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Control and Stabilization of Systems with Homoclinic Orbits

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Abstract:

In this paper we consider the control of two physical systems, the near wall region of a turbulent boundary layer and the rigid body, using techniques from the theory of nonlinear dynamical systems. Both these systems have saddle points linked by heteroclinic orbits. In the fluid system we show how the structure of the phase space can be used to keep the system near an (unstable) saddle. For the rigid body system we discuss passage along the orbit as a possible control manouver, and show how the Energy-Casimir method can be used to analyze stabilization of the system about the saddles.

Introduction:

Our goal in this paper is to analyze control and stability of two physical systems of great practical importance using some recently developed techniques from the theory of nonlinear dynamical systems. The first system is a model of the near wall region of a turbulent boundary layer. The second system is the rigid body. Both these systems have saddle points connected by heteroclinic orbits and here we wish to consider and exploit this structure in carrying out the control analysis for these problems. A somewhat more detailed analysis of the boundary layer system is given by the authors in Bloch and Mardsen [1989]. This analysis is based on the paper of Aubry, Holmes, Lumley and Stone [1988], which showed how the near wall region of of a turbulent boundary layer could be analyzed as a finite-dimensional dynamical system. The reduction to finite dimensions relies on the proper orthogonal decomposition of Lumley [1967, 1970, 1981]. It was shown in this paper that a possible description of the so called bursting events of turbulence was as passage along a heteroclinic orbit. In Bloch and Marsden [1989] we suggested in this light a possible mechanism for controlling the frequences of bursting and thus the degree of turbulence. We shall discuss this hereunder.

Secondly we consider the rigid body problem. There have been many interesting recent developments in analysing the stability of the rigid body with rigid and flexible attachments. Two seminal papers in this regard are Krishnaprasad and Marsden [1987] and Baillieul and Levi [1987]. The former paper discussed the Energy-Casimir method for analysing stability. More recently, the Energy-Momentum method which analyzes stability in the spatial (as opposed to the body) frame has been developed. A series of papers has appeared on this topic including Marsden and Simo [1989], Marsden, Simo, Lewis and Posbergh [1989], Simo, Posbergh and Marsden [1989] and Simo, Lewis and Marsden [1989]. (A variant of these methods was used in Bloch [1989] and Bloch and Ryan [1989].) In this paper we show how the Energy-Casimir method can be used to analyze *stability* of a *controlled* rigid body system about its unstable saddle. We also discuss possible use of the heteroclinic cycle structure as a control tool.

2. Controlling Chaos in the Near Wall Region of a Turbulent Boundary Layer

We begin by considering a rather general result which we shall apply to the specific dynamic model which we discuss below.

Theorem 2.1. Consider the C^r ($r \geq 2$) affine nonlinear control system given by

$$\dot{z} = f(z) + \sum_{i=1}^{m} u_i(t) g_i(z), \qquad z \in \mathbb{R}^n, \tag{2.1}$$

where the u_i are piecewise continuous scalar functions and f and the g_i are C^r functions from $\mathbb{R}^n$ to $\mathbb{R}^n$. Suppose that the free system $\dot{z} = f(z)$ has a hyperbolic fixed point at $z = z_0$ and that z_0 has a homoclinic orbit connecting z_0 to itself. Let $\mathcal{L}_0(x)$ be given by

$$\mathcal{L}_0(x) = \operatorname{Span}\{g_i, [f, g_i], [f, [f, g_i]], \ldots, \; i = 1, \ldots, m\}. \tag{2.2}$$

If $\dim \mathcal{L}_0(0) = n$, then a control u may be found such that the system spends an arbitrarily long time in a neighborhood U of the fixed point z_0 after the control force is removed. In particular, if all trajectories of the free system near the homoclinic orbit are periodic, or if the orbit is stable, a control may be found such that the system exhibits arbitrarily long periods when the control force is removed.

Proof: The proof rests on some observations about the free systems that may be found, for example, in Silnikov [1967] and Wiggins [1988]. Details are given in Bloch and Marsden [1989]. We give a brief sketch here.

