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CHERENKOV-CURVATURE RADIATION AND PULSAR RADIO EMISSION GENERATION

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ABSTRACT

Electromagnetic processes associated with a charged particle moving in a strong circular magnetic field are considered in cylindrical coordinates. We investigate the relation between the vacuum curvature emission and Cherenkov emission and argue that, for the superluminal motion of a particle in the inhomogeneous magnetic field in a dielectric, the combined effects of magnetic field inhomogeneity and the presence of a medium give rise to the synergetic Cherenkov-curvature emission process. We find the conditions under which the operator relations between electric field and electric displacement in cylindrical coordinates may be approximated by algebraic relations. For nonresonant electromagnetic waves, the interaction with particles streaming along the curved magnetic field may be described in the WKB approximation. For resonant waves interacting with superluminal particles we use a plane-wave approximation to compute the local dielectric tensor of a plasma in a weakly inhomogeneous magnetic field. We find in this approximation the polarization of normal modes in the plasma, Cherenkov-curvature and Cherenkov-drift emissivities and growth rates.

Subject headings: MHD — pulsars: general — radiation mechanisms: nonthermal — radiative transfer — waves

1. INTRODUCTION

Despite much theoretical effort, there is still no widely accepted explanation of how pulsars emit the high brightness radio emission by which they were first discovered over 30 years ago. Difficulties with "antenna" mechanisms have led to a resurgence of interest in "maser" mechanisms where an inverted population of electrons and/or positrons amplifies an outgoing wave mode (e.g., Melrose 1995). Two such maser process that are particularly promising are the Cherenkov-drift mechanism (which is Cherenkov-curvature emission with a drift) and cyclotron-Cherenkov emission at the anomalous Doppler resonance (Kazbegi, Machabeli, & Melikidze 1991). This paper is concerned with elucidating the physics of the Cherenkov-curvature and Cherenkov-drift processes and developing a new mathematical description of it using cylindrical coordinates. If we use this approach, its close relationship to the pure Cherenkov emission processes becomes clear.

Pulsar radio emission is believed to originate on the open magnetic field lines that trace a path from the neutron star surface to the light cylinder at $r = c/\Omega$ and beyond. The field geometry is complex close to the star at $r \approx R_* \approx 10$ km and near the light cylinder. However, it should be primarily dipolar for $R_* \ll r \ll c/\Omega$. The characteristic length scale, essentially the radius of curvature of the field lines, $R_C \approx (cr/\Omega)^{1/2}$, is very large compared with the wavelength of interest, λ, and so we should be able to adopt a WKB approach following individual wave packets as they propagate out through the magnetosphere. However, in order to compute the local emission and absorption it is necessary that the magnetic field remains curved. A simple model problem that retains the essential ingredients comprises a set of circular concentric cylindrical magnetic surfaces. (Fig. 1). We show that it is possible to ignore the radial variations in the strength of $\boldsymbol{B}$ in computing the local interaction, provided that this variation is not very rapid. In this model problem the plasma circulates continually along the circular trajectories.

It is generally presumed that electrons and positrons are in their ground gyrational state and follow the curved field lines. This is a good approximation for the bulk of the plasma, which we will suppose travels with a Lorentz factor $\gamma_p \approx 100$. However, we believe that there is also a population of ultraenergetic particles with $\gamma = 10^5$–10^7, and, in the outer magnetosphere, these will experience a curvature drift relative to the bulk plasma. This has the interesting consequence that their main electrodynamic interaction is with the waves propagating in the bulk plasma at a finite angle to the magnetic field $\boldsymbol{B}$. It is these waves that, we assert, become the high brightness electromagnetic waves that escape from the magnetosphere.

When considering the wave amplification in this problem, we assume that fluctuating currents, present on the field lines at radii smaller than the radius of the considered region, produce electromagnetic fluctuations that are subsequently amplified due to the interaction with resonant particles (Fig. 1). This is a convective type instability, when a wave is amplified as it propagates through an active medium.

A first attack on this problem was made by Blandford (1975). In this paper the energy transfer from a plane infinite electromagnetic wave to a single electron moving with ultrarelativistic speed along a curved trajectory was found to be always positive, independent of the distribution function, and so there is no possibility of wave growth. The subsequent works (Zheleznyakov & Shaposhnikov 1979; Lou & Melrose 1992; Melrose 1978) basically followed this approach, which emphasizes the analogy between curvature emission and conventional cyclotron emission. This approach, although formally correct, has limited applicability and ignores two important features of the emission mechanism. The first is that, in adopting a plane-wave formalism, the interaction length for an individual electron, $\approx R_C/\gamma_b$, was essentially coextensive with the region

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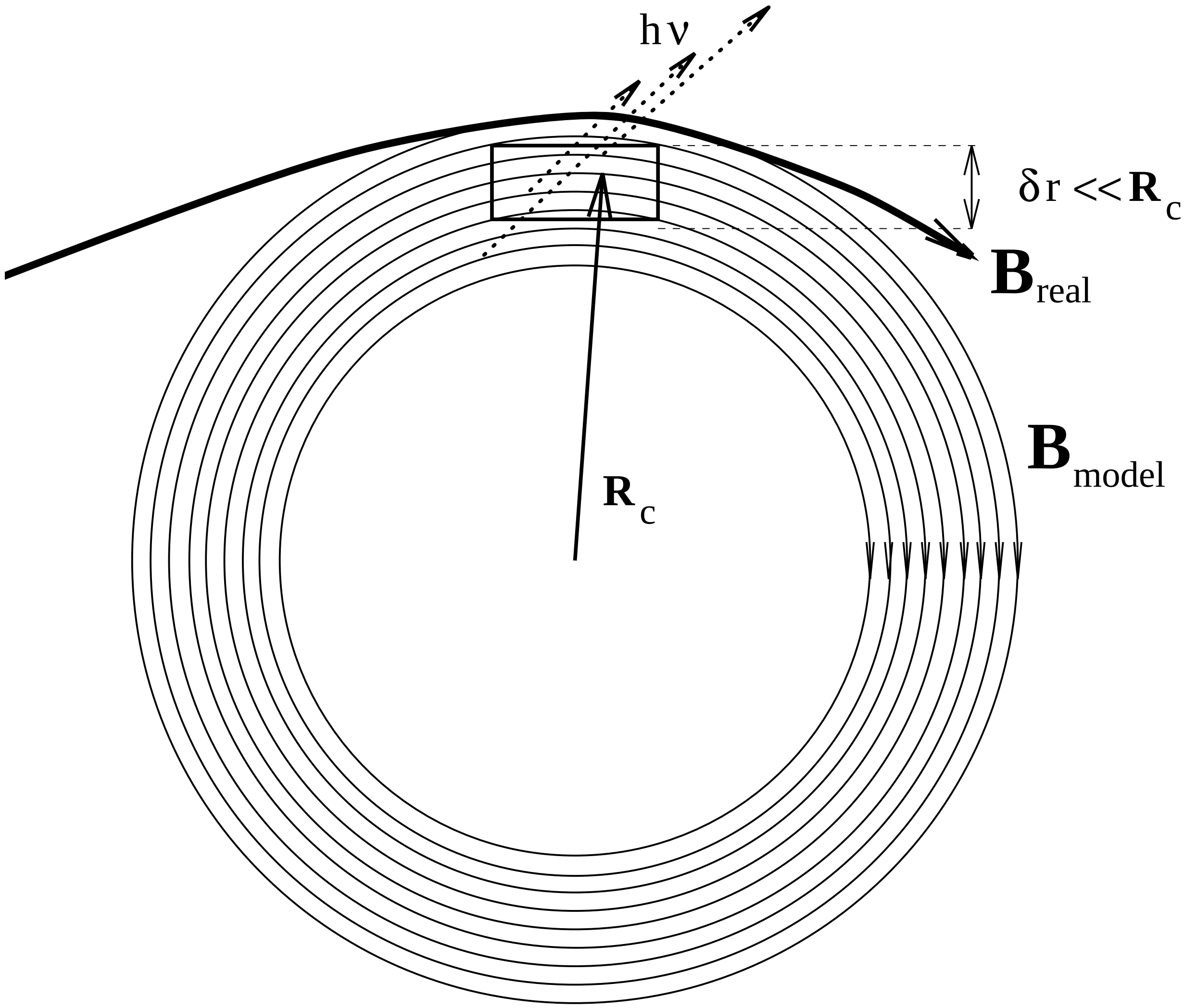

FIG. 1.—Geometry of the considered problem. The magnetic field lines of the model problem $\boldsymbol{B}_{\rm model}$ are concentric coplanar circular arcs with the radius of curvature much larger than the size of the region. They are local approximation to the $\boldsymbol{B}_{\rm real}$ curved field lines.

over which the waves could interact with any electron. This approach precludes a strong amplification under all circumstances because the wave would have to grow substantially during a single interaction. The second problem was that a dispersion of the waves was neglected. We address the first shortcoming by expanding the electromagnetic field in cylindrical waves centered on $r = 0$ and the second explicitly by considering general plasma modes.

In a separate approach developed by Beskin, Gurevich, & Istomin (1986) an attempt was made to incorporate the collective effects of a plasma in an inhomogeneous magnetic field. The fundamentals of that approach have been seriously criticized (Nambu 1989; Machabeli 1995).

One of the key new elements in our approach is that we expand the electromagnetic fields in cylindrical coordinates and consider *resonant* interaction between a particle and these modes. A common procedure in calculating the energy emitted by a given current distribution is to find the power emitted into a given normal mode of the medium and sum over all the modes. A power emitted into a normal mode by a given charge distribution is proportional to the square of the expansion of this current in normal modes. In a homogeneous or weakly inhomogeneous stratified medium considered in Cartesian coordinates the normal modes are plane waves (with slowly changing parameters), so that the power emitted into a given normal mode turns out to be proportional to the Fourier transform of the current. To find a similar expression for a single particle emissivity into a cylindrical mode in an inhomogeneous medium is a difficult problem for two reasons: there is a complicated, radius-dependent, form of the vector cylindrical waves, and the dielectric response in an inhomogeneous medium is nonlocal. We address these complications by finding the conditions under which the interaction of a particle with cylindrical waves can be approximated by an interaction of a particle with plane waves. We find two cases when this can be done. First, this may be a true approximation for the *nonresonant* modes. This is equivalent to the WKB approximation to the radial dependence of normal modes (mathematically, this corresponds to the tangent expansion of Bessel functions when the argument is significantly larger than the order). In this case the response of a medium becomes local. Second, we find a particular case in which the *resonant* modes can be approximated as local plane waves. A resonant interaction of a relativistic particle with a cylindrical mode occurs near the point when the argument of Bessel functions is close to the order. The WKB approximation, or expansion in tangents, is not applicable in this case, and we have to use the Airy function approximation to Bessel function, which has a plane-wave approximation for the interaction of *subluminous* waves with the particles moving with speed larger than the speed of light in a medium. This corresponds to the Airy function expansion argument being larger than the order.

We should note here that the electrodynamics of the interaction of a particle moving along the curved magnetic field with speed larger than the speed of light in the medium is quite unusual and can be considered as a new, Cherenkov-curvature, emission mechanism which differs from conventional Cherenkov, cyclotron, or curvature emission and includes, to some extent, the features of each of these mechanisms. In an extension of ideas of Schwinger, Tsai, & Erber (1976), conventional synchrotron emission and Cherenkov radiation may be regarded as respective limiting cases of $|n-1| \to 1$ and $B \to 0$ of a "synergetic" cyclotron-Cherenkov radiation, the Cherenkov-curvature radiation necessary includes effects of the magnetic field gradients.

When the dielectric response of a medium to a cylindrical wave is *local* it is possible to calculate a simplified dielectric tensor. Using this dielectric tensor, we find the normal modes of strongly magnetized electron-positron plasma and show that a beam of particles propagating along the curved magnetic field can amplify the electromagnetic waves. The amplification occurs at the Cherenkov-drift resonance $\omega - k_\phi v_\phi - k_x u_d = 0$ (ω is the frequency of the wave, k_ϕ, k_x, and v_ϕ are the corresponding projections of the wavevector and velocity and, u_d is a curvature drift velocity). The fact that this is a Cherenkov-type resonance immediately implies that the presence of a subluminous wave with the phase velocity smaller than the speed of light is essential. The presence of a drift provides a coupling between the electric field of the electromagnetic wave particle's motion. Our estimates show that this instability can grow fast enough to account for the observed pulsar radio emission. This instability may be regarded as new type of a curvature maser. An interesting and peculiar feature of this mechanism is that, unlike with conventional curvature emission, the emitted waves have a polarization almost perpendicular to the oscillating plane of the magnetic field.

The overview of this work is the following. In § 2 we analyze the properties of electromagnetic waves in cylindrical coordinates in vacuum and the interaction of the vacuum waves with a charged particle. In § 3 we analyze in cylindrical coordinates the electromagnetic fields of a particle moving along a spiral trajectory using the dyadic Green's function for the vector wave equation, derive the emissivity of a particle in ground gyrational level into a cylindrical mode, and rederive the curvature emissivity in cylindrical coordinates. This is followed in § 4 by the generalization to a dispersive isotropic medium, and the importance of waves with the phase speed less than the speed of light in vacuum is brought out. In § 5 we consider *anisotropic* plasma in an infinitely strong magnetic field and derive the curvature emissivity using the Vlasov approach. In § 6 we discuss the various regimes of the Airy function approximation to the Cherenkov-curvature emission and find the conditions for the plane-wave approximation. In § 7 we investigate the features of the electromagnetic waves in the asymptotic regime $z \gg 1$ and in § 8 we calculate the response tensor for a one-dimensional plasma in a strong curved magnetic field taking into account the drift velocity. Finally, in § 9 we analyze the polarization properties of electromagnetic waves in the plane-wave approximation and calculate the increment of the Cherenkov-drift instability.

