4) to the case of incident elliptical polarization is straightforward. If we introduce perpendicular unit polarization vectors, $\mathbf{e}_\parallel$, $\mathbf{e}_\perp$, with $\mathbf{e}_\parallel$ lying in the orbital plane, the electric vector of an elliptically polarized plane wave can be written

$$\mathbf{E} = E\{\mu \mathbf{e}_\parallel \cos(\chi+\phi) + \lambda \mathbf{e}_\perp \sin(\chi+\phi)\}; \quad \mu^2+\lambda^2 = 1$$

and (4) still holds.

Thus, adapting Jackson (1962, eq. 14.83), we have for radiation incident in the spontaneously emitted polarization,

$$\begin{aligned} \langle \Delta\gamma^2 \rangle &= 8\pi^2 \left(\frac{eE}{mc\omega}\right)^2 \left(\frac{\omega\rho}{2c}\right)^{2/3} \left\{ A_i'^2(z) + \theta^2 \left(\frac{\omega\rho}{2c}\right)^{2/3} A_i^2(z) \right\}, \\ \langle \Delta\gamma \rangle &= -\gamma^{-3} \left(\frac{\omega\rho}{2c}\right)^{2/3} \frac{d}{dz} \langle \Delta\gamma^2 \rangle, \end{aligned} \tag{5}$$

where $z = (\omega\rho/2c)^{2/3}(\gamma^{-2}+\theta^2)$ and $A_i(z)$ is the Airy function. As both $A_i^2(z)$ and $A_i'^2(z)$ are monotonically decreasing functions of positive z, we see that $\langle \Delta\gamma \rangle > 0$.*

We have therefore demonstrated that electrons streaming along circular magnetic field lines, provided that they are randomly distributed, are incapable of amplifying electromagnetic radiation in a given state of polarization.

* If the radiation has a different polarization (to that spontaneously emitted), $\langle \Delta\gamma \rangle$ can be negative. Nevertheless this is not genuine maser action because, although photons are being added to the radiation field, they are not in the same sense of polarization as the incident radiation. The effect arises solely because the emitted polarization (at a given, ω, θ) is itself energy dependent.

3. NEGATIVE ABSORPTION AT A POINT OF INFLECTION

The conclusion of the previous section applies only to electrons on circular arcs. It is interesting to ask whether or not this will always occur for any one-dimensional trajectory and if not what further conditions must be satisfied if maser action is to follow. The former question is now answered in the negative simply by producing a counter example (there are others, less easily analysed) but as yet we have been unable to extend the investigation to include the latter question.

The trajectory we use consists simply of two co-planar equal circular arcs matched tangentially at the origin where there is a point of inflection. Because there is a discontinuity in the curvature, this probably does not represent a good model of a magnetic field line, but it has the advantage that it can be analysed in terms of tabulated functions.

In fact we must make a further specialization and only consider observer directions lying in a plane normal to the orbital plane and containing the tangent at the point of inflection. Still defining θ as the angle the observer direction makes with the orbital plane, we see that χ has the same form as in 2 and that (4) still applies. The emitted radiation is now linearly polarized and the perpendicular polarized component of $I_{\omega\Omega}$ is unchanged. The parallel polarized component is however given by,

$$I_{\omega\Omega}{}^{\parallel} = 2\frac{e^2}{c}\left(\frac{\omega\rho}{2c}\right)^{2/3} G_i'^2(z),$$

where

$$G_i(z) = \pi^{-1}\int_0^\infty \sin\left(uz+\tfrac{1}{3}u^3\right) du.$$

$G_i'(z)$ has been tabulated by Scorer (1950). Hence,

$$\langle\Delta\gamma\rangle = -8\pi^2\left(\frac{eE}{mc\omega}\right)^2 \gamma^{-3}\left(\frac{\omega\rho}{2c}\right)^{4/3}\frac{d}{dz}\left\{G_i'(z)^2+\theta^2\left(\frac{\omega\rho}{2c}\right)^{2/3} A_i^2(z)\right\}.$$

Now, $G_i'(z)\,G_i''(z)>0$ when $0{\cdot}61<z<1{\cdot}64$, and so $\gamma\sim(\omega\rho/2c)^{1/3}\ll\theta^{-1}$ are sufficient conditions for negative absorption with this geometry. If there are several electrons, we see that in order for them to amplify a particular frequency there must be at most a few of energy lower than $\sim(\omega\rho/2c)^{1/3}\,mc^2$ and so a definite population inversion must be achieved if maser action is to be possible.

4. APPLICATION TO PULSARS

For convenience we introduce a ' standard ' pulsar model in which the neutron star radius is 10^6 cm and the magnetic field is predominantly dipolar with surface strength 10^{12} Gauss. Electrons (and possibly positrons) are accelerated over a fraction $\alpha/4\pi$ of the surface to energies $\gamma m_e c^2$ and stream out along the open field lines, so that their mean density at radius $10^6\,R_6$ cm is given by

$$n_e \sim 3\times10^7\,\dot{N}_{30}\alpha^{-1}R_6{}^{-3}\ \mathrm{cm}^{-3}$$

where $10^{30}\,\dot{N}_{30}\ \mathrm{s}^{-1}$ is the total particle discharge from the pulsar.