Consider the free system $\dot{z} = f(z)$ and suppose that $Df(z_0)$ has s eigenvalues with negative real part and u with positive real part. The system may be transformed to the system

$$\begin{aligned} \dot{x} &= Ax + f_1(x, y) \\ \dot{y} &= By + f_2(x, y) \end{aligned} \tag{2.3}$$

where $(x, y) \in \mathbb{R}^s \times \mathbb{R}^u$, A is an $s \times s$ Jordan block with all diagonal entries having negative real parts and B is a $u \times u$ block with diagonal entries having positive real parts.

One then considers the neighborhood of the origin $N = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u \mid |x| \leq \varepsilon, \; |y| \leq \varepsilon\}$ whose boundary is given by

$$C_\varepsilon^s = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u \mid |x| = \varepsilon, \; |y| < \varepsilon\}$$

$$C_\varepsilon^u = \{(x, y) \in \mathbb{R}^s \times \mathbb{R}^u \mid |x| < \varepsilon, \; |y| = \varepsilon\}. \tag{2.4}$$

C_ε^s and C_ε^u give cross sections to the vector field (2.3). We denote by S_ε^s and S_ε^u the intersection of the stable manifold with C_ε^s and the intersection of the unstable manifold with C_ε^u respectively. The key idea, as discussed in Silnikov [1967] and Wiggins [1988] is to divide the Poincaré map into two parts, one restricted to the interior of N, which we call P_0, and the other restriced to the exterior, which we call P_1.

P_0 thus maps $C_\varepsilon^s \backslash S_\varepsilon^s$ to $C_\varepsilon^u \backslash S_\varepsilon^u$, and we denote by $T = T(x_0, y_0)$ the time taken for a point $(x_0, y_0) \in C_\varepsilon^s \backslash S_\varepsilon^s$ to reach $C_\varepsilon^u \backslash S_\varepsilon^u$. Then one can show that P_0^L, the Poincaré map for the vector field linearized about the origin, approximates P_0 to within an error $O(\varepsilon^2)$. Since P_0^L is given explicitly by $(x_0, y_0) \to (e^{AT'} x_0, e^{BT'} y_0)$ where T' solves $|e^{BT'} y_0| = \varepsilon$, one can show that $T(x_0, y_0) \to \infty$ logarithmically as $y_0 \to 0$.

Returning to the controlled system, by virtue of the condition on $\mathcal{L}_0$, we know the linearized system at z_0 is controllable. Hence we find (explicitly for the linearized system) a control that takes the system to a point on $C_\varepsilon^s \backslash S_\varepsilon^s$ choosing the point so that y_0 is as close to zero as we wish. We then remove the control and the theorem follows. □

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Now, of course, the above scheme is clearly not robust, as one would need infinite accuracy to get infinitely close to the stable manifold and trajectories obviously can be sensitive to outside perturbations. A more practical controller would thus be one which drives the system to within a small distance δ of the stable manifold, and when it senses the system has drifted a certain distance from the equilibrium reactivates. To make good physical sense, one would also like structural stability and asymptotic stability of the homoclinic orbit.

Consider then the system

$$\dot{x} = f(x) + \sum_{i=1}^{m} u_i g_i(x) + \delta w(t) \tag{2.5}$$

where $w(t)$ is a vector white noise process and δ is a small parameter. We assume that for $(\delta, u_i) = (0,0)$ the system has an asymptotically stable homoclinic orbit to a hyperbolic saddle point p. The free system was analyzed by Stone and Holmes [1988], who utilized the following conditions for asymptotic stability of the homoclinic orbit:

1) $W^u(p) \subset W^s(p)$ where W^s and W^u are the stable and unstable manifolds of p respectively, and

2) $\lambda_s > \lambda_u$ where the eigenvalues of $Df(p)$ are given by

$$\begin{aligned}\lambda_u &= Re(\lambda_1) \geq Re(\lambda_2) \geq \cdots \geq Re(\lambda_{k-1}) > 0 > Re(\lambda_k) \\ &= -\lambda_s \geq \cdots \geq Re(\lambda_n).\end{aligned}$$

The behavior of the general n-dimensional system is captured essentially by the 2-dimensional system

$$dx = -\lambda_s x\, dt + \delta dw_x$$

$$dy = \lambda_u y\, dt + \delta dw_y + u \tag{2.6}$$

where w_x and w_y are zero mean, independent, Wiener processes.