2. VACUUM SOLUTIONS

2.1. *Vacuum Normal Modes*

We expand the fundamental solutions of the wave equation

$$\mathrm{curl}\ \mathrm{curl}\ \boldsymbol{E} - \frac{\omega^2}{c^2}\boldsymbol{E} = 0 \tag{1}$$

in terms of Fourier amplitudes in x and ϕ coordinates and time,

$$\boldsymbol{E}(\boldsymbol{r}, t) = \sum_{\nu=-\infty}^{\nu=\infty} \int d\omega \int \frac{dk_x}{2\pi} \boldsymbol{E}(r, k_x, \nu, \omega) \exp\{-i(\omega t - \nu\phi - k_x x)\} . \tag{2}$$

The wave equation (1) then takes the form

$$\frac{i\nu}{r^2}\frac{\partial}{\partial r}(rE_\phi) + ik_x \frac{\partial}{\partial r}E_x - \frac{k_x \nu}{r}E_\phi + \left[\frac{\nu^2}{r^2} - \left(\frac{\omega^2}{c^2} - k_x^2\right)\right]E_r = 0 , \tag{3}$$

$$-\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + i\nu\frac{\partial}{\partial r}\left(\frac{E_r}{r}\right) - \frac{k_x \nu}{r}E_x - \left(\frac{\omega^2}{c^2} - k_x^2\right)E_\phi = 0 , \tag{4}$$

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}E_x\right) + \frac{ik_x}{r}\frac{\partial}{\partial r}(rE_r) - \left(\frac{\omega^2}{c^2} - \frac{\nu^2}{r^2}\right)E_x = 0 . \tag{5}$$

Using the fact that in a wave the magnetic field is related to the electric field,

$$\mathrm{curl}\ \boldsymbol{E} = -\frac{1}{c}\frac{\partial \boldsymbol{B}}{\partial t} = i\frac{\omega}{c}\boldsymbol{B} , \tag{6}$$

we find the amplitudes $\boldsymbol{E}(r, k_x, \nu, \omega)$, subject to condition of being finite at $r = 0$

$$\boldsymbol{E}^{(lt)}(r, k_x, \nu, \omega) = E^{(lt)}\left[\frac{ik_x}{\lambda}J'_\nu(\lambda r)\boldsymbol{e_r} - \frac{\nu k_x}{\lambda^2 r}J_\nu(\lambda r)\boldsymbol{e}_\phi + J_\nu(\lambda r)\boldsymbol{e_x}\right] , \tag{7}$$

$$B^{(lt)}(r, k_x, \nu, \omega) = \frac{\omega}{\lambda^2 c} E^{(lt)}\left[\frac{\nu J_\nu(\lambda r)}{r} e_r + i\lambda J'_\nu(\lambda r) e_\phi\right], \tag{8}$$

$$E^{(t)}(r, k_x, \nu, \omega) = E^{(t)}\left[\frac{i\nu}{\lambda r} J_\nu(\lambda r) e_r - J'_\nu(\lambda r) e_\phi\right], \tag{9}$$

$$B^{(t)}(r, k_x, \nu, \omega) = E^{(t)}\left[\frac{k_x c}{\omega} J'_\nu(\lambda r) e_r + \frac{i k_x c \nu}{\lambda r \omega} J_\nu(\lambda r) e_\phi - i \frac{c\lambda}{\omega} J_\nu(\lambda r) e_x\right], \tag{10}$$

where $\lambda = (\omega^2/c^2 - k_x^2)^{1/2}$, $J_\nu(\lambda r)$ are Bessel functions, which satisfy the equation

$$\frac{1}{r}\frac{1}{\partial r}\left(r \frac{1}{\partial r} J_\nu\right) - \left(\frac{\nu^2}{r^2} + k_x^2 - \frac{\omega^2}{c^2}\right) J_\nu = 0 , \tag{11}$$

and e_r, e_ϕ, and e_x are unit vectors along the corresponding axes, $E^{(lt)}$ and $E^{(t)}$ are amplitudes of the modes. The superscripts denote polarization: E^{lt} is perpendicular to the coordinate surface e_x and $E^{(t)}$ lies on the coordinate surface e_x.

2.1.1. *Short Radial Wavelength Expansion of Vacuum Solutions*

For large orders $\nu \gg 1$, arguments larger than the order $\lambda r > \nu$, and those not very close to it, $\lambda r - \nu \geq \nu$, we can use expansion in tangents of the Bessel functions,

$$J_\nu(\nu \sec \zeta) \approx \sqrt{\frac{2}{\pi \nu \tan \zeta}} \cos\left[\nu(\tan \zeta - \zeta) + \frac{\pi}{4}\right], \qquad \zeta \geq \nu^{-1/2} , \tag{12}$$

in our case $\cos \zeta = \nu/(\lambda r)$.

Introducing

$$k_r^2 = \lambda^2 - \frac{\nu^2}{r^2} , \tag{13}$$

the equation (12) for the Bessel function may be viewed as a dispersion relation,

$$k_r^2 + k_\phi^2 + k_x^2 = \frac{\omega^2}{c^2} . \tag{14}$$

Then the expansion (eq. [12]) reads

$$J_\nu(\lambda r) \approx \sqrt{\frac{2}{\pi k_r r}} \cos(\pm k_r r - \phi_\nu) , \tag{15}$$

where ϕ_ν is insignificant phase shift. The limits of applicability of this expansion are that $k_r r \gg 1$ and ζ is not very small. The normal modes in this limit are

$$E^{(lt)}(r, t) \propto e^{-i(\omega t - \nu\phi - k_x x \pm k_r r)}\left(-\frac{k_x k_r}{\lambda^2} e_r - \frac{k_x k_\phi}{\lambda^2} e_\phi + e_x\right),$$
$$E^{(t)}(r, t) \propto e^{-i(\omega t - \nu\phi - k_x x \pm k_r r)}\left(\frac{k_\phi}{\lambda} e_r - \frac{k_r}{\lambda} e_\phi\right). \tag{16}$$

2.1.2. *WKB Solution* (*Short Radial Wavelength*)

We can obtain solutions (eq. [16]) of equations (3)–(5) using the WKB procedure, when the effective radial wavelength is much shorter than the characteristic scale of the problem. Assuming that the solutions of equations (3)–(5) have a form

$$E \propto \frac{1}{\sqrt{r}} e^{iS} , \qquad \frac{\partial S}{\partial r} \gg \frac{1}{r} , \tag{17}$$

we find

$$S = \pm \int dr \sqrt{\lambda^2 - \frac{\nu^2}{r^2}} = \pm\left(k_r r - \nu \arccos \frac{\nu}{\lambda r}\right), \tag{18}$$

with k_r defined by equation (14).

The dispersion equation, which is obtained by substituting equation (17) in equations (3)–(5), takes a form

$$\epsilon_{ij}^{(0)} E_i = 0 , \tag{19}$$

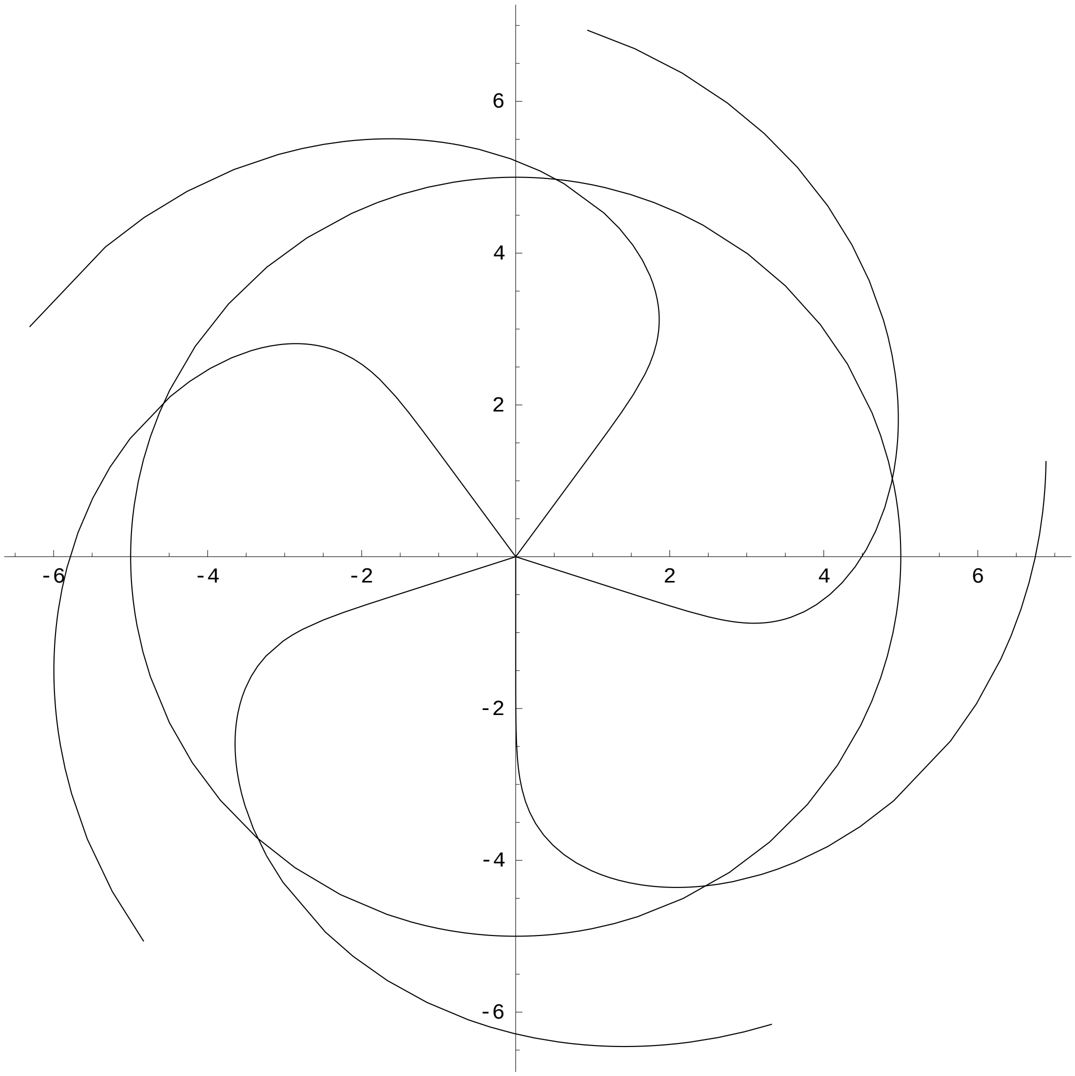

FIG. 2.—Surfaces of constant phase of Hankel function $H_{\nu}^{(1)}$ for $\nu = 5$. Circle $\lambda r = \nu$ is an analog of a light cylinder. For radii much smaller than the "light cylinder" radius $r_{\nu} = \nu/\lambda$, the rotation is similar to solid body rotation, whereas for the radii much larger than the light cylinder radius the surface of a constant phase has a form of unwinding spiral with a wavelength $\lambda r/\nu$.

where

$$\epsilon_{ij}^{(0)} = \begin{pmatrix} k_{\phi}^2 + k_x^2 - \frac{\omega^2}{c^2} & -k_r k_{\phi} + \frac{ik_{\phi}}{2r} & -k_x k_r - \frac{ik_x}{2r} \\ -k_r k_{\phi} - \frac{3ik_{\phi}}{2r} & k_x^2 + k_r^2 + \frac{3}{4r^2} - \frac{\omega^2}{c^2} & -k_x k_{\phi} \\ -k_x k_r + \frac{ik_x}{2r} & -k_x k_{\phi} & k_{\phi}^2 + k_r^2 - \frac{1}{4r^2} - \frac{\omega^2}{c^2} \end{pmatrix} . \tag{20}$$

For $k_r \gg 1/r$ we can drop the complex part of the tensor $\epsilon_{ij}^{(0)}$ (which is not even Hermitian since k_r is not a Killing vector). The dispersion equation det $||\epsilon_{ij}^{(0)}|| = 0$ then gives again

$$k_r^2 + k_x^2 + k_{\phi}^2 = \frac{\omega^2}{c^2} . \tag{21}$$

The Bessel equation (11) may be viewed as a Schrödinger-type equation for a wave function of a particle moving in a two-dimensional potential given by the ν^2/r^2 term in equation (11). The characteristic scale of the Schrödinger-type equation (11) is $r_{\nu} = \nu/\lambda$. A point $r = r_{\nu}$ may be viewed as a classical reflection point. Solutions are exponentially decaying for smaller radii and have a wavelike structure at larger radii. This analogy extends even further. When approaching the classical reflection point a common approach in quantum mechanics is to use the Airy function approximation to the solutions of the Schrödinger equation. This corresponds to the transition region in the Bessel function expansion, when relations (eq. [12]) are no longer valid and one should use the Airy function expansion. We will discuss the various regimes of the Airy function expansion in § 6.

The WKB method breaks down when the radial wavelength becomes comparable to the inhomogeneity scale of the "potential." For a given radius this occurs for

$$\nu \approx \frac{\omega r}{c} \approx \frac{r}{\bar{\lambda}} = 10^8 \tag{22}$$

for $r = 10^9$ cm and the wavelength $\bar{\lambda} = 10$ cm.

The breakdown of the WKB method may be illustrated graphically using a different analogy. On a two-dimensional plane consider the surfaces of a constant phase of Hankel function $H_\nu^{(1)}$

$$\phi = \arctan\left[\frac{Y_\nu(\lambda r)}{J_\nu(\lambda r)} + \frac{2\pi l}{\nu}\right], \qquad l = 0, 1, 2, \ldots \nu - 1 \tag{23}$$

(Fig. 2). This whole picture is rotating with a frequency ω. The points when the argument equals order, i.e., the "classical reflection point," may be regarded as light cylinder. For radii much larger than the "light cylinder" radius, $r_\nu = \nu/\lambda$, the surface of a constant phase has a form of an unwinding spiral with a wavelength λ/ν. Near the light cylinder the radial wavelength becomes comparable to the light cylinder radius.

From the quantum mechanical point of view, the kinetic energy, given by the derivative term in equation (11), becomes zero at the light cylinder radius. This is the classical reflection point. For larger radii the kinetic energy is real and positive: the solution of the "Schrödinger equation" (eq. [11]) has a wavelike structure. For smaller radii the kinetic energy is complex: the solution of the "Schrödinger equation" (eq. [11]) has exponentially decaying form, modified by the presence of the of the reflection point at $r = 0$.

2.2. *Wave-Particle Interaction*

A considerable simplification of the wave-particle interaction in cylindrical coordinates can be made when the interaction occurs in the WKB limit, or, equivalently, when we can use the expansion in tangent for the Bessel functions. From equation (12) we find a lower limit on the azimuthal wavenumber,

$$\nu \geq (\lambda r)^{2/3} \approx 10^5 , \tag{24}$$

for the observed wavelength 10 cm and the curvature radius 10^9 cm. From equations (22) and (24) it follows that for a given radius and given wavelength cylindrical waves with azimuthal wavenumbers in the range $(\lambda r)^{2/3} \leq (\lambda r)$ can be considered in the WKB limit, i.e., in the plane-wave approximation.

It is possible to picture graphically the resonant and nonresonant interaction of a particle moving along the circular trajectory. The pattern in Figure 2 may be thought of as rotating with a frequency $\lambda c/\nu$. A particle at a given distance r_0 with a given velocity v_ϕ is rotating with an angular frequency $\Omega = v_\phi/r_0$. A resonant interaction occurs when these two frequencies are equal. In vacuum,

$$r_0 = \nu\beta_\phi\,\lambda \approx r_\nu\left(1 - \frac{1}{2\gamma^2}\right), \tag{25}$$

where $r_\nu = \nu/\lambda$ is the light cylinder radius for the mode ν. So, in vacuum, the resonance between a particle and a wave always happens inside the light cylinder.

3. SYNCHROTRON EMISSION CONSIDERED IN CYLINDRICAL COORDINATES

In this section we find electromagnetic fields from a particle executing helical motion in vacuum and rederive the conventional expressions for the cyclotron (or curvature) emissivities using cylindrical coordinates. The results of this section show how the conventional expressions for the synchrotron emissivity in vacuum can be obtained using our approach.

We first find the expansion of electromagnetic fields due to a particle executing helical motion in vacuum and in a homogeneous dielectric in terms of normal vector modes in cylindrical coordinates. This is accomplished by the use of dyadic Green's function that gives a response of a vector field to a vector source. We then show how conventional expressions for synchrotron or curvature emissivities can be obtained by two different methods: first by calculating an emissivity into a given cylindrical mode using an expansion of a current in terms of the normal vector modes and secondly by calculating Poynting flux through cylindrical surface. These calculations show how the resonant wave-particle interaction can be reformulated for cylindrical waves for a particular choice of helical motion.