A single line of force, as well as displaying curvature, may also display points of inflection or torsion, resulting from either higher, unaligned multipole moments or rotational distortion. (The latter will probably not be appreciable until the speed of light cylinder is approached.) Under these circumstances, it is possible that particles streaming along the field lines are capable of amplifying radiation propagating within a finite solid angle and frequency range, in a similar manner to that analysed in Section 3. In addition the injection of particles at the surface or in pairs as the result of a γ-ray interacting with the magnetic field (*cf.* Ruderman & Sutherland 1975) will, for certain observer directions, result in the particle interacting with the radiation field for incomplete segments of circular arcs. A further possibility is that there may be a component of electric field parallel to the magnetic field and so the energy of the electron might change significantly within the time that it interacts with a wave. Preliminary investigations indicate that these electrodynamic configurations can also have the property (for specific observer directions and frequencies) of amplifying the radiation field.

It is not however sufficient that particles of a given energy on one particular field line be capable of amplification. There must also be an integration over all particle energies and all field lines along a line of sight, and only if the *total* optical depth is negative will the number of photons in the radiation field be increased. This obviously requires a detailed knowledge of the field geometry—it is clearly inadequate to integrate the total absorption from a single particle over solid angle. If the particles are accelerated electrostatically it might be expected that they should be approximately mono-energetic and so population inversion can arise quite naturally within a pulsar magnetosphere.

For this type of emission mechanism to be possible, it must further be shown that the conditions necessary to achieve optical depths < -1 are compatible with observational constraints. Guided by the results of Section 3 we estimate the optical depth under optimal conditions for negative absorption as

$$\begin{aligned} -\tau &\sim n_e r_e \gamma \lambda^2 \\ &\sim 0.1\, \dot{N}_{30} \alpha^{-1} R_6^{-3} \gamma \lambda_2^2, \\ &\sim 10^5\, L_{e30} \alpha^{-1} R_6^{-3} \lambda_2^2 \end{aligned}$$

where λ_2 is the wavelength in metres and $10^{30}\, L_{e30}$ erg s^{-1} is the energy carried away by relativistic electrons. The lowest upper limit on L_{e30} is ~ 1 for the slowest pulsars and a characteristic wavelength is ~ 1 m. Thus we see that there need be no difficulty in achieving optical depths $\lesssim -1$ even if the emission arises at a radius $R_6 \sim 10^2$, which is the maximum at which the motion can be considered as one-dimensional.

There is an important distinction to be drawn between this type of maser action and that more usually encountered in physics and that is that the length scale over which an individual particle interacts with the radiation field is comparable with the length over which the interacting particles are located. Therefore, calculations similar to those presented in Sections 2 and 3 can only strictly be valid as long as $|\tau| \ll 1$. As pulsars require $|\tau| \gg 1$, the maser would be operating in the non-linear regime, and this might, for instance, seriously alter the frequency and polarization of the most rapidly growing mode as well as the conditions for amplification. A further indication of this non-linearity is that a perturbation expansion such as that used in Section 2 is only valid as long as $\Delta\gamma \ll \gamma$, or $(eE/mc\omega) \ll 1$. Observations indicate that this inequality is not always satisfied. Nevertheless, as long as L_e exceeds the radio power, and for all pulsars for which slowing down times are at present available, it can do so and still be much less than the total power loss, it is in principle possible for the electrons to generate the pulsed luminosity.

The calculations so far have all referred to individual particles moving *in vacuo*. The simplest estimate of the influence of collective effects (using (2)) gives an effective plasma frequency

$$\omega'_{\mathrm{p}} \sim \omega_{\mathrm{p}} \langle \gamma^{-3} \rangle^{1/2}$$

where ω_{p} is the formal plasma frequency corresponding to an electron density n_{e}. Razin suppression should occur for frequencies $\omega < \omega_{\mathrm{p}}^{6/7}(c/\rho)^{1/7}$ and so we can set an approximate upper limit on n_{e} from $n_{\mathrm{e}} \lesssim \gamma\lambda^{-2} r_{\mathrm{e}}^{-1}$, a condition which does not preclude $|\tau| \gg 1$. This limit might also be applicable to models in which the radio emission arises spontaneously from bunches of particles. As with synchrotron emission, Razin suppression provides a further method for achieving negative absorption.

In summary, we conclude that coherent pulsar emission could arise from maser-like action by relativistic electrons (or positrons) confined to one-dimensional motion in strong magnetic fields. Necessary conditions are that there be a population inversion in particle energies (possibly resulting from electrostatic acceleration), and an electrodynamical situation at least more complex than free particles streaming along a strong magnetic field curved in a circular arc. Even for those situations that have been analysed, the calculations must be extended into the non-linear regime before any reliable relation of the spectral and polarization properties of the emitted radiation to the magnetospheric structure can be made.

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