Now for $u = 0$, Stone and Holmes show that the expected mean passage time across the region N defined by (2.4) is given by

$$\tau \sim \frac{1}{\lambda_u} \ln\left(\frac{\varepsilon}{\delta}\right) + O(1). \tag{2.7}$$

Hence, if we set $u = -ky\,dt$, $0 < k < \lambda_u$, we decrease the expected passage time.

Now of course for k sufficiently large, the system is stabilized, but we assume here that we do not have sufficient control force to do this. This is precisely the situation we expect to have in controlling turbulence in the near wall region of a turbulent boundary layer. We now briefly describe the model of this system developed by Aubry et.al. [1988].

In this model the instantaneous field is expanded in a basis of eigenfunctions using the proper orthogonal decomposition of Lumley mentioned above. This expansion is particularly useful for flows in which large coherent structures contain a major fraction of the energy. The wall region of a turbulent boundary layer exhibits such structures, called large eddies. These large eddies undergo intermittent jumps between fixed points called bursting events.

Now the proper orthogonal decomposition together with Fourier analysis and Galerkin projection yields a truncated set of ODE's which captures the maximum amount of kinetic energy among all possible truncations of the same order. In Aubry et.al. [1988] models of various orders are examined.

In this paper we consider a model of 2 complex dimensions or 4 real dimensions which was analyzed in Armbruster, Guckenheimer and Holmes [1988]. While this model is of too low an order for really good physical representation, it does contain many of the features of the higher order models, in particular exhibiting asymptotically stable and structurally stable heteroclinic cycles in certain regions of the phase space. The key idea is that bursting corresponds to passage close to the heteroclinic cycle, while no bursting corresponds to remaining close to a given hyperbolic point. A further important part of this model is the presence of pressure fluctuations in the outer layer which can trigger a bursting event; noise can be used to model these fluctations in the manner of Stone and Holmes discussed above.

Our purpose here in controlling such a system is to control the frequency of bursting events, which hopefully can be used to control the amount of turbulence in the boundary layer. In general one wishes to reduce the frequency of bursting, but in other instances it might be advisable to encourage a burst or regularize its period. We consider here a model with controls that could be heatable patches (combined with hot film sensors) or welts raised by piezoelectric effects. Classical drag reduction by polymer addition may be analyzed also in our framework but we omit discussion of this here (see Bloch and Marsden [1989]).

The free system (which is $O(2)$ equivariant) may be written in complex form as (see Aubry et.al. [1988] or Armbruster et.al. [1988])

$$\begin{aligned}\dot{z}_1 &= z_1(\mu_1 + d_{11}|z_1|^2 + d_{12}|z_2|^2) + c_{12}\bar{z}_1 z_2 + O(4) \\ \dot{z}_2 &= z_2(\mu_2 + d_{21}|z_1|^2 + d_{22}|z_2|^2) + c_{11}\bar{z}_1^2 + O(4)\end{aligned} \tag{2.8}$$

where the z_i are complex variables and μ_i, d_i, and c_i are parameters.

Assuming $c_{12}, c_{11} \neq 0$, one can rescale (2.8) to

$$\dot{z}_1 = \bar{z}_1 z_2 + z_1(\mu_1 + e_{11}|z_1|^2 + e_{12}|z_2|^2)$$

$$\dot{z}_2 = \mp z_1^2 + z_2(\mu_2 + e_{21}|z_1|^2 + e_{22}|z_2|^2).$$

Now we assume that through a "checkerboard" of heating elements or piezoelectric controls one can essentially change the magnitude of all eigenvalues of the system, and through sensors one can monitor the amplitude of all modes.