3.1. *Dyadic Green's Function*

In order to find the electromagnetic fields produced by a given charge distribution in a medium is it necessary to know the Green's function for the vector d'Alembert's equation,

$$\Box \boldsymbol{F}(\boldsymbol{r}, t) = \boldsymbol{Q}(\boldsymbol{r}, t) . \tag{26}$$

The solution of the equation (26) may be written using *dyadic* Green's function $\mathcal{G}(\boldsymbol{r}, t, \boldsymbol{r}', t')$

$$\boldsymbol{F}(\boldsymbol{r}, t) = \int d\boldsymbol{r}'\, dt'\, \mathcal{G}(\boldsymbol{r}, t, \boldsymbol{r}', t') \cdot \boldsymbol{Q}(\boldsymbol{r}', t') . \tag{27}$$

In what follows we limit ourselves to the sources changing with time $\propto e^{i\omega t}$. Then in vacuum d'Alembert's equation reduces to Helmholtz equation

$$\text{curl curl } \boldsymbol{F}(\boldsymbol{r}, \omega) - \frac{\omega^2}{c^2} \boldsymbol{F}(\boldsymbol{r}, \omega) = \boldsymbol{Q}(\boldsymbol{r}, \omega) , \tag{28}$$

where $\boldsymbol{F}$ is electric or magnetic field and $\boldsymbol{Q}$ is a source function. For a single particle source, which is a delta function in coordinates, equation (28) becomes an equation for the Green's function. To find Green's function we first have to find the solutions to the homogeneous equation with $\boldsymbol{Q} = 0$. The solutions to the homogeneous vector wave equation can be obtained from the solution of the scalar wave equation

$$\Delta\Psi - \frac{\omega^2}{c^2} \Psi = 0 , \tag{29}$$

$$\Psi(\nu, k_x, \lambda) = e^{i(\nu\phi + k_x x)} Z_\nu(\lambda r) \tag{30}$$

(Z_ν is any Bessel function) using the relations

$$\boldsymbol{E}^{(t)} = \text{curl } (\boldsymbol{e_x} \Psi) , \qquad \boldsymbol{E}^{(lt)} = \text{curl curl } (\boldsymbol{e_x} \Psi) . \tag{31}$$

Equations (31) are eigenfunction for the scalar wave equation. The eigenfunctions for the vector wave equation (28) are (e.g., eqs. [6.5]–[6.6] of Tai 1994)

$$\boldsymbol{L} = \frac{1}{\lambda} \text{ grad } \Psi = e^{i(\nu\phi + k_x x)} \left(Z'_\nu \boldsymbol{e_r} + \frac{i\nu}{\lambda r} Z_\nu \boldsymbol{e_\phi} + \frac{ik_x}{\lambda} Z_\nu \boldsymbol{e_x} \right) ,$$

$$\boldsymbol{M} = \frac{1}{\lambda} \text{ curl } (\boldsymbol{e_x} \Psi) = e^{i(\nu\phi + k_x x)} \left(i \frac{\nu}{\lambda r} Z_\nu \boldsymbol{e_r} - Z'_\nu \boldsymbol{e_\phi} \right) ,$$

$$\boldsymbol{N} = \frac{c}{\omega\lambda} \text{ curl curl } (\boldsymbol{e_x} \Psi) = \frac{c\lambda}{\omega} e^{i(\nu\phi + k_x x)} \left(i \frac{k_x}{\lambda} Z'_\nu \boldsymbol{e_r} - \frac{k_x \nu}{\lambda^2 r} Z_\nu \boldsymbol{e_\phi} + Z_\nu \boldsymbol{e_x} \right) . \tag{32}$$

We note that curl $\boldsymbol{L} = 0$. $\boldsymbol{L}$ is an expansion in eigenfunctions of the longitudinal electric field from the point source and $\boldsymbol{M}$ and $\boldsymbol{N}$ are expansions in eigenfunctions of the transverse fields.

The orthogonality properties of functions (eq. [32]) are

$$\int d\boldsymbol{r} \begin{pmatrix} \boldsymbol{L}^*(r, \lambda', k'_x, \nu') \\ \boldsymbol{M}^*(r, \lambda', k'_x, \nu') \\ \boldsymbol{N}^*(r, \lambda', k'_x, \nu') \end{pmatrix} \begin{pmatrix} \boldsymbol{L}(r, \lambda, k_x, \nu) \\ \boldsymbol{M}(r, \lambda, k_x, \nu) \\ \boldsymbol{N}(r, \lambda, k_x, \nu) \end{pmatrix} = (2\pi)^2 (1 + \delta_{\nu,0}) \frac{\delta(\lambda - \lambda')}{\lambda} \delta(k_x - k'_x) \delta_{\nu - \nu'} . \tag{33}$$

For the particle moving along trajectory $\boldsymbol{r}_0(t)$ the eigenfunction expansion of the scalar Green's function for the scalar wave equation

$$\text{curl curl } \Psi - \frac{\omega^2}{c^2} \Psi = -4\pi \int dt\, e^{i\omega t} \frac{\delta[r - r_0(t)]}{r} \delta[x - x_0(t)] \delta[\phi - \phi_0(t)] \tag{34}$$

is

$$G(r, r_0, \lambda, \nu, k_x) = i\pi\delta(\omega - \nu\Omega - k_x v_x) \begin{cases} J_\nu(\lambda r) H_\nu^{(1)}(\lambda r_0) & \text{if } r \le r_0 , \\ J_\nu(\lambda r_0) H_\nu^{(1)}(\lambda r_0) & \text{if } r \ge r_0 . \end{cases} \tag{35}$$

This particular choice of Bessel and Hankel functions ensures that the solution is regular at zero and corresponds to the outgoing wave at infinity.

The corresponding dyadic Green's functions for the vector wave equation (28) can be found using the relations

$$\mathscr{G}_B(\boldsymbol{r}, \boldsymbol{r}_0) = \text{curl } \mathscr{I} G(\boldsymbol{r}, \boldsymbol{r}_0) , \tag{36}$$

$$\mathscr{G}_E(\boldsymbol{r}, \boldsymbol{r}_0) = \left(\mathscr{I} + \frac{\omega^2}{c^2} \nabla\nabla \right) G(\boldsymbol{r}, \boldsymbol{r}_0) , \tag{37}$$

where $\mathscr{I}$ is a unity matrix, $G(\boldsymbol{r}, \boldsymbol{r}_0)$ is the scalar Green's function, and $\mathscr{G}_B(\boldsymbol{r}, \boldsymbol{r}_0)$ and $\mathscr{G}_E(\boldsymbol{r}, \boldsymbol{r}_0)$ are magnetic and electric dyadic Green's functions.

The eigenfunction expansion of the electric dyadic Green's function is

$$\mathscr{G}_E(r, r_0, \lambda, \nu, k_x) = i\pi(2 - \delta_{\nu,0}) \delta(\omega - k_\phi v_\phi - k_x u_d) \times [\boldsymbol{L}^*(\lambda r) \otimes \boldsymbol{L}(\lambda r_0) + \boldsymbol{M}^*(\lambda r) \otimes \boldsymbol{M}(\lambda r_0) + \boldsymbol{N}^*(\lambda r) \otimes \boldsymbol{N}(\lambda r_0)] , \tag{38}$$

which, using equations (32), gives

$$\begin{aligned}\mathcal{G}_E(r, r_0, \lambda, \nu, k_x) = i\pi(2-\delta_{\nu,0})\Bigg\{&\left[H_\nu^{(1)\prime}(\lambda r)\boldsymbol{e_r} - \frac{i\nu}{\lambda r} H_\nu^{(1)}(\lambda r)\boldsymbol{e}_\phi - \frac{ik_x}{\lambda} H_\nu^{(1)}(\lambda r)\boldsymbol{e_x}\right]\\ &\otimes\left[J'_\nu(\lambda r_0)\boldsymbol{e_r} + \frac{i\nu}{\lambda r_0} J_\nu(\lambda r_0)\boldsymbol{e}_\phi + \frac{ik_x}{\lambda} J_\nu(\lambda r_0)\boldsymbol{e_x}\right]\\ &+\frac{c^2\lambda^2}{\omega^2}\left[-i\frac{k_x}{\lambda} H_\nu^{(1)\prime}(\lambda r)\boldsymbol{e_r} - \frac{k_x\nu}{\lambda^2 r} H_\nu^{(1)}(\lambda r)\boldsymbol{e}_\phi + H_\nu^{(1)}(\lambda r)\boldsymbol{e_x}\right]\\ &\otimes\left[i\frac{k_x}{\lambda} J'_\nu(\lambda r_0)\boldsymbol{e_r} - \frac{k_x\nu}{\lambda^2 r_0} J_\nu(\lambda r_0)\boldsymbol{e}_\phi + J_\nu(\lambda r_0)\boldsymbol{e_x}\right]\\ &+\left[-i\frac{\nu}{\lambda r} H_\nu^{(1)}(\lambda r)\boldsymbol{e_r} - H_\nu^{(1)\prime}(\lambda r)\boldsymbol{e}_\phi\right]\otimes\left[i\frac{\nu}{\lambda r_0} J_\nu(\lambda r_0)\boldsymbol{e_r} - J'_\nu(\lambda r_0)\boldsymbol{e}_\phi\right]\Bigg\}\,. \end{aligned} \tag{39}$$

The first term here is just the eigenfunction expansion of the Green's dyadic for the point source.

The magnetic Green's function (the Fourier transform in ϕ, x, and t) for the vector wave equation is

$$\mathcal{G}_B(r, r_0, \lambda, \nu, k_x) = i\pi(2-\delta_{\nu,0})[\boldsymbol{N}^*(\lambda r)\otimes\boldsymbol{M}(\lambda r_0) + \boldsymbol{M}^*(\lambda r)\otimes\boldsymbol{N}(\lambda r_0)]\,, \tag{40}$$

which, using equations (37) gives

$$\begin{aligned}\mathcal{G}_B(r, r_0, \lambda, \nu, k_x) = {}& i\pi(2-\delta_{\nu,0})\frac{c\lambda}{\omega}\\ &\times\Bigg\{\left[-i\frac{\nu}{\lambda r} H_\nu^{(1)}(\lambda r)\boldsymbol{e_r} - H_\nu^{(1)\prime}(\lambda r)\boldsymbol{e}_\phi\right]\otimes\left[i\frac{k_x}{\lambda} J'_\nu(\lambda r_0)\boldsymbol{e_r} - \frac{k_x\nu}{\lambda^2 r_0} J_\nu(\lambda r_0)\boldsymbol{e}_\phi + J_\nu(\lambda r_0)\boldsymbol{e_x}\right]\\ &+\left[-i\frac{k_x}{\lambda} H_\nu^{(1)\prime}(\lambda r)\boldsymbol{e_r} - \frac{k_x\nu}{\lambda^2 r} H_\nu^{(1)}(\lambda r)\boldsymbol{e}_\phi + H_\nu^{(1)}(\lambda r)\boldsymbol{e_x}\right]\otimes\left[i\frac{\nu}{\lambda r_0} J_\nu(\lambda r_0)\boldsymbol{e_r} - J'_\nu(\lambda r_0)\boldsymbol{e}_\phi\right]\Bigg\}\,. \end{aligned} \tag{41}$$

The solution to the inhomogeneous vector wave equation (28) is given by the integral with a kernel given by the dyadic Green's function

$$\boldsymbol{F}(r, \lambda, \nu, k_x) = \int r'dr'\mathcal{G}(r, r', \lambda, \nu, k_x)\cdot\boldsymbol{Q}(r', \lambda, \nu, k_x, \omega)\,, \tag{42}$$

where $\boldsymbol{Q}(r', \lambda, \nu, k_x)$ is a Fourier transform of $\boldsymbol{Q}(\boldsymbol{r})$.

3.2. *Synchrotron Emissivity into Cylindrical Mode*

In this subsection we show how conventional emissivities into plane waves can be modified to obtain similar relations for the emissivities into cylindrical modes. To simplify the analysis we consider in detail only emissivity of a charge moving along a spiral trajectory. An important factor that simplifies our calculations and allows one to obtain the emissivity in a simple closed form is that for helical motion a particle always stays at the same radial coordinates. If the particles were not on the ground gyrational level, then in calculating the emissivity it would be necessary to average the gyrational motion of particles taking into account the radial dependence of the vector normal modes.

Following the conventional approach in the theory of electromagnetic wave-particle interaction (e.g., Melrose 1978), we can identify the source of energy in the wave with the work done by the extraneous current,

$$P = \int d\boldsymbol{r}\,dt\,\boldsymbol{E}(\boldsymbol{r}, t)\cdot\boldsymbol{j}(\boldsymbol{r}, t)\,. \tag{43}$$

Using the power theorem for the Fourier transform, i.e.,

$$\int dx f(x)g(x) = \int\frac{dk}{2\pi} f(k)g(k) \tag{44}$$

[where $f(k)$ and $g(k)$ are Fourier transforms of $f(t)$ and $g(t)$], we can rewrite equation (43) for the Fourier transforms of $\boldsymbol{E}$ and $\boldsymbol{j}$ in time, ϕ and x, to be[4]

$$P = \sum_\nu\int r\,dr\,\frac{d\omega\,dk_x}{(2\pi)^3}\boldsymbol{E}(r, \nu, k_x, \omega)\cdot\boldsymbol{j}^*(r, \nu, k_x, \omega)\,. \tag{45}$$

[4] It is also possible to make an expansion of the vector fields $\boldsymbol{E}$ and $\boldsymbol{j}$ in terms of the normal modes $\boldsymbol{L}$, $\boldsymbol{N}$, $\boldsymbol{M}$ and use later the dyadic Green's function $\mathcal{G}(\lambda, \nu, k_x, \omega)$.

The electric field in equation (45) may be expressed in terms of the dyadic Green's function and the current (see eq. [42]). For a particular case of a particle executing helical motion with velocity $\boldsymbol{v} = \{0, v_\phi, v_x\}$, the Fourier transform of the current density $\boldsymbol{j}(\boldsymbol{r}) = qv\delta(\boldsymbol{r} - \boldsymbol{r}_0)$ is

$$\begin{aligned} \boldsymbol{j}(r, \lambda, \nu, k_x, \omega) &= \int dx\, e^{-ik_x x} \int dt\, e^{i\omega t} \int d\phi\, e^{-i\nu\phi} \boldsymbol{j}(\boldsymbol{r}) , \\ &= q\boldsymbol{v}\delta(\omega - \nu\Omega - k_x v_x) \frac{\delta(r - r_0)}{r} , \end{aligned} \tag{46}$$

so that the electric field is given by ($\nu \neq 0$)

$$\boldsymbol{E}(r, \lambda, \nu, k_x, \omega) = \frac{2\pi q\omega}{c} \sum_i \boldsymbol{N}_i^*(\lambda r)[\boldsymbol{N}_i(\lambda r_0) \cdot \boldsymbol{v}]\delta(\omega - \nu\Omega - k_x v_x) , \tag{47}$$

where $\boldsymbol{N}_i = \boldsymbol{L}, \boldsymbol{N}, \boldsymbol{M}$. Inserting equation (47) in equation (45) we find the radiated energy

$$P = \frac{2\pi q^2\omega}{c} \sum_i |\boldsymbol{N}_i(\lambda r_0) \cdot \boldsymbol{v}|^2 , \tag{48}$$

which gives the cyclotron emissivity

$$\eta(\omega, k_x) = q^2\omega \sum_\nu \left\{ J_\nu'^2(\lambda r_0) v_\phi^2 + \left[\frac{k_x c}{\lambda} - \frac{\omega v_x}{\lambda c} \right]^2 J_\nu^2(\lambda r_0) \right\} \delta(\omega - \nu\Omega - k_x v_x) , \tag{49}$$

which is exactly the cyclotron emissivity per unit interval $d\phi\, d\omega\, dk_x$.

3.3. *Alternative Derivation of Cyclotron Emissivity*

We can use an alternative method to derive the cyclotron emissivity into cylindrical modes by finding the electromagnetic fields produced by the given charge distribution and integrate the Pointing vector over the cylindrical surface. Using equations (39), (42), and (46), we find the transverse component of the electric field,

$$\begin{aligned} \boldsymbol{E}_\perp = \pi q(2 - \delta_{\nu,0}) \frac{\omega}{c} \delta(\omega - \nu\Omega - k_x v_x) \Big\{ &-J_\nu'(\lambda r_0) v_\phi \left[-i \frac{\nu}{\lambda r} H_\nu^{(1)}(\lambda r) \boldsymbol{e}_r - H_\nu^{(1)\prime}(\lambda r) \boldsymbol{e}_\phi \right] \\ &- \frac{c^2\lambda^2}{\omega^2} \left(\frac{k_x \nu\Omega}{\lambda^2} - v_x \right) J_\nu(\lambda r_0) \left[-i \frac{k_x}{\lambda} H_\nu^{(1)\prime}(\lambda r) \boldsymbol{e}_r - \frac{k_x \nu}{\lambda^2 r} H_\nu^{(1)}(\lambda r) \boldsymbol{e}_\phi + H_\nu^{(1)}(\lambda r) \boldsymbol{e}_x \right] \Big\} . \end{aligned} \tag{50}$$

In the wave zone, $\lambda r \gg 1$, and for $\nu \gg 1$ this can be simplified

$$\boldsymbol{E}_\perp = iq\pi \sqrt{\frac{2}{\pi\lambda r}} \frac{\omega}{c} e^{i\lambda r} \left[J_\nu'(\lambda r_0) v_\phi \boldsymbol{e}_\phi - \frac{c^2\lambda^2}{\omega^2} \left(-\frac{k_x \nu\Omega}{\lambda^2} + v_x \right) J_\nu(\lambda r_0) \left(i \frac{k_x}{\lambda} \boldsymbol{e}_r + \boldsymbol{e}_x \right) \right] \delta(\omega - \nu\Omega - k_x v_x) . \tag{51}$$

Similarly, we find for the magnetic field

$$\begin{aligned} \boldsymbol{B} = i\pi q(2 - \delta_{\nu,0})\lambda \Big\{ &\left[-i \frac{\nu}{\lambda r} H_\nu^{(1)}(\lambda r) \boldsymbol{e}_r - H_\nu^{(1)\prime}(\lambda r) \boldsymbol{e}_\phi \right] \left(-\frac{k_x \nu}{\lambda^2 r_0} v_\phi + v_x \right) J_\nu(\lambda r_0) \\ &- \left[-i \frac{k_x}{\lambda} H_\nu^{(1)\prime}(\lambda r) \boldsymbol{e}_r - \frac{k_x \nu}{\lambda^2 r} H_\nu^{(1)}(\lambda r) \boldsymbol{e}_\phi + H_\nu^{(1)}(\lambda r) \boldsymbol{e}_x \right] J_\nu'(\lambda r_0) v_\phi \Big\} \delta(\omega - \nu\Omega - k_x v_x) , \end{aligned} \tag{52}$$

which in the limit $\lambda r \gg 1$ and $\nu \gg 1$ gives

$$\boldsymbol{B} = iq\pi \sqrt{\frac{2}{\pi\lambda r}} \lambda e^{i\lambda r} \delta(\omega - \nu\Omega - k_x v_x) \left[iJ_\nu(\lambda r_0) \left(\frac{k_x \nu\Omega}{\lambda^2} - v_x \right) \boldsymbol{e}_\phi + \left(\frac{k_x}{\lambda} \boldsymbol{e}_r + \boldsymbol{e}_x \right) J_\nu'(\lambda r_0) v_\phi \right] . \tag{53}$$

The Pointing flux, $\boldsymbol{S} = c\boldsymbol{E}^* \times \boldsymbol{B}/(4\pi)$ for $r \to \infty$ is then

$$\boldsymbol{S} = \frac{q^2\omega}{r} \left[J_\nu'^2 v_\phi^2 + \left(\frac{k_x c}{\lambda} - \frac{\omega v_x}{\lambda c} \right)^2 J_\nu^2(\lambda r_0) \right] \left(\boldsymbol{e}_r + \frac{k_x}{\lambda} \boldsymbol{e}_x \right) \delta(\omega - \nu\Omega - k_x v_x)\, dl\, d\omega\, dk_x , \tag{54}$$

where $dl = r\, d\phi$ is a unit arc length of a cylinder.

We are interested only in the radial component of the flux. Equating the Pointing flux to the emissivity we find the cyclotron emissivity (eq. [49]).

4. WAVES IN AN ISOTROPIC DIELECTRIC

We expect that the dielectric properties of a medium, i.e., collective effects, will play an important role in defining the properties of the wave-particle interaction. So, as a next step, we consider in brief the waves in an isotropic, frequency-dispersive medium. In such a medium the wave equation is

$$\text{curl curl } \boldsymbol{E} - \frac{\partial^2 \boldsymbol{D}}{\partial t^2} = 0 \, . \tag{55}$$

In an isotropic, dispersive medium the relation between the electric induction $\boldsymbol{D}$ and electric field may be written as

$$\boldsymbol{D}(\omega) = \epsilon(\omega)\boldsymbol{E}(\omega) \, . \tag{56}$$

From equations (2) and (56) it follows that in a dielectric, the solutions of the wave equation are

$$\boldsymbol{E}^{(lt)}(r, k_x, v, \omega) = E^{(lt)}\left[\frac{ik_x}{\hat{\lambda}^2} J'_v(\hat{\lambda}r)\boldsymbol{e_r} - \frac{vk_x}{\hat{\lambda}^2 r} J_v(\hat{\lambda}r)\boldsymbol{e}_\phi + J_v(\hat{\lambda}r)\boldsymbol{e_x}\right] , \tag{57}$$

$$\boldsymbol{B}^{(lt)}(r, k_x, v, \omega) = \frac{\omega\epsilon}{\hat{\lambda}^2 c} E^{(lt)}\left[\frac{vJ_v(\hat{\lambda}r)}{r} \boldsymbol{e_r} + iJ'_v(\hat{\lambda}r)\boldsymbol{e}_\phi\right] , \tag{58}$$

$$\boldsymbol{E}^{(t)}(r, k_x, v, \omega) = E^{(t)}\left[\frac{iv}{r} J_v(\hat{\lambda}r)\boldsymbol{e_r} - J'_v \boldsymbol{e}_\phi\right] , \tag{59}$$

$$\boldsymbol{B}^{(t)}(r, k_x, v, \omega) = E^{(t)}\left[\frac{k_x c}{\omega} J'_v(\hat{\lambda}r)\boldsymbol{e_r} + \frac{ik_x v}{r\omega} J_v(\hat{\lambda}r)\boldsymbol{e}_\phi - i\,\frac{c\hat{\lambda}^2}{\omega} J_v(\hat{\lambda}r)\boldsymbol{e_x}\right] , \tag{60}$$

where $\hat{\lambda} = (\omega^2/c^2\epsilon - k_x^2)^{1/2}$.

4.1. *Cherenkov Emission in Cylindrical Coordinates*

As another didactic example we illustrate the features of the fields in the case of Cherenkov-type emission taking as an example scalar waves. The solution for the vector waves can then be obtained using the general relations between the solutions of the scalar and vector wave equations.

In cylindrical coordinates the equation for the eigenfunction expansion of the scalar Green's function is given by equation (35) with $r_0 = 0$. The only remaining term is

$$G(r, r_0, \lambda, v = 0, k_x) = i\pi\delta(\omega - k_x v_x)H_0(\lambda r) \, . \tag{61}$$

Using the argument of the δ function, we find

$$\lambda = \sqrt{\frac{\omega^2\epsilon}{c^2} - k_x^2} = k_x\sqrt{\epsilon\beta_x^2 - 1} \, , \tag{62}$$

where $\beta_x = v_x/c$. For $\epsilon\beta_x^2 < 1$ the argument of the Hankel function is complex, so that the fields decay exponentially at large distances. For $\epsilon\beta_x^2 > 1$ the argument of the Hankel function is real, which corresponds to outgoing waves. For $\epsilon\beta_x^2 = 1$ the Green's function (eq. [61]) has a discontinuity, this is the shock front corresponding to the Cherenkov cone.

5. DISPERSION RELATION IN AN INFINITELY STRONG CURVED MAGNETIC FIELD: VLASOV APPROACH

The next problem that we will consider is a dielectric tensor in cylindrical coordinates for plasma in the *infinitely strong* circular magnetic field. In this particular case, it is possible to relate the electric field and the electric induction in cylindrical coordinates through a conventional dielectric tensor and not the dielectric tensor operator, in other words the dielectric tensor becomes local.

5.1. *One-dimensional Plasma in Inhomogeneous Magnetic Field*

The equations governing the electrodynamics of a plasma in a strong, weakly inhomogeneous magnetic field are the wave equation

$$\text{curl curl } \boldsymbol{E}(\boldsymbol{r}, t) + \frac{\partial^2 \boldsymbol{E}(\boldsymbol{r}, t)}{\partial t^2} = \frac{4\pi}{c}\frac{\partial \boldsymbol{j}(\boldsymbol{r}, t)}{\partial t} \tag{63}$$

and the collisionless Vlasov equation

$$\frac{\partial f(\boldsymbol{r}, t, \boldsymbol{p})}{\partial t} + \left(\boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right) f(\boldsymbol{r}, t, \boldsymbol{p}) + q\left[\boldsymbol{E} + \frac{1}{c}\,\boldsymbol{v}\times(\boldsymbol{B} + \boldsymbol{B}^0)\right]\frac{\partial f(\boldsymbol{r}, t, \boldsymbol{p})}{\partial \boldsymbol{p}} = 0 \, , \tag{64}$$

where $\boldsymbol{B}^0$ is the external magnetic field, $f(\boldsymbol{r}, t, \boldsymbol{p})$ is a particle distribution function, $\boldsymbol{j}(\boldsymbol{r}, t)$ is a current, and q is a charge of particles.

We represent the distribution function as a sum of the unperturbed distribution and fluctuating part, $f(\boldsymbol{r}, t) = f^{(0)}(\boldsymbol{r}) + f^{(1)}(\boldsymbol{r}, t)$. We assume that the unperturbed distribution function is stationary, i.e., $\partial f^{(0)}/\partial t = 0$. A real plasma present on the open field lines of the pulsar magnetosphere may not satisfy this condition. It can be hydrodynamically unstable with a growth rate much smaller than the dynamical time (which is equal to the pulsar period). But since in our model problem the plasma circulates infinitely long along the circular magnetic field, this condition is necessary to impose. Then the unperturbed distribution function satisfies the equation

$$v \cdot \frac{\partial}{\partial \boldsymbol{r}} f^{(0)} + \left(\frac{e}{c} \boldsymbol{v} \times \boldsymbol{B}^0\right) \frac{\partial f^{(0)}}{\partial \boldsymbol{p}} = 0 . \tag{65}$$

In other words, $f^{(0)}$ must be a function of the integrals of motion. The integrals of motion for a particle in azimuthally symmetrical static magnetic field have been considered by Mikhailovski (1992). The relevant constants of motion are the radial coordinate of the gyration center

$$r_g = r + \frac{v_\phi}{\omega_B} , \tag{66}$$

parallel velocity of the gyration center V_ϕ, which, up to terms linear in the small parameter of the adiabatic approximation coincides with the parallel velocity of the particle $V_\phi = v_\phi$ and the quantity $V_\perp$ (Mikhailovski 1992, eq. [15.11]), which, in turn, up to terms linear in the same small parameter coincides with the gyrational part of the particle velocity $V_\perp = v_\perp$ (see below). The most general form of the distribution function is then

$$f^{(0)} = F(V_\perp, V_\phi, r_g) , \tag{67}$$

where F is an arbitrary function. We also note that since the distribution function is independent of x and ϕ by our choice and cannot depend on the gyrational phase of the particle, the first term in equation (65) is zero.

Following Mikhailovski (1992), we find that the inhomogeneity of the external magnetic field results in an equilibrium electric current (diamagnetic current) given by (Mikhailovski 1992)

$$j_x = \frac{c}{B}\left(\frac{\partial P_\perp}{\partial r} + \frac{P_\perp - P_\parallel}{r}\right) , \tag{68}$$

where $P_\perp$ and $P_\parallel$ are transverse and parallel plasma pressures, defined by

$$P_\perp = \sum \int v_\perp p_\phi \, f^{(0)} \, d\boldsymbol{v} ,$$

$$P_\parallel = \sum \int v_\phi p_\phi \, f^{(0)} \, d\boldsymbol{v} . \tag{69}$$

Here $v_\perp^2 = v_r^2 + (v_x - u_d)^2$, u_d is a drift velocity.

If one assumes Maxwell equations and equation (68), the condition of the transverse equilibrium becomes (Mikhailovski 1992)

$$\frac{\partial}{\partial r}\left(P_\perp + \frac{B^2}{8\pi}\right) + \frac{B^2}{4\pi r} + \frac{P_\perp - P_\parallel}{r} = 0 . \tag{70}$$

In the case $P_\perp = 0$ and $f^{(0)} \propto n^{(0)} \delta(p_\phi - p_\phi^{(0)})$ the diamagnetic current (eq. [68]) may be written as

$$j_x = q n^{(0)} u_d , \qquad u_d = \frac{\gamma_\phi^{(0)} v_\phi^{(0)2}}{\omega_B r} , \tag{71}$$

where u_d *turns out* to be equal to the drift velocity directed along the binormal to the magnetic field. We note here that there is an alternative way of deriving equation (71) from the single particle equation of motion using averaging over fast rotations around magnetic field. After averaging over fast rotation the resulting equation of motion for the location of the gyration center contains inertial forces due to the inhomogeneity of the magnetic field. These inertial forces result in a drift motion that produces diamagnetic current. These results imply that our initial state, consisting of plasma circulating along magnetic field and drifting along the binormal, is in a transverse hydrodynamic equilibrium.

Returning to the Vlasov equation, we make a Fourier transform in time, ϕ and x [$\propto \exp\{i(v\phi + k_x x - \omega t)\}$] and find an equation for $f^{(1)}(r, m, k_x, \omega)$,

$$i\left(-\omega + \frac{v_\phi v}{r} + k_x v_x + v_r \frac{\partial}{\partial r}\right) f^{(1)}(r, m, k_x, \omega, \boldsymbol{p}) + q\left[\boldsymbol{E}(r, m, k_x, \omega) + \frac{1}{c} \boldsymbol{v} \times \boldsymbol{B}(r, m, k_x, \omega)\right] \frac{\partial f^{(0)}}{\partial \boldsymbol{p}} = 0 . \tag{72}$$

5.2. *Infinitely Strong Magnetic Field*

Equation (72) is a *differential* equation for $f^{(1)}(r, m, k_x, \omega, \boldsymbol{p})$. This reflects the fact that in cylindrical coordinates (any curvilinear coordinates) the relation between electric field and electric induction involves a dielectric tensor *operator*.[5] The case of infinitely strong magnetic field allows a considerable simplification: in this case the operator relation is reduced to algebraic, i.e., to the conventional dielectric tensor.

[5] Mathematically, the difference is that dielectric tensor acts in a space tangent to the vector field at some point, whereas the dielectric tensor operator acts on a vector field itself.