In Cartesian form the system with controls is thus

$$\dot{x}_1 = x_1 x_2 + y_1 y_2 + x_1(\mu_1 + e_{11} r_1^2 + e_{12} r_2^2) + u_1$$

$$\dot{y}_1 = x_1 y_2 - y_1 x_2 + y_1(\mu_1 + e_{11} r_1^2 + e_{12} r_2^2) + u_2$$

$$\dot{x}_2 = \mp(x_1^2 - y_1^2) + x_2(\mu_2 e_{21} r_1^2 + e_{22} r_2^2) + u_3$$

$$\dot{y}_2 = \pm 2x_1 y_1 + y_2(\mu_2 + e_{21} r_1^2 + e_{22} r_2^2) + u_4$$

when $r_1^2 = x_1^2 + y_1^2$.

In the "-" case, one can show that in certain regions of the parameter space, a locally asymptotically stable heteroclinic cycle links the fixed points at $(x_1, y_1, x_2, y_2) = (0, 0, \pm(-\mu_2/e_{22})^{1/2}, 0)$ (see Armbruster et.al. [1988]).

One can also check that the linearized system about the fixed points is controllable (see Bloch and Marsden [1989]). Thus our earlier results apply to this system. Thus the suggested control strategy *in the presence of noise* is 1) *decrease the magnitude of the largest unstable eigenvalue of the system linearized about the given fixed point*, thus increasing the time the system is expected to remain near this point and 2) if a perturbation drives the system a sufficiently large distance from the fixed point, return it to a point as close as possible to the stable manifold using an explicit control for the linearized system.

3. Stability and Stabilization for the Rigid Body Problem

Another dynamical system which contains heteroclinic orbits is that of the rigid body. It is a classical result that a rigid body rotates stably about its major and minor principal axes, but unstably about its intermediate axis. Elegant proof of nonlinear stability about the major or minor axes can be given by the Energy-Casimir method (see Holm, Marsden, Ratiu and Weinstein [1985]) or the more recently developed Energy-Momentum method (see Simo, Posborgh and Marsden [1989]). Now, corresponding to rotation about the intermediate axis, are saddle points on the momentum sphere linked by four heteroclinic orbits. A treatment of this may be found in Holmes and Marsden [1983] (see also Abraham and Marsden [1978]).

Recall that the rigid body equations are given by

$$\begin{aligned} I_1\dot{w}_1 &= (I_2 - I_3)w_2w_3 \\ I_2\dot{w}_2 &= (I_3 - I_1)w_3w_1 \\ I_3\dot{w}_3 &= (I_1 - I_2)w_1w_2 , \end{aligned} \tag{3.1}$$

where I_i are the principal moments of inertia.

In Hamiltonian form the free rigid body is a left-invariant Hamiltonian system on $T^*SO(3)$. By reduction, we can write the system as a Lie-Poisson system on $so(3)^*$, the dual of the Lie algebra of $so(3)$. Now $so(3)^*$ may be identified with $so(3)$ (by the killing form) and $so(3)$ may be identified with $\mathbb{R}^3$ by mapping $v = (p, q, r) \in \mathbb{R}^3$ to $\hat{v} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & r & 0 \end{bmatrix} \in so(3)$.

The Lie bracket is then mapped to cross product in the sense that $[\hat{v}, \hat{w}] = (v \times w)^\wedge$.

Now elements $m \in so(3)^*$ may be taken to represent the body angular momentum of the rigid body, and viewing $m \in \mathbb{R}^3$, we have $m_i = I_i w_i$, $i = 1, 2, 3$, and the equations of motion are

$$\begin{aligned} \dot{m}_i &= \frac{I_2 - I_3}{I_2 I_3} m_2 m_3 \\ \dot{m}_2 &= \frac{I_3 - I_1}{I_1 I_3} m_1 m_3 \\ \dot{m}_3 &= \frac{I_1 - I_2}{I_1 I_2} m_1 m_2 . \end{aligned} \tag{3.2}$$

With Hamiltonian $H(m) = \frac{1}{2}\sum_{i=1}^{3} \frac{m_i^2}{I_i}$, these equations of motion are given by $\dot{F} = \{\{F, H\}\}$ where the Lie-Poisson bracket $\{\{F, G\}\}(m) = -m \cdot (\nabla F \times \nabla G)$. Conservation of momentum, which is here equivalent to preserving co-adjoint orbits, is given by constancy of $\ell^2 = m_1^2 + m_2^2 + m_3^2$. Flow lines are given by intersecting these momentum spheres with the ellipsoids H =constant. There are saddle points at $(0, \pm\ell, 0)$ which are connected by four heteroclinic orbits. As discussed in Holmes and Marsden [1983], these orbits lie in the invariant planes $m_3 = \pm\sqrt{\frac{a_3}{a_1}} m_1$ where $a_1 = \frac{I_2 - I_3}{I_2 I_3} > 0$, $a_2 = \frac{I_3 - I_1}{I_1 I_3} < 0$ and $a_3 = \frac{I_1 - I_2}{I_1 I_2} > 0$. Explicitly the orbits are given by

$$\begin{aligned} m_1^+(t) &= \pm\ell\sqrt{\frac{a_1}{-a_2}}\operatorname{sech}(-\sqrt{a_1 a_3}\ell t) \\ m_2^+(t) &= \pm\ell\tanh(-\sqrt{a_1 a_3}\ell t) \\ m_3^+(t) &= \pm\ell\sqrt{\frac{a_3}{-a_2}}\operatorname{sech}(-\sqrt{a_1 a_3}\ell t) \end{aligned} \tag{3.3}$$

for $m_3 = +(\sqrt{a_3/a_1})m_1$ and by

$$m_1^-(t) = m_1^+(-t), \quad m_2^-(t) = m_2^+(-t), \quad m_3^-(t) = -m_3^+(-t) \tag{3.4}$$

for $m_3 = -(\sqrt{a_3/a_1})m_1$.

Here we assume $I_1 > I_2 > I_3$.

As far as control is concerned, the situation which is closest to that discussed in the previous section, is when we have three independent torques (e.g. gas jets) about the principal axes, i.e. we have the system

$$\begin{aligned} \dot{m}_1 &= a_1 m_2 m_3 + u_1 \\ \dot{m}_2 &= a_2 m_3 m_1 + u_2 \\ \dot{m}_3 &= a_3 m_1 m_2 + u_3 \end{aligned} \tag{3.5}$$

In this case, the linearized system is controllable (see Crouch [1984]), and we can apply Theorem 2.1, thus driving the system as close to the stable manifold as we wish, before removing the control forces. One situation where the techniques discussed in the previous section might be useful in this context is when one does not have sufficient control power to stabilize the system. We may also, in fact, wish to exploit rapid passage close to the heteroclinic orbits as a control manouver. This leads not only to a rapid change in the sign of the momentum component m_2, but to a rapid "tumble" in configuration space also.

In case where one does have enough control power, it is natural to consider stabilizing the rigid body about the unstable principal axis.

One natural way to stabilize the system is by altering I_2. Suppose, for example, one had a unit mass attached to the rigid body along the intermediate axis of inertia at a distance x from the center of mass. then $I_2 \to I_2 + x^2$.

Hence for x sufficiently large, $I_2 + x^2 > I_1 > I_3$ and the system is stabilized about the intermediate axis, as one can check by the Energy-Casimir or Energy-Momentum methods. Letting x go to zero will lead to destabilization and rapid passage near a heteroclinic. We remark that this model is similar to that of Levi [1989], about which we comment further later.