In the case of infinitely strong magnetic fields the velocities across the field are zero: v_x, $v_r = 0$ and equation (72) can be solved for $f^{(1)}(r, m, k_x, \omega)$,

$$f^{(1)}(r, m, k_x, \omega, \boldsymbol{p}) = \frac{iqE_\phi}{c(\omega - \Omega v)} \frac{\partial f^{(0)}}{\partial p_\phi} . \tag{73}$$

The corresponding current density is

$$j_\phi = q \int f^{(1)}(r, m, k_x, \omega, p_\phi) v_\phi \, dp_\phi = \frac{q^2 i}{c} \int dp_\phi \frac{v_\phi}{c(\omega - \Omega v)} \frac{\partial f^{(0)}}{\partial p_\phi} E_\phi , \tag{74}$$

where

$$f^{(1)}(r, m, k_x, \omega, p_\phi) = \int dp_x \, dp_r \, f^{(1)}(r, m, k_x, \omega, \boldsymbol{p}) \tag{75}$$

is a one-dimensional distribution function.

We can now introduce a dielectric tensor

$$\epsilon_{ij}(r, v, k_x, \omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 - K & 0 \\ 0 & 0 & 1 \end{pmatrix} , \tag{76}$$

where

$$K = \frac{4\pi q^2}{m_e} \int \frac{dp_\phi}{\gamma^3} \frac{f^{(0)}}{(\omega - \Omega v)^2} = \frac{4\pi q^2}{\omega} \int dp_\phi \frac{v_\phi}{\omega - \Omega v} \frac{\partial f^{(0)}}{\partial p_\phi} . \tag{77}$$

Note that both $j_i(r, v, k_x, \omega)$ and $E_j(r, v, k_x, \omega)$ are conventional Fourier transforms, so that the Hermitian, $\epsilon^{\rm H}_{ij}(r, v, k_x, \omega)$, and anti-Hermitian, $\epsilon^{A}_{ij}(r, v, k_x, \omega)$, parts of the dielectric tensor (eq. [76]) satisfy the usual conditions

$$\begin{aligned} \epsilon^{\rm H}_{ij}(r, v, k_x, \omega) &= \epsilon^{\rm H}_{ij}(r, -v, -k_x, -\omega) , \\ \epsilon^{A}_{ij}(r, v, k_x, \omega) &= -\epsilon^{\rm H}_{ij}(r, -v, -k_x, -\omega) , \\ \epsilon_{ij}(r, -v, -k_x, -\omega) &= \epsilon^{*}_{ij}(r, -v, -k_x, -\omega) . \end{aligned} \tag{78}$$

The wave equation now reads

$$\begin{aligned} \frac{iv}{r^2} \frac{\partial}{\partial r} (rE_\phi) + ik_x \frac{\partial}{\partial r} E_x + \left[\frac{v^2}{r^2} - \left(\frac{\omega^2}{c^2} - k_x^2\right)\right] E_r = 0 , \\ -\frac{\partial}{\partial r}\left[\frac{1}{r}\frac{\partial}{\partial r}(rE_\phi)\right] + iv \frac{\partial}{\partial r}\left(\frac{E_r}{r}\right) - \frac{k_x v}{r} E_x - \left[\frac{\omega^2}{c^2}(1 - K) - k_x^2\right] E_\phi = 0 , \\ -\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} E_x\right) + \frac{ik_x}{r}\frac{\partial}{\partial r}(rE_r) - \left(\frac{\omega^2}{c^2} - \frac{v^2}{r^2}\right) E_x = 0 . \end{aligned} \tag{79}$$

To illustrate the solutions of equations (79) we show how, in the WKB approximation, the solutions can be obtained for a given dependence $K(r)$ in the case $k_x = 0$. Then we find exact solutions for $k_x = 0$ and constant (independent of r) K. Unfortunately, there seems to be no simple way of solving equations (79) in known functions for $k_x \neq 0$.

5.3. *Propagation with* $k_x = 0$

For $k_x = 0$ the equation for E_x separates from the equations for E_r and E_ϕ. From equation (79) we find

$$\boldsymbol{E}^{(\rm lt)}(r, t) = \exp\{i(v\phi - \omega t)\} J_v\left(\frac{\omega r}{c}\right) \boldsymbol{e}_x , \tag{80}$$

$$\boldsymbol{B}^{(\rm lt)}(r, t) = -\exp\{i(v\phi - \omega t)\} \frac{c}{\omega}\left[\frac{v}{r} J_v\left(\frac{\omega r}{c}\right)\boldsymbol{e}_r + iJ'_v\left(\frac{\omega r}{c}\right)\boldsymbol{e}_\phi\right] . \tag{81}$$

By the use of equations (80) and (81) it is possible to show that the following relation holds,

$$(1 - K) v E_\phi - i \frac{\partial}{\partial r}(rEr) = 0 , \tag{82}$$

which can be derived directly from the relation div $\boldsymbol{D} = 0$, where $D_i = \epsilon_{ij} E_j$ is the electric induction.

Using equation (82), we find

$$\frac{\partial^2}{\partial r^2} E_r + \frac{1}{r}\left(3 + \frac{r}{1 - K}\frac{\partial K}{\partial r}\right)\frac{\partial E_r}{\partial r} + \left[\left(\frac{\omega^2}{c^2} - \frac{v^2}{r^2}\right)(1 - K) + \frac{1}{r^2} + \frac{1}{r(1 - K)}\frac{\partial K}{\partial r}\right] = 0 . \tag{83}$$

Equation (83) can be solved using the WKB method for a given dependence $K(r)$. If $K(r)$ is a constant (as a functions of r), then it is possible to find exact solutions,

$$\boldsymbol{E}^{(t)}(\boldsymbol{r}, t) = \exp\{i(\nu\phi - \omega t)\}\left[\frac{i\nu}{r} J_{\nu\sqrt{1-K}}\left(\frac{\omega r}{c}\sqrt{1-K}\right)\boldsymbol{e}_r - J'_{\nu\sqrt{1-K}}\left(\frac{\omega r}{c}\sqrt{1-K}\right)\boldsymbol{e}_\phi\right], \tag{84}$$

$$\boldsymbol{B}^{(t)}(\boldsymbol{r}, t) = \exp\{i(\nu\phi - \omega t)\}\frac{ic}{\omega}\left[\frac{\omega^2}{c^2}(1-K) + \frac{K\nu^2}{r^2}\right] J_{\nu\sqrt{1-K}}\left(\frac{\omega r}{c}\sqrt{1-K}\right)\boldsymbol{e}_x . \tag{85}$$

5.4. *Curvature Emission in Vacuum and in the Isotropic Medium*

In this section we will show how the conventional emissivities for curvature emission can be obtained in this approach. We will calculate the single particle emissivity as the energy dissipated by the optically active medium. This can be done by relating in equation (45) the local current $\boldsymbol{j}(r, \nu, k_x, \omega)$ and the electric field through the anti-Hermitian part of the dielectric tensor (eq. [76]),

$$j_i(r, \nu, k_x, \omega) = -i\frac{\omega}{4\pi}\epsilon^A_{ij} E_j(r, \nu, k_x, \omega)\frac{\delta(r-r_0)}{r} . \tag{86}$$

To proceed further we define the wave amplitude, $E_{\boldsymbol{N}_i}$, as

$$\boldsymbol{E}(r, \nu, k_x, \omega) = 2\pi \sum_i E_{\boldsymbol{N}_i}\boldsymbol{N}_i(r, \nu, k_x, \lambda)\delta(\lambda - \sqrt{\omega^2/c^2 - k_x^2}) . \tag{87}$$

Then the energy density in the wave W is ($V = \pi l^2 H$)

$$\begin{aligned} W &= \lim_{T\to\infty}\lim_{l\to\infty}\lim_{H\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt\,\frac{1}{H}\int_{-H/2}^{H/2} dx\int_0^{2\phi} d\phi\,\frac{1}{\pi l^2}\int_0^l r\,dr\,\frac{|\boldsymbol{E}(\boldsymbol{r}, t)|^2}{8\pi} , \\ &= \lim_{l\to\infty}\int\frac{r\,dr}{\pi l^2}\frac{1}{(2\pi)^3}\sum_\nu\int dk_x\,d\omega\,\frac{|\boldsymbol{E}(r, \nu, k_x, \omega)|^2}{8\pi} , \\ &= \lim_{l\to\infty}\sum_i\frac{1}{(2\pi)^3}\sum_\nu\int dk_x\,d\omega\,\frac{\pi|E_{\boldsymbol{N}_i}|^2}{2}\int\frac{r\,dr}{\pi l^2}|\boldsymbol{N}_i|^2\left[\delta\left(\lambda - \sqrt{\frac{\omega^2}{c^2} - k_x^2}\right)\right]^2 . \end{aligned} \tag{88}$$

Using the relations

$$\begin{aligned} \lim_{l\to\infty}\int r\,dr J^2(\lambda r) &= \lim_{l\to\infty}\frac{l^2}{2}\left[J_\nu'^2(\lambda r) + \left(1 - \frac{\nu^2}{\lambda^2 l^2}\right)J_\nu^2(\lambda r)\right] = \frac{l}{\pi\lambda} , \\ \lim_{l\to\infty}\frac{1}{l}\left[\delta\left(\lambda - \sqrt{\frac{\omega^2}{c^2} - k_x^2}\right)\right]^2 &= \frac{1}{2\pi}\delta\left(\lambda - \sqrt{\frac{\omega^2}{c^2} - k_x^2}\right) , \end{aligned} \tag{89}$$

we find

$$W = \frac{1}{(2\pi)^3}\sum_\nu\int dk_x\,d\omega\,\frac{|E_{\boldsymbol{N}_i}|^2}{4\pi^2}\delta\left(\lambda - \sqrt{\frac{\omega^2}{c^2} - k_x^2}\right) = \frac{1}{(2\pi)^3}\sum_\nu\int dk_x\,d\omega\,W_{\boldsymbol{N}_i} , \tag{90}$$

where we identified an energy in a vector mode $\boldsymbol{N}_i$ as

$$W_i = \frac{|E_{\boldsymbol{N}_i}|^2}{4\pi^2}\delta\left(\lambda - \sqrt{\frac{\omega^2}{c^2} - k_x^2}\right) . \tag{91}$$

Defining the energy gain of the wave as $-\Gamma_i W_i$ we find using equation (45)

$$-\sum_i\frac{1}{(2\pi)^3}\sum_\nu\int dk_x\,d\omega\,\Gamma_i W_i = -i\frac{\omega}{4\pi}\frac{1}{(2\pi)^3}\sum_\nu\int dk_x\,d\omega\,|E_{\boldsymbol{N}_i}|^2\int r\,dr\,\boldsymbol{N}_i^*(\lambda r)\cdot\epsilon^A\cdot\boldsymbol{N}_i(\lambda r)\frac{\delta(r-r_0)}{r} , \tag{92}$$

so that

$$\Gamma_i = \frac{\omega}{\pi}\boldsymbol{N}_i^*(\lambda r_0)\cdot\epsilon^A\cdot\boldsymbol{N}_i(\lambda r_0) . \tag{93}$$

This expression provides a growth rate for a vector cylindrical mode $\boldsymbol{N}_i = \boldsymbol{N}, \boldsymbol{M}$ due to a current circulating at a radius r_0.

To find the single particle emissivity we assume that the growth of the wave is due to the resonant interaction with plasma particles that emit and absorb wave quanta. We then can identify the single particle probability of emission $w(p_\phi, \nu, k_x, \lambda)$ as

$$\Gamma = \sum_i\int dp_\phi\,w(p_\phi, \nu, k_x, \lambda)\hbar k_\phi\frac{\partial f(\boldsymbol{p})}{\partial p_\phi} . \tag{94}$$

Using the anti-Hermitian part of the dielectric tensor

$$\epsilon''_{\phi\phi} = -i\,\frac{4\pi^2 q^2}{\omega}\int dp_\phi\, v_\phi \frac{\partial f(p_\phi)}{\partial p_\phi}\,\delta(\omega - \nu\Omega)\,, \tag{95}$$

we find

$$w(p_\phi, \nu, k_x, \lambda) = \frac{4\pi q^2 v_\phi^2}{\hbar}\left[J_\nu'^2(\lambda r) + \frac{\nu^2 k_x^2 c^2}{\lambda^2 r^2 \omega^2}\, J_\nu^2(\lambda r)\right]\delta(\omega - \nu\Omega) \tag{96}$$

and emissivity $\eta(\nu, k_x, \omega)$

$$\eta(\omega, k_x) = \sum_\nu \frac{\omega^3 q^2 r_0^2}{\nu^2}\left[J_\nu'^2(\lambda r_0) + \frac{\nu^2 k_x^2 c^2}{\lambda^2 r_0^2 \omega^2}\, J_\nu^2(\lambda r)\right]\delta(\omega - \nu\Omega)\,. \tag{97}$$

This is emissivity per unit range $dk_x\, d\omega\, d\phi$.

Effectively, we calculated the induced energy loss of a particle due to the interaction with a wave and then, using Einstein relations between the induced and spontaneous emission, we find the spontaneous emissivity.

Introducing notations $k_x = \omega/c \sin\theta$, $dk_x = \omega/c \cos\theta\, d\theta$, $\lambda = \omega/c \cos\theta$, and $\beta_\phi = \Omega r/c$ the expression for the curvature emissivity may be reduced to[6]

$$\eta(\omega) = \frac{\omega^2 q^2 v_\phi^2}{2\pi c}\left[J_\nu'^2(\nu\beta_\phi \cos\theta) + \frac{\tan^2\theta}{\beta_\phi^2}\, J_\nu^2(\nu\beta_\phi\cos\theta)\right]\delta\left(\omega - \frac{\nu\beta_\phi c}{r}\right). \tag{98}$$

This is emissivity per unit solid angle $d\Omega$. This is exactly the expression for the single particle curvature emissivity in vacuum (Melrose 1986, eq. [13.62]). We stress that equation (98) was obtained in cylindrical coordinates. It gives a single particle emissivity per unit frequency, which is, of course, independent of the coordinate system used.

As the waves propagate outward from the emitting region, the approximation of an infinitely long cylinder will cease to be true. Then, at some transition region the cylindrical waves will become spherical. It should be possible to consider the details of such transition using Kirchoff integrals over a remote surface where dispersive properties of a medium are not important. Here we only note that the parameter θ introduced above corresponds in a vacuum case to the polar angle in spherical coordinates. This provides a simple formal relation between the spherical and cylindrical coordinates.

In our approach, which treats the wave-particle interaction as resonant, the impossibility of the amplified curvature emission in vacuum, first derived by Blandford (1975), follows from the fact that in vacuum there are no subluminous waves, so that no waves have a phase velocity that could fall into the region where the derivative $\partial f(p_\phi)/\partial p_\phi$ is positive.

Similarly to vacuum case, we can obtain a curvature emissivity in an isotropic medium,

$$\eta(\omega) = \frac{\sqrt{\epsilon}\,\omega^2 q^2 v_\phi^2}{2\pi c}\left[J_\nu'^2(\nu\sqrt{\epsilon}\,\beta_\phi\cos\theta) + \frac{\tan^2\theta}{\epsilon\beta_\phi^2}\, J_\nu^2(\nu\sqrt{\epsilon}\,\beta_\phi\cos\theta)\right]\delta\left(\omega - \frac{\nu\beta_\phi c}{r}\right) \tag{99}$$

(emissivity per unit solid angle $d\Omega$).

6. AIRY FUNCTION APPROXIMATION

In this section we consider the Airy function approximation to the curvature emissivities of ultrarelativistic particles *in a medium*, which is a common way of approximating the cyclotron and curvature emissivity of ultrarelativistic particles in vacuum (e.g., Melrose 1978). We show that for a particles moving with the velocity larger than the speed of light in a medium that has $\epsilon > 1$ a qualitatively different expansion of the Airy function should be used.