There has been a great deal of work over the past decade analyzing the problem of stabilizing both the angular momentum equations for a rigid body and the full attitude (configuration space) equations. We mention in particular in this regard the work of Baillieul [1981], Bonnard [1981], Brockett [1983], Crouch [1984], Aeyels [1985 a,b], Aeyels and Szafranski [1988] and Byrnes and Isidori [1989].

In the latter paper, Byrnes and Isidori show that with two torques (gas jets) the full attitude equations may be asymptotically stabilized to revolute motion about a principal axis.

In Brockett [1983], it is shown by finding a Liapunov function that the null solution of the angular velocity equations may be stabilized by two control torques. In Aeyels [1985a], the same result is demonstrated by Lyapunov theory. In Aeyels [1985b], it is shown that the angular velocity equation, may be "robustly" stabilized (though not asymptotically stabilized) by a single torque aligned with either a major or minor principal axis. This result is tight in that it is shown in Aeyels and Szafianski [1988], that the equations cannot be asymptotically stabilized by a single torque about a principal axis.

We show here, via the Energy-Casimir method, that we can stabilize the rigid body equations about the intermediate axis of inertia by a single torque about the minor or major axis. More precisely, we show

Theorem 3.1: The rigid body equations (3.2) with a single torque about the minor (or major) axis:

$$\begin{aligned} \dot{m}_1 &= a_1 m_2 m_3 \\ \dot{m}_2 &= a_2 m_1 m_3 \\ \dot{m}_3 &= a_3 m_1 m_2 + u_3 \end{aligned} \tag{3.6} \quad \left(\text{or} \quad \begin{aligned} \dot{m}_1 &= a_1 m_2 m_3 + u_1 \\ \dot{m}_2 &= a_2 m_1 m_3 \\ \dot{m}_3 &= a_3 m_1 m_2 \end{aligned}\right)$$

may be stabilized about the relative equilibrium $(m_1, m_2, m_3) = (0, M, 0)$ by the control $u_3 = -k m_1 m_2$ (or $u_1 = -k m_2 m_3$).

Proof: Consider first the system linearized about $(0, M, 0)$. Its eigenvalues are given by the solution of

$$\lambda(\lambda^2 - a_1(a_3 - k)M^2) = 0 .$$

Hence for $k = 0$, the system is unstable, but for k sufficiently large we have two imaginary and one zero eigenvalue. Is the system stable? We prove that it is via the Energy-Casimir method.

Recall that the Energy-Casimir method (see, e.g. Krishnaprasad and Marsden [1987]), requires finding a constant of motion for the system, E, usually the energy, and a family of constants of motion C, such that for some C, $E+C$ has a critical point at the (relative) equilibrium of interest. (Often the C's are taken to be Casimirs - functions that commute with all other functions under the Poisson bracket). Then, in finite dimensions, definiteness of $\delta^2(E+C)$ at the critical point is sufficient to prove stability.

Now here we have

Lemma 3.2:

$$E_C = \frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\frac{a_3}{a_3 - k}\right) \tag{3.7}$$

and

$$M_C^2 = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2\frac{a_3}{a_3 - k}\right) \tag{3.8}$$

are conserved for the system (3.6) with $u = -km_1m_2$.

Proof: $\frac{1}{2}\dot{(m_3)^2} = m_3\dot{m}_3 = m_3(a_3 - k)m_1m_2$ and then the calculations reduce to the standard rigid body calculations. □

We remark that the system (3.6) with $u_3 = -km_1m_2$ is a Lie-Poisson system (see Krishnaprasad [1985] and Alvarez-Sanchez [1986] or Holmes and Marsden [1983]), with respect to the non-canonical Lie-Poisson bracket $\{F,G\} = -\nabla\left(\frac{a_3-k}{a_3}M_C^2\right)\cdot(\nabla F \times \nabla G)$. In fact, it is Lie-Poisson in a number of different ways, see Bloch and Marsden [1989b] for further details.