In the transition region, when the argument of the Bessel functions is large and close to their orders it is possible to use the Airy function approximation

$$J_\nu(\nu + z\nu^{1/3}) \approx \left(\frac{2}{\nu}\right)^{1/3}\mathrm{Ai}(-2^{1/3}z) = \begin{cases} \dfrac{2^{2/3}}{3\nu^{1/3}}\left[J_{1/3}\left(\dfrac{2^{3/2}}{3}\, z^{3/2}\right) + J_{-1/3}\left(\dfrac{2^{3/2}}{3}\, z^{3/2}\right)\right] & \text{if } z > 0\,, \\ \dfrac{2^{2/3}\sqrt{z}}{\sqrt{3}\,\pi\nu^{1/3}}\, K_{1/3}\left(\dfrac{2^{3/2}}{3}\,|z|^{3/2}\right) & \text{if } z < 0, \end{cases} \tag{100}$$

where Ai(x) is Airy function.

In vacuum, for a resonant wave-particle interaction the argument of the Bessel functions is always smaller than the order $\lambda r = \nu\beta_\phi \cos\theta < \nu$. In a medium, the argument of the Bessel functions can become larger than the order. In a dielectric, the argument of the Bessel functions is $\hat{\lambda} r = (\omega^2\epsilon/c^2 - k_x^2)^{1/2} r$. If we introduce a notation $k_x = \omega\epsilon^{1/2}/c \sin\theta$ (this is a definition of angle θ, the condition of regularity at infinity, $\hat{\lambda}^2 > 0$ ensures that $\sin\theta < 1$), then we have $\hat{\lambda} r = \nu\epsilon^{1/2}\beta_\phi \cos\theta$, which can be larger than ν if $\beta_\phi > 1/\epsilon^{1/2}$ for the superluminal motion of a particle. Using these notations, we find

$$z = (\sqrt{\epsilon}\,\beta_\phi\cos\theta - 1)\nu^{2/3} \approx \left(\delta - \frac{1}{2\gamma^2} - \frac{\theta^2}{2}\right)\nu^{2/3} \tag{101}$$

[6] Since angle θ is measured from the plane $k_x = 0$ the unit solid angle is $d\Omega = d\phi \cos\theta\, d\theta$.

for $\epsilon = 1 + 2\delta$, $\delta \ll 1$, $\gamma \gg 1$, and $\theta \ll 1$. It is clear from equation (101) that in vacuum $z < 0$ and in a medium z becomes positive for superluminal particles.

Following the discussion of § 2.1.2, we can identify the light cylinder radius for the mode ν as $r_\nu = \nu/\hat{\lambda}$. We can argue that in the case $z > 0$ the resonant interaction of a particle with a wave occurs *outside* the light cylinder

$$r_0 = r_\nu \left(1 + \frac{z}{\nu^{2/3}}\right) . \tag{102}$$

A transition through point $z = 0$ ("light cylinder" or "classical reflection point") is nontrivial. It resembles phase transition (Schwinger et al. 1976) in a sense that correlation length for thermal or quantum fluctuations is very large near the transition. The physical conditions beyond and above the transition point are essentially different.

For superluminal motion the collective effects of the medium play an important role. In this case, the corresponding emission process can be called Cherenkov-curvature emission, stressing the fact that both inhomogeneous magnetic field and a medium are important for the emission. In the vacuum limit, $n \to 1$, Cherenkov-curvature emission reduces to the conventional curvature emission. Conventional Cherenkov radiation may be obtained in the limit $r \to \infty$ after integration over ν (see Schwinger et al. 1976 for the corresponding transition for the cyclotron-Cherenkov radiation).

The emissivity for the Cherenkov-curvature process is

$$\eta(\omega) = \frac{2^{2/3}\sqrt{\epsilon}\,\omega^2 q^2 v_\phi^2}{2\pi c \nu^{2/3}} \left[\frac{\tan^2\theta}{\epsilon\beta_\phi^2}\,\mathrm{Ai}^2(-2^{1/3}z) + \left(\frac{2}{\nu}\right)^{2/3} \mathrm{Ai}'^2(-2^{1/3}z)\right]\delta(\omega - \nu\Omega) . \tag{103}$$

For $z < 0$ this reduces to the conventional representation for synchrotron emission in terms of MacDonald functions $K_{1/3}$. For $z > 0$ equation (103) gives

$$\eta(\omega) = \frac{2^{1/3}\sqrt{\epsilon}\,\omega^2 q^2 v_\phi^2}{9\pi c \nu^{2/3}} \left\{\frac{\tan^2\theta}{\epsilon\beta_\phi^2}\left[J_{1/3}(\xi) + J_{-1/3}(\xi)\right]^2 + \frac{2^{2/3}z^2}{\nu^{2/3}}\,[J_{2/3}(\xi) - J_{-2/3}(\xi)]^2\right\}\delta(\omega - \nu\Omega) , \tag{104}$$

where we used

$$J'_\nu(\nu + z\nu^{1/3}) \approx \left(\frac{2}{\nu}\right)^{2/3} \mathrm{Ai}'(-2^{1/3}z) ,$$

$$\mathrm{Ai}'(-z) = -\frac{z}{3}\,[J_{-2/3}(-2^{1/3}z) - J_{2/3}(-2^{1/3}z)] . \tag{105}$$

Emissivity (eq. [104]) may be simplified in the case $\xi \gg 1$. Then we can use the asymptotic expansion for Airy functions

$$\mathrm{Ai}(-z) \approx \frac{1}{\sqrt{\pi}\,z^{1/4}} \sin\left(\frac{2}{3}\,z^{3/2} + \frac{\pi}{4}\right) ,$$

$$\mathrm{Ai}'(-z) \approx \frac{z^{1/4}}{\sqrt{\pi}} \cos\left(\frac{2}{3}\,z^{3/2} + \frac{\pi}{4}\right) , \tag{106}$$

to find

$$\eta(\omega) = \frac{\sqrt{2}\sqrt{\epsilon}\,\omega^2 q^2 \beta_\phi^2 \sqrt{z}}{\pi^2 c \nu^{4/3}} \left[\cos^2\left(\xi + \frac{\pi}{4}\right) + \frac{\sin^2\,(\xi + \pi/4)\nu^{2/3}\,\tan^2\theta}{2z\epsilon\beta_\phi^2}\right]\delta(\omega - \nu\Omega) , \tag{107}$$

where $\xi = (2z)^{3/2}/3$.

For $\delta \gg 1/\gamma^2$ and $\delta \gg \theta^2$ the condition $z \gg 1$ implies $\delta\nu^{2/3} \gg 1$. Then equation (107) can be further simplified,

$$\eta(\omega) \approx \frac{\sqrt{2}\sqrt{\epsilon}\sqrt{\delta}\,\omega^2 q^2 \beta_\phi^2}{\pi^2 c \nu} \cos^2\left(\frac{2^{3/2}\nu}{3}\,\delta^{3/2} + \frac{\pi}{4}\right)\delta(\omega - \nu\Omega) . \tag{108}$$

In this context we note that the total spectral power for the curvature emission in a medium, viz.

$$\eta(\omega) = \frac{q^2\omega}{n^2 r}\left[2n^2\beta_\phi^2 J'_{2\nu}(2\nu n\beta_\phi) - (1 - n^2\beta_\phi^2)\int_0^{2\nu n\beta_\phi} dx\,J_{2\nu}(x)\right] , \tag{109}$$

for the case of superluminal motion, $n\beta_\phi > 1$, can be reduced to the explicitly Cherenkov-type emission form,

$$\eta(\omega) = \frac{q^2\omega\beta_\phi}{4\pi c}\left(1 - \frac{1}{n^2\beta_\phi^2}\right)\Lambda(z) , \tag{110}$$

with $\Lambda(z)$ of the order of unity for $z \geq 1$ (Schwinger et al. 1976).

7. RESONANT ELECTROMAGNETIC WAVES IN THE ASYMPTOTIC REGIME $z \gg 1$

We wish to simplify the equations (57)–(60) in the limit $z \gg 1$. Instead of standing waves, described by Bessel J functions, we consider propagating waves, described by Hankel $H^{(1)}$ and $H^{(2)}$ functions. For the outgoing waves, corresponding to $H^{(1)}$, we

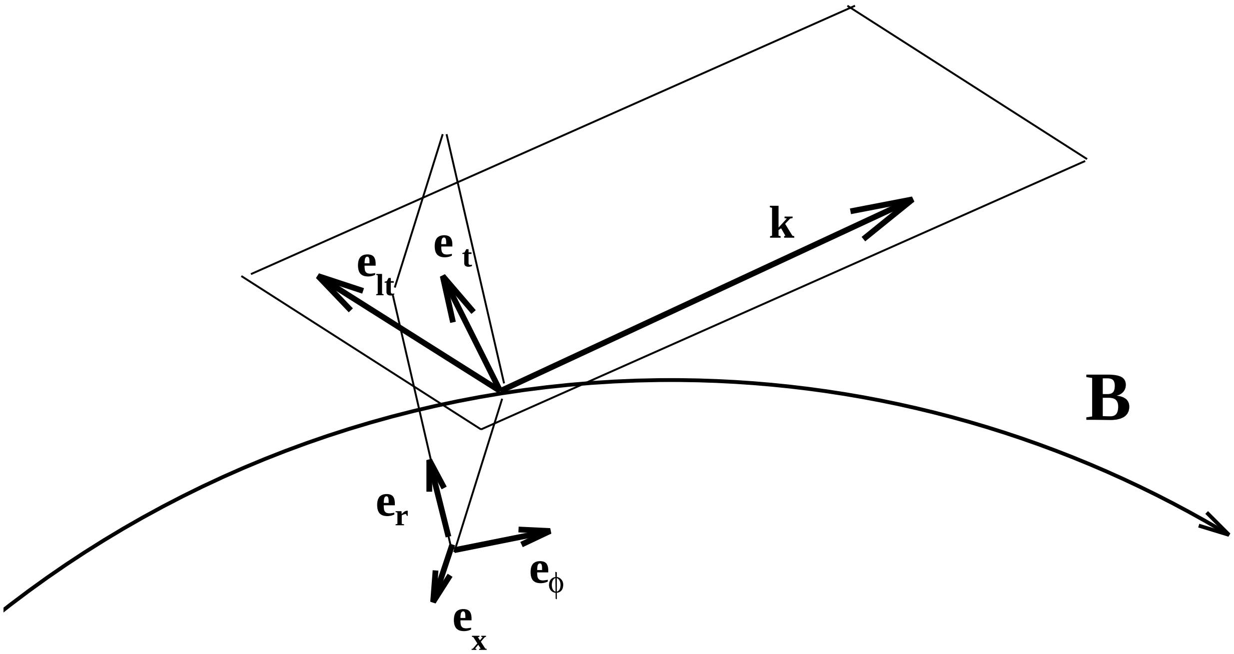

FIG. 3.—Polarization of normal modes in the limit of very strong magnetic field. The electric field vector of the t mode is in the plane $\boldsymbol{e_r}$-$\boldsymbol{e_x}$ and the electric vector of the lt mode is orthogonal to $\boldsymbol{e_t}$-$\boldsymbol{k}$ plane.

find in the limit $z \gg 1$,

$$H^{(1)} \approx -\frac{2^{1/4}}{\sqrt{\pi}\, \nu^{1/3} z^{1/4}}\, e^{i(2^{3/2}/3)z^{3/2} + i(\pi/4)} ,$$

$$H^{(1)\prime} \approx -i\, \frac{2^{3/4} z^{1/4}}{\sqrt{\pi}\, \nu^{2/3}}\, e^{i(2^{3/2}/3)z^{3/2} + i(\pi/4)} . \tag{111}$$

Next we define the radial wavenumber

$$k_r = -i\, \frac{\partial \ln H^{(1)}}{\partial r} . \tag{112}$$

Using equations (111) and (112), we find

$$k_r = \frac{\sqrt{2z}\, \hat{\lambda}}{\nu^{1/3}} \approx \sqrt{2\delta}\, \hat{\lambda} . \tag{113}$$

Introducing a notation $k_\phi = \nu/r$ the waves in the isotropic dielectric in the limit $z \gg 1$ may be written

$$\boldsymbol{E}^{(lt)}(\boldsymbol{k}, \omega) = E^{(lt)}\left(- \frac{k_x k_r}{\hat{\lambda}^2}\, \boldsymbol{e_r} - \frac{k_\phi k_x}{\hat{\lambda}^2}\, \boldsymbol{e}_\phi + \boldsymbol{e_x} \right) , \tag{114}$$

$$\boldsymbol{B}^{(lt)}(\boldsymbol{k}, \omega) = \frac{\omega \epsilon}{\hat{\lambda}^2 c}\, E^{(lt)}(k_\phi \boldsymbol{e_r} - k_r \boldsymbol{e}_\phi) , \tag{115}$$

$$\boldsymbol{E}^{(t)}(\boldsymbol{k}, \omega) = E^{(t)}(k_\phi \boldsymbol{e_r} - k_r \boldsymbol{e}_\phi) , \tag{116}$$

$$\boldsymbol{B}^{(t)}(\boldsymbol{k}, \omega) = E^{(t)}\left(\frac{k_r k_x c}{\omega}\, \boldsymbol{e_r} + \frac{k_x k_\phi c}{\omega}\, \boldsymbol{e}_\phi - \frac{c \hat{\lambda}^2}{\omega}\, \boldsymbol{e_x} \right) , \tag{117}$$

with $\boldsymbol{k} = \{k_r, k_\phi, k_x\}$ and a phase dependence $\exp\{-i(\omega t - k_r r - k_\phi \phi - k_x x)\}$.

With these notations the dispersion relation is

$$k_x^2 + k_r^2 + k_\phi^2 = \frac{\omega^2 \epsilon}{c^2} . \tag{118}$$

Equations (114)–(117) look similar to the equations (16), although they were obtained in quite a different manner. Equations (16) are valid for $\lambda r - \nu \geq \nu$ with the radial wavevector defined by equation (14). Equations (114)–(117) are valid for $\lambda r \approx \nu$ and $\xi \gg 1$ (the radial wavevector in this case is defined by eq. [113]).

The importance of these results is that in these two limits, the electromagnetic wave in cylindrical coordinates look like plane waves. This is a considerable simplification. It allows one to implement a well-developed technique of Fourier transforms in considering the wave propagation. This representation of the electromagnetic fields requires (1) the presence of a medium with the index of refraction $n > 1$, (2) superluminal motion of the resonant particle $\delta > 1/(2\gamma^2)$, (3) large harmonic number $\nu > \delta^{-3/2}$.

Another important feature is that in this limit there is an additional freedom in the choice of wave polarizations in an isotropic dielectric. For example, we can introduce the polarization vector corresponding to the waves (eq. [114]–[117]),

$$e^{(t)} = \frac{1}{\lambda} \{k_\phi, -k_r, 0\} = \{\sin \psi, -\cos \psi, 0\} ,$$

$$e^{(lt)} = \{-\cos \psi \sin \theta, -\sin \psi \sin \theta, \cos \theta\} , \tag{119}$$

which satisfy relations

$$e^{(t)} = e_k \times e_x ,$$

$$e^{(lt)} = e_k \times e^{(t)} . \tag{120}$$

The special role played by the e_x in defining the polarizations of the normal modes comes from the fact that in the isotropic homogeneous medium the solutions of the vector wave equations may be chosen to be tangential to the coordinate surfaces e_x. In the nonisotropic medium this is not true (see § 9).

Alternatively, *in the limit* $z \gg 1$ we can chose the following polarizations

$$e^{(t)} = \frac{1}{k_\perp} \{-k_x, 0, k_r\} ,$$

$$e^{(lt)} = \frac{1}{kk_\perp} \{k_\phi k_r, -k_\perp^2, k_x k_\phi\} ; \tag{121}$$

(see Fig. 3). This particular choice of polarizations has an advantage that it may be related to the polarizations in the straight field line geometry.

8. RESPONSE TENSOR FOR A ONE-DIMENSIONAL PLASMA IN A CURVED MAGNETIC FIELD

We will calculate the response tensor of a one-dimensional plasma in cylindrical coordinates using the analogy of the forward-scattering method (see, e.g., Melrose 1986) adopted to cylindrical coordinates. We represent the current as a sum of currents due to the each single particle moving along its trajectory,

$$j(r, t) = \int dr^0\, dp\, j_{\rm sp}(r, t) f[p, r^0(t)] . \tag{122}$$

We expand the single particle current (denoted by the subscript sp)

$$j_{\rm sp}(r, t) = q\dot{r}\, \delta[r - r^0(t)] \tag{123}$$

in Fourier amplitudes in time, k_x and ϕ and in Hankel amplitudes in r

$$j_{\rm sp}(r, m, k_x, \omega) = q \int dt \exp\{-i\omega t\} \int dx \exp\{ik_x x\} \int d\phi \exp\{i\nu\phi\} j_{\rm sp}(r, t) ,$$

$$= \int dt \exp[i(\nu\phi^0(t) + k_x x_0(t) - \omega t)]\dot{r} \int \xi\, d\xi\, J_\nu(\xi r) J_\nu[\xi r^0(t)] , \tag{124}$$

where we used a Hankel transform of the delta function[7]

$$\int \xi\, d\xi\, J_\nu(\xi r) J_\nu(\xi r^0) = \frac{\delta(r - r^0)}{r} . \tag{125}$$

[7] A Hankel transform is defined as $F(x) = \int_0^\infty J_\nu(\xi x)\xi\, d\xi \int_0^\infty y\, dy\, J_\nu(\xi y) F(y)$.

We expand the orbit of the particle in powers of wave amplitudes,

$$\boldsymbol{r}(t) = \boldsymbol{r}^0(t) + \boldsymbol{r}^{(1)}(t) . \tag{126}$$

The first-order Fourier transform of the single particle current is then

$$\boldsymbol{j}_{\rm sp}^{(1)}(r, m, k_x, \omega) = q \int dt \exp\{i[\nu\phi^0(t) + k_x x_0(t) - \omega t]\} \int \xi\, d\xi\, J_\nu(\xi r) \times \left(\dot{\boldsymbol{r}}^{(1)} J_\nu[\xi r^0(t)] + \boldsymbol{r}^0(t)\left\{i\nu\phi^{(1)}(t) + ik_x x^{(1)}(t) J_\nu[\xi r^0(t)] + \frac{\partial J_\nu[\xi r^0(t)]}{\partial r^0}\, r^{(1)}(t)\right\}\right) . \tag{127}$$

The orbit of a particle is found by solving the equation of motion

$$\frac{d\boldsymbol{p}}{dt} = \boldsymbol{F}^0(t, \boldsymbol{r}, \boldsymbol{v}) + \boldsymbol{F}^{(1)}(t, \boldsymbol{r}, \boldsymbol{v}) , \tag{128}$$

where $\boldsymbol{F}^0(t, \boldsymbol{r}, \boldsymbol{v})$ is a force acting on a particle when no wave is present and $\boldsymbol{F}^{(1)}(t, \boldsymbol{r}, \boldsymbol{v})$ is a force acting on a particle due to the presence of a waves. Expanding equation (128) in powers of waves amplitudes we find

$$\frac{d\boldsymbol{p}^0(t)}{dt} = \boldsymbol{F}^0(t, \boldsymbol{r}^0, \boldsymbol{v}^0) , \tag{129}$$

$$\frac{d\boldsymbol{p}^{(1)}(t)}{dt} = \boldsymbol{F}^{(1)}(t, \boldsymbol{r}^0, \boldsymbol{v}^0) . \tag{130}$$

The force $\boldsymbol{F}^{(1)}(t, \boldsymbol{r}^0, \boldsymbol{v}^0)$ can be expanded in Fourier amplitudes

$$\boldsymbol{F}^{(1)}(t, \boldsymbol{r}^0, \boldsymbol{v}^0) = \sum_{\nu'} \int d\omega'\, dk'_x \exp - \{i[\omega' t - k'_x x^0(t) - \nu'\phi^0(t)]\} \boldsymbol{F}^{(1)}[\omega, m, k_x, r^0(t), t] . \tag{131}$$

Equation (130) can be solved for the first-order velocity perturbation $\dot{\boldsymbol{r}}^{(1)}(t)$ and first-order trajectory perturbations $x^{(1)}(t)$, $\phi^{(1)}(t)$, and $r^{(1)}(t)$,

$$\begin{pmatrix} \dot{\boldsymbol{r}}^{(1)}(t) \\ x^{(1)}(t) \\ \phi^{(1)}(t) \\ r^{(1)}(t) \end{pmatrix} = \sum_{\nu'} \int d\omega'\, dk'_x \exp\{-i[\omega' t - k'_x x^{0+}(t) - \nu'\phi^0(t)]\} \begin{pmatrix} \tilde{\boldsymbol{V}}[\omega', \nu', k'_x, r^0(t), t] \\ \tilde{x}^{(1)}[\omega', \nu', k'_x, r^0(t), t] \\ \tilde{\phi}^{(1)}[\omega', \nu', k'_x, r^0(t), t] \\ \tilde{r}^{(1)}[\omega', \nu', k'_x, r^0(t), t] \end{pmatrix} , \tag{132}$$

where tilde denotes Fourier transforms.

The first-order single particle current then becomes

$$\boldsymbol{j}_{\rm sp}^{(1)}(r, m, k_x, \omega) = q \sum_{\nu'} \int d\omega'\, dk'_x \int dt\, e^{\{i[(\omega-\omega')t - (k_x - k_{x'})x^0(t) - (\nu-\nu')\phi^0(t)\ ]\}} \int \xi\, d\xi\, J_\nu(\xi r) \times \left(\tilde{\boldsymbol{V}} J_\nu[\xi r^0(t)] + \boldsymbol{r}^0(t)\left\{(i\nu\tilde{\phi}^{(1)} + ik_x \tilde{x}^{(1)}) J_\nu[\xi r^0(t)] + \frac{\partial J_\nu[\xi r^0(t)]}{\partial r^0}\, \tilde{r}^{(1)}\right\}\right) . \tag{133}$$

Using this single particle current, we can calculate the current (eq. [122]). In the general case, current (eq. [133]) may be related to the electric field of the perturbing wave through a generalized dielectric tensor,

$$\boldsymbol{j}(r, m, k_x, \omega) = \mathscr{E}(r, m, k_x, \omega) \cdot \boldsymbol{E}(r, m, k_x, \omega) , \tag{134}$$

where $\mathscr{E}$ is a dielectric tensor operator that involves a Hankel transform of the external current and partial derivatives with respect to r. This is different from Cartesian coordinates, where the electric induction is related to the electric field through a dielectric tensor. To simplify the calculations we assume that the radial dependence of both the Hankel transform of the external current and of the perturbing electric field can be approximated by the plane-wave form. For the nonresonant term in the dielectric tensor operator this is justified in the WKB limit, whereas for the resonant terms this is justified for subluminous waves in the limit $\delta\nu^{2/3} \gg 1$.

Thus, in the Hankel transform of the external current and in the derivatives of the perturbing electric field, we can identify

$$\frac{\partial}{\partial r^0} \to ik_{r_0}\ \frac{\partial}{\partial r} \to ik_r . \tag{135}$$

In this approximation the Hankel transform of the external current will reduce to the Fourier transform in r and the derivatives of the perturbing electric field are replaced by ik_r. The dielectric tensor operator $\mathscr{E}$ then becomes a conventional tensor.

The integration of the induced current over $d\phi^0\,dx^0$ then gives delta functions $\delta(v - v')\,\delta(k_x - k'_x)$ that are subsequently removed by the corresponding integrations. For $r_0 = \text{const}$, the time and $d\omega'$ integrations ensure that only secular terms are retained. The integration over $d\xi$ and dr^0 are removed using equation (125).

8.1. *Perturbed Trajectory*

The equations of motions for a particle in a circular magnetic field when the Larmor radius is much smaller than the radius of curvature are the following:

$$\frac{dp_r}{dt} = eE_r(\boldsymbol{r}^0) + \frac{q}{c}\,\{[B_\phi(\boldsymbol{r}^0) + B_0]v_x - v_\phi\,B_x(\boldsymbol{r}^0)\}\ , \tag{136}$$

$$\frac{dp_x}{dt} = eE_x(\boldsymbol{r}^0) + \frac{q}{c}\,\{B_r(\boldsymbol{r}^0)v_\phi - v_r[B_\phi(\boldsymbol{r}^0) + B_0]\}\ , \tag{137}$$

$$\frac{dp_\phi}{dt} = eE_\phi(\boldsymbol{r}^0) + \frac{q}{c}\,[B_x(\boldsymbol{r}^0)v_r - v_x\,B_r(\boldsymbol{r}^0)]\ . \tag{138}$$

We solve equations (136)–(138) by expanding in powers of the wave amplitudes. We assume that initially the particles are in the ground gyration state. Then in the zeroth order we find $v_r^0 = 0$, $\phi^0 = \Omega t = v_\phi\,t/r$, $v_x^0 = u_d$, with $u_d = \gamma v_\phi^2/R_C\,\omega_B$. The first order in wave amplitudes gives

$$\frac{dv_r^{(1)}}{dt} = \frac{q}{\gamma m_e\,c}\left[E_r(\boldsymbol{r}^0) - \frac{v_\phi}{c}\,B_x(\boldsymbol{r}^0) + \frac{u_d}{c}\,B_\phi(\boldsymbol{r}^0)\right] + \frac{\omega_B}{\gamma}\,v_x^{(1)}\ , \tag{139}$$

$$\frac{dv_x^{(1)}}{dt} = \frac{q}{\gamma m_e\,c}\left\{E_x(\boldsymbol{r}^0) + \frac{v_\phi}{c}\,B_r(\boldsymbol{r}^0) - \frac{u_d[v_\phi\,E_\phi(\boldsymbol{r}^0) + u_d\,E_x(\boldsymbol{r}^0)]}{c^2}\right\} + \frac{\omega_B}{\gamma}\,v_r^{(1)}\ , \tag{140}$$

$$\frac{dv_\phi^{(1)}}{dt} = \frac{q}{\gamma m_e\,c}\left\{E_\phi(\boldsymbol{r}^0) - \frac{u_d}{c}\,B_r(\boldsymbol{r}^0) - \frac{v_\phi[v_\phi\,E_\phi(\boldsymbol{r}^0) + u_d\,E_x(\boldsymbol{r}^0)]}{c^2}\right\}\ . \tag{141}$$

Expanding equations (139)–(141) in Fourier amplitudes we find the solutions,

$$\tilde{v}_r^{(1)} = \frac{q}{m_e\,c\gamma(\omega_B^2/\gamma^2 - \Omega^{0,2})}\left\{i\Omega^0\left(E_r + \frac{B_\phi\,u_d - B_x\,v_\phi}{c}\right) + \frac{\omega_B}{\gamma}\left[E_x + \frac{v_\phi\,B_r}{c} - \frac{u_d(v_\phi\,E_\phi + u_d\,E_x)}{c^2}\right]\right\}, \tag{142}$$

$$\tilde{v}_x^{(1)} = \frac{q}{m_e\,c\gamma(\omega_B^2/\gamma^2 - \Omega^{0,2})}\left\{i\Omega^0\left[E_x + \frac{v_\phi\,B_r}{c} - \frac{u_d(v_\phi\,E_\phi + u_d\,E_x)}{c^2}\right] - \frac{\omega_B}{\gamma}\left(E_r + \frac{B_\phi\,u_d - B_x\,v_\phi}{c}\right)\right\}, \tag{143}$$

$$\tilde{v}_\phi^{(1)} = i\,\frac{q}{m_e\,\Omega^0\gamma}\left[-\left(E_\phi - \frac{u_d}{c}\,B_r\right) + \frac{v_\phi(v_\phi\,E_\phi + u_d\,E_x)}{c^2}\right], \tag{144}$$

where $\Omega^0 = \omega - k_\phi\,v_\phi - k_x\,u_d$, $k_\phi = v/r$.

The first-order variations in trajectory are

$$x^{(1)} = \frac{v_x^{(1)}}{i\hat{\omega}}\ , \qquad \phi^{(1)} = \frac{v_\phi^{(1)}}{i\hat{\omega}r}\ , \qquad r^{(1)} = \frac{r^{(1)}}{i\hat{\omega}}\ . \tag{145}$$

In cylindrical coordinates, the relation between the electric and magnetic field is

$$B_r = \frac{k_x\,c}{\omega}\,E_\phi - \frac{k_\phi\,c}{\omega}\,E_x\ ,$$

$$B_\phi = -\frac{k_x\,c}{\omega}\,E_r - i\,\frac{c\partial E_x}{\omega\partial r}\ ,$$

$$B_x = \frac{k_\phi\,c}{\omega}\,E_r + i\,\frac{c}{\omega r}\,\frac{\partial}{\partial r}\,(rE_\phi)\ . \tag{146}$$

8.2. *Simplified Response Tensor*

Using equations (142)–(146), we can find the dielectric tensor operator $\mathscr{E}$. In §§ 2.1.1 and 2.1.2 we discussed when the operator relations between electric field and electric displacement can be simplified to algebraic relations for the nonresonant waves, and in § 7 we found a limiting case $z \gg 1$ when this can be done for the resonant waves. In these limits it is possible to change all the radial derivatives $\partial_r \to ik_r$. The relations between magnetic and electric field then simplify

$$B_\phi \approx -\frac{k_x c}{\omega} E_r + \frac{k_r c}{\omega} E_x ,$$

$$B_x \approx \frac{k_\phi c}{\omega} E_r - \frac{k_r c}{\omega} E_\phi . \tag{147}$$