Now use the "modified" Energy-Casimir function

$$\begin{aligned} E_C + \phi(M_C^2) = &\frac{1}{2}\left(\frac{m_1^2}{I_1} + \frac{m_2^2}{I_2} + \frac{m_3^2}{I_3}\frac{u_3}{a_3 - k}\right) \\ &+ \frac{1}{2}\phi\left(m_1^2 + m_2^2 + m_3^2\frac{a_3}{a_3 - k}\right) \end{aligned} \tag{3.9}$$

where ϕ is an arbitrary smooth function. Now

$$\begin{aligned} \delta(E_C + \phi(M_C^2)) = &\left(\frac{m_1}{I_1}\delta m_1 + \frac{m_2}{I_2}\delta m_2 + \frac{m_3}{I_3}\delta m_3\frac{a_3}{a_3 - k}\right) \\ &+ \phi'(M_C^2)\left\{m_1\delta m_1 + m_2\delta m_2 + m_3\delta m_3\frac{a_3}{a_3 - k}\right\} \end{aligned} \tag{3.10}$$

This equals zero if

$$\begin{aligned} &\frac{m_1}{I_1} + \phi' m_1 = 0 \\ &\frac{m_2}{I_2} + \phi' m_2 = 0 \\ &\frac{m_3}{I_3}\frac{a_3}{a_3 - k} + \phi' m_3\frac{a_3}{a_3 - k} = 0 \end{aligned} \tag{3.11}$$

Now at equilibrium $(m_1, m_2, m_3) = (0, M, 0)$ this will be zero if $\phi' = \frac{-1}{I_2}$. Then

$$\begin{aligned} \delta^2(E_C + \phi(M_C^2)) = &\frac{(\delta m_1)^2}{I_1} + \frac{(\delta m_2)^2}{I_2} + \frac{(\delta m_3)^2}{I_3}\frac{a_3}{a_3 - k} \\ &- \frac{1}{I_2}\left\{(\delta m_1)^2 + (\delta m_2)^2 + (\delta m_3)^2\frac{a_3}{a_3 - k}\right\} + \phi''(M_C^2)M^2(\delta m_2)^2 \end{aligned} \tag{3.12}$$

at the equilibrium of interest.

Now, since $I_1 > I_2 > I_3$ and $a_3 = \frac{I_1 - I_2}{I_1 I_2}$, for k sufficiently large that $a_3 - k < 0$ and choosing $\phi'' < 0$, the second variation is negative definite and we have nonlinear stability. (A similar argument holds for $u_1 = -km_2m_3$.) □

Finally, we make some remarks on stabilizing more complex systems than the rigid body. We have in mind the problem of stabilizing systems of coupled rigid and elastic bodies. Stability of coupled rigid bodies and flexible rods was analyzed in Krishnaprasad and Marsden [1987] and Baillieul and Levi [1987]. See also Bloch and Ryan [1989]. A prototype finite dimensional model of a rigid body with elastic appendage - a mass on a spring - has been analyzed recently by Levi [1989]. Stability of motion of two coupled rigid bodies has been analyzed by Patrick [1989]. More recent work on analyzing the stability of coupled systems may be found in Simo, Posbergh and Marsden [1989], Marsden, Simo, Lewis and Posbergh [1989] and Simo, Lewis and Marsden [1989].

While in the Energy-Casimir method discussed earlier, the analysis takes place in the body frame, in the latter work the Energy-Momentum method, which takes place in the spatial frame, is used. More importantly, in this context, the papers alluded to above, prove the existence of a block-diagonalization of the second variation of the energy-momentum function, thus vastly simplifying the analysis of stability for complex coupled systems.

These results can be formulated quite generally for the Hamiltonians of mechanical systems with symmetry. The test for stability of equilibrium in this context reduces to a test for definiteness of the second variation of the Energy-Momentum function on a linear subspace lying in the kernel of the derivative of the momentum map arising from the symmetry group action. This second variation can then be shown to decouple into "rigid body" variations and "internal vibration" variations.

Our goal is to apply some of these techniques to the stability analysis of complex controlled systems. The rigid body analysis we have carried out here is at least suggestive that this line of investigation might be fruitful.

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