The resulting dielectric tensor is still complicated. It can be simplified according to the following procedure: (1) for the nonresonant parts the drift velocity is small and can be neglected, (2) for Cherenkov-type resonances (resonances that do not involve ω_B) we retain all the terms, (3) for cyclotron-type resonances (resonances that do involve ω_B) we assume that $k_x/k_\phi \approx u_d/c$, $u_d/c \gg 1/\gamma$ and expand in small drift velocity retaining only the first terms in u_d and k_x. Implicit in this expansion in small u_d is the assumption that $\gamma\omega^2 \gg \omega_B^2 u_d^2/c^2$. In this limit, in the cyclotron-type resonant terms involving drift velocity we can approximate $v_\phi \approx c$. The dielectric tensor is then

$$\epsilon_{xx} = 1 - \frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma}[(\omega - k_\phi v_\phi)A_\alpha^+ f_\alpha] - \sum_\alpha \omega_{p\alpha}^2 \int \frac{dp_\phi}{\gamma}\frac{f_\alpha}{\Omega_\alpha^{o\,2}}\frac{k_\phi^2 u_\alpha^2}{\omega^2}\left(1 - \frac{\omega v_\phi}{k_\phi c^2}\right),$$

$$\epsilon_{rr} = 1 - \frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma}(\omega - k_\phi v_\phi - k_x u_\alpha)A_\alpha^+ f_\alpha ,$$

$$\epsilon_{\phi\phi} = 1 - \sum_\alpha \omega_{p\alpha}^2 \int \frac{dp_\phi}{\gamma}\frac{f_\alpha}{\Omega_\alpha^{o2}}\left\{\left(1 - \frac{k_x u_\alpha}{\omega}\right)\left[\left(1 - \frac{k_x u_\alpha}{\omega}\right) - \frac{v_\phi^2}{c^2}\right]\right\}$$

$$- \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma} f_\alpha \frac{(k_x^2 + k_r^2)v_\phi^2}{\Omega_\alpha^+\Omega_\alpha^-} - \frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma} k_r u_\alpha A_\alpha^- f_\alpha ,$$

$$\epsilon_{xr} = -\frac{i}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma}[(\omega - k_\phi v_\phi)A_\alpha^- + ik_r u_d A_\alpha^+]f_\alpha ,$$

$$\epsilon_{rx} = \frac{i}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma}[(\omega - k_\phi v_\phi)A_\alpha^- - ik_r u_\alpha A_\alpha^+]f_\alpha ,$$

$$\epsilon_{x\phi} = -\frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma} v_\phi\left[\left(k_x - \frac{\omega u_\alpha}{c^2}\right)A_\alpha^+ + ik_r A_\alpha^-\right]f_\alpha - \sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma} v_\phi u_\alpha \frac{(k_x^2 + k_r^2)}{\Omega_\alpha^+\Omega_\alpha^-} f_\alpha$$

$$- \sum_\alpha \omega_{p\alpha}^2 \int \frac{dp_\phi}{\gamma}\frac{f_\alpha}{\Omega_\alpha^{o2}}\frac{k_\phi u_\alpha}{\omega}\left[\left(1 - \frac{k_x u_\alpha}{\omega}\right) - \frac{v_\phi^2}{c^2}\right],$$

$$\epsilon_{\phi x} = \frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha 2}}{\omega^2}\int \frac{dp_\phi}{\gamma} v_\phi(k_x A_\alpha^+ - ik_r A_\alpha^-)f_\alpha - \sum_\alpha \omega_{p\alpha}^2 \int \frac{dp_\phi}{\gamma}\frac{f_\alpha}{\Omega_\alpha^{o\,2}}\frac{k_\phi u_\alpha}{\omega}\left(1 - \frac{k_x u_\alpha}{\omega}\right)\left(1 - \frac{\omega v_\phi}{k_\phi c^2}\right),$$

$$\epsilon_{r\phi} = -\frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma}\frac{v_\phi}{c}\left[k_r A_\alpha^+ - i\left(k_x c - \frac{\omega u_\alpha}{c}\right)A_\alpha^-\right]f_\alpha ,$$

$$\epsilon_{\phi r} = -\frac{1}{2}\sum_\alpha \frac{\omega_{p\alpha}^2}{\omega^2}\int \frac{dp_\phi}{\gamma}\frac{v_\phi}{c}(k_r A_\alpha^+ + ik_x A_\alpha^-)f_a . \tag{148}$$

Here

$$A_\alpha^+ = \left(\frac{1}{\Omega_\alpha^+} + \frac{1}{\Omega_\alpha^-}\right), \qquad A_\alpha^- = \left(\frac{1}{\Omega_\alpha^-} - \frac{1}{\Omega_\alpha^+}\right),$$

$$\Omega_\alpha^\pm = \omega - k_\phi v_\phi - k_x u_\alpha \pm \omega_B \gamma^{-1} , \qquad \Omega_\alpha^o = \omega - k_\phi v_\phi - k_x u_\alpha , \tag{149}$$

where f_α are one-dimensional distribution functions of the components α.

This dielectric tensor reduces to the dielectric tensor for plasma in straight magnetic fields for $u_\alpha = 0$. It takes a correct account of the Cherenkov-curvature emission and gives the drift corrections to the cyclotron emission. We also note that this dielectric tensor is non-Hermitian, since k_r is not a Killing vector.

The dielectric tensor in the infinitely strong magnetic field (eq. [76]) may be obtained from equation (148) by setting $A^\pm$ and u_α to zero.

9. WAVES IN CYLINDRICAL COORDINATES IN ANISOTROPIC DIELECTRIC

In this section we consider polarization of waves in anisotropic dielectric in cylindrical coordinates. Since the dielectric properties of a medium are determined by the nonresonant wave-particle interaction, we assume that it can be considered in the WKB approximation, so we can use the dielectric tensor (eq. [148]). The properties of a inhomogeneous medium are determined by the very strong, circular magnetic field.

9.1. *Infinitely Strong Magnetic Field*

In the infinitely strong magnetic field the dielectric tensor is equation (76). The dispersion relation is then

$$[(1-K)(1-n_\phi^2)-n_r^2-n_x^2](1-n_\phi^2-n_r^2-n_x^2)=0\,, \tag{150}$$

where $n_i = k_i c/\omega$. Equation (150) has solutions

$$n^2=1\,,\qquad e^{(t)}=\frac{1}{n_\perp}\{-n_x,0,n_r\}\,,\qquad n_\phi^2=1-\frac{n_\perp^2}{1-K}\,,$$
$$e^{(lt)}=\frac{1}{\sqrt{(1-K)^2+Kn_\perp^2}}\left[\sqrt{(1-K)(1-K-n_\perp^2)}\,\frac{n_r}{n_\perp}\,,\,-n_\perp,\,\sqrt{(1-K)(1-K-n_\perp^2)}\,\frac{n_x}{n_\perp}\right], \tag{151}$$

where $e^{(t)}$ and $e^{(lt)}$ are polarization vectors of the t and lt mode according to classification of Kazbegi et al. (1991). The electric field of the t wave is always perpendicular to the magnetic field and the wavevector.

In the limit $K \to 0$ the polarization vector for the lt wave reduces to

$$e^{(lt)}=\frac{1}{n_\perp}\{-n_r n_\phi,\,n_\perp,\,-n_x n_\phi\}\,. \tag{152}$$

9.2. *Finite Magnetic Field*

In the finite magnetic field the dielectric tensor can be found from equation (148). For a cold plasma in the center of momentum frame the dielectric tensor is

$$\epsilon_{ij}=\begin{pmatrix}1+2d & 0 & 0\\ 0 & 1-K & 0\\ 0 & 0 & 1+2d\end{pmatrix}, \tag{153}$$

where $d=\omega_p^2/\omega_B^2$. The dispersion relation is then

$$[-(1-K)n_\phi^2+(1+2d)(1-K-n_r^2-n_x^2)](1+2d-n_\phi^2-n_r^2-n_x^2)=0\,. \tag{154}$$

Equation (154) has solutions

$$n^2=(1+2d)\,,\qquad e^{(t)}=\frac{1}{n_\perp}\{-n_x,0,n_r\}\,,\qquad n_\phi^2=(1+2d)\left(1-\frac{n_\perp^2}{1-K}\right),$$
$$e^{(lt)}\approx\frac{1}{\sqrt{(1-K)^2+Kn_\perp^2}}\left[\sqrt{(1-K)(1-K-n_\perp^2)}\,\frac{n_x}{n_\perp}\,,\,-n_\perp,\,\frac{n_r}{n_\perp}\right]+O(d)\,. \tag{155}$$

So that the polarization vectors are the same as in the case of infinitely strong magnetic field within factors ω_p^2/ω_B^2.

9.3. *Growth Rate of the Cherenkov-Drift Instability*

Next we calculate the growth rate of the Cherenkov-drift instability of a beam propagating through a medium. In the plane-wave approximation the resonance condition for the Cherenkov-drift instability

$$\omega-\nu\Omega-k_x v_x\equiv\omega-k_\phi v_\phi-k_x u_d=0 \tag{156}$$

takes the form

$$\frac{1}{2\gamma_{\rm res}^2}-\delta+\frac{k_r^2}{2k_\phi^2}+\frac{1}{2}\left(\frac{k_x}{k_\phi}-\frac{u_d}{c}\right)^2=0\,, \tag{157}$$

where we used $v_{\rm res}=c[1-(1/2\gamma_{\rm res}^2)-(u_d^2/2c^2)]$ and $\epsilon=1+2\delta$, $\delta\ll 1$. From equation (157) it is clear that Cherenkov-drift resonance can be satisfied only for superluminal particles with $\gamma>1/(2\delta)^{1/2}$. The emission geometry at the Cherenkov-drift resonance is shown in Figures 4 and 5.

Growth rate of the Cherenkov-drift instability may be calculated from

$$\Gamma=-2i\,\frac{e_i^*\epsilon_{ij}''e_j}{(1/\omega^2)(\partial/\partial\omega)(\omega^2 e_i^*\epsilon_{ij}'e_j)}\,, \tag{158}$$

where ϵ' and ϵ'' are Hermitian and anti-Hermitian parts of the dielectric tensor and e_i are the polarization vectors. Using dielectric tensor (eq. [148]) and polarization vectors (eqs. [151] and [152]), we find the growth rate in the limit $K\approx 0$. Growth

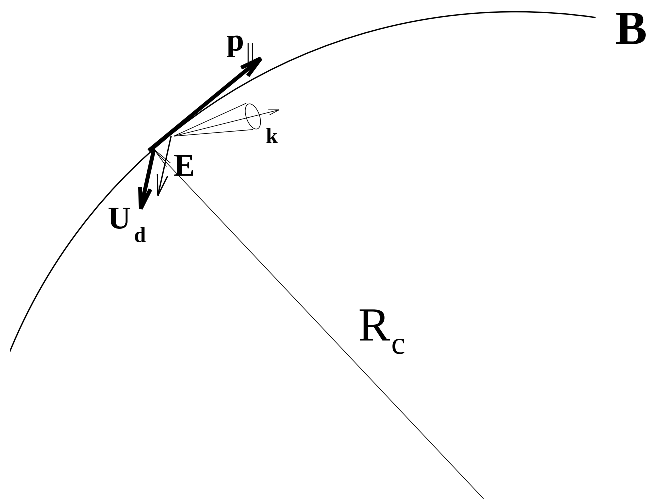

FIG. 4.—Cherenkov-drift emission in the case $\delta \ll u_d^2/c^2$. Drift velocity $\boldsymbol{u}_d$ is perpendicular to the plane of the curved field line ($\boldsymbol{B}$-$\boldsymbol{R}_c$ plane, $\boldsymbol{R}_c$ is a local radius of curvature). The emitted electromagnetic waves are polarized along $\boldsymbol{u}_d$. The emission is generated in the cone centered at the angle $\theta^{\rm em} = u_d/c$ and with the opening angle $(2\delta)^{1/2} \ll \theta^{\rm em}$.

rate for the *lt* mode is

$$\Gamma^{lt} = \frac{4\pi^2 q^2}{m} \int dp_\phi \left(\frac{k_\phi k_x u_d}{ckk_\perp} - \frac{v_\phi}{c} \frac{k_\perp}{k} \right)^2 \frac{\partial f(p_\phi)}{\partial p_\phi} \delta(\omega - k_\phi v_\phi - k_x u_d) , \tag{159}$$

and growth rate for the *t* mode is

$$\Gamma^{t} = \frac{4\pi^2 q^2}{m} \int dp_\phi \left(\frac{k_r u_d}{k_\perp c} \right)^2 \frac{\partial f(p_\phi)}{\partial p_\phi} \delta(\omega - k_\phi v_\phi - k_x u_d) . \tag{160}$$

The maximum growth rate for the *t* mode is reached when $k_x/k_\phi = u_d/c$ and the maximum growth rate for the *lt* mode is reached when $k_r = 0$. We also note that in the excitation of both *lt* and *t* wave it is the *x* component of the electric field that is growing exponentially.

It is clear from equations (159) and (160) that the growth rate of the *t* wave is proportional to the drift velocity and becomes zero in the limit of vanishing drift velocity. As for the *lt* wave, it can be excited in the limit of vanishing drift by the conventional Cherenkov mechanism that does not rely on the curvature of the magnetic field lines. We recall that in the limit of a strong magnetic field and *oblique* propagation *lt* wave has two branches: one superluminous (*O*-mode) and one subluminous (Alfvén mode) (Arons & Barnard 1986). On the conventional Cherenkov resonance it is possible to excite only subluminous Alfvén waves. In our approach a proof by Blandford (1975) that curvature emission cannot be amplified follows from the fact that in vacuum there are no subluminous waves that could resonate with particles.

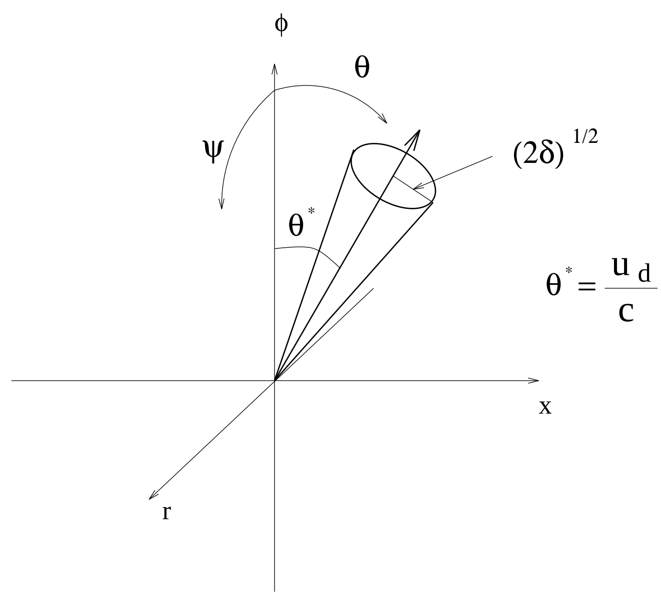

FIG. 5.—Emission geometry of the Cherenkov-drift resonance

Estimating the growth rates for the distribution function of the resonant particles having a Gaussian form

$$f(p_\phi) = \frac{1}{\sqrt{2\pi}\, p_t} \exp\left[-\frac{(p_\phi - p_b)^2}{2p_t^2} \right] \tag{161}$$

(here p_b is the momentum of the bulk motion of the beam and p_t is the dispersion of the momentum) and assuming that $u_d \gamma_b / c \gg 1$ we find the growth rates

$$\Gamma^t = \Gamma^{lt} \approx \sqrt{\frac{2}{\pi}} \frac{\omega_{p,\mathrm{res}}^2 \delta\gamma_b}{\omega \gamma_t^2} \frac{c^2}{u_d^2} , \tag{162}$$

where $\gamma_b = p_b/(mc)$, $\gamma_t = p_t/(mc)$. Numerical estimates show that the growth rate (eq. [162]) may be large enough to account for the high brightness radiation emission generation in pulsars (Lyutikov, Blandford, & Machabeli 1999).

10. CONCLUSION

In this work we investigated electromagnetic processes associated with a charged particle moving in a dielectric in a strong circular magnetic field. We derived a simple expression for the growth rate of the Cherenkov-drift instability that, we believe, may be responsible for the generation of the pulsar radio emission. We leave the discussion of the astrophysical applications of our results for a subsequent paper (Lyutikov et al. 1999). Here we just note that the Cherenkov drift developing on the open field lines in the outer parts of pulsar magnetosphere can explain various features of the cone-type (Rankin 1992) emission patterns observed in pulsars.

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