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General relativity in $(1 + 1)$ dimensions

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Abstract

We describe a theory of gravity in $(1 + 1)$ dimensions that can be thought of as a toy model of general relativity. The theory should be a useful pedagogical tool, because it is mathematically much simpler than general relativity but shares much of the same conceptual structure; in particular, it gives a simple illustration of how gravity arises from spacetime curvature. We derive the theory from fundamental physical principles using two different methods, one based on extrapolating from Newtonian gravity and one based on the equivalence principle, and present several exact solutions.

1. Introduction

General relativity is a difficult subject to teach to beginning students because a great deal of mathematics needs to be introduced, and the complexity of this mathematics can sometimes obscure the underlying physical concepts. The complexity of the mathematics reflects the complexity of describing spacetime curvature in $(3 + 1)$ dimensions; in lower dimensions, the mathematics needed to describe spacetime curvature is much simpler. For example, whereas in $(3 + 1)$ dimensions 20 different parameters are needed to characterize the spacetime curvature, in $(1+1)$ dimensions a single parameter will suffice[1]. In this paper, we show that by working in $(1 + 1)$ dimensions one can construct a toy model of gravity that is much simpler than general relativity, yet retains much of its conceptual structure. The model is a useful pedagogical tool because it illustrates many of the ideas of general relativity in a simplified context; in particular, it shows how gravity arises from spacetime curvature. The paper should be accessible to advanced undergraduates and beginning graduate students, and is intended to supplement an introductory class in general relativity taught at the level of Misner, Thorne and Wheeler [1].

It is not obvious how to construct a theory of gravity in $(1+1)$ dimensions that is analogous to general relativity; simply writing the Einstein field equations in $(1 + 1)$ dimensions does not work [2], as can be understood from the following considerations. In $(1 + 1)$ dimensions the

[1] The number of parameters needed to characterize the spacetime curvature is given by the number of independent components of the Riemann tensor, which in d spacetime dimensions is $(d^2/12)(d^2 - 1)$.

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Riemann tensor can be expressed as[2]

$$R^{\gamma}{}_{\nu\alpha\beta} = \frac{R}{2} g^{\gamma\mu}(g_{\mu\alpha} g_{\nu\beta} - g_{\mu\beta} g_{\nu\alpha}), \tag{1}$$

where R is the curvature scalar and $g_{\mu\nu}$ is the metric tensor. Thus, the Ricci tensor is given by

$$R_{\nu\beta} = R^{\gamma}{}_{\nu\gamma\beta} = \frac{R}{2} g_{\nu\beta}. \tag{2}$$

Equation (2) implies that in (1 + 1) dimensions the Einstein tensor vanishes identically,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = 0, \tag{3}$$

and the Einstein field equations reduce to

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} = 0, \tag{4}$$

where $T_{\mu\nu}$ is the energy–momentum tensor. Thus, the Einstein field equations do not constrain the metric tensor, and simply state that the energy–momentum tensor vanishes.

Instead of writing the Einstein field equations in (1 + 1) dimensions, we will consider an alternative theory of gravity that is described by the field equation

$$R = 4G g_{\alpha\beta} T^{\alpha\beta}. \tag{5}$$

In the 1980s this theory was proposed as a lower-dimensional model of gravity [4–6], and its dynamical content has been investigated by a number of authors [7–11]. The theory is the direct (1 + 1)-dimensional analog of a theory of gravity in (3 + 1) dimensions that was proposed by Nordstrøm in 1913. The Nordstrøm field equations are[3]

$$R = 24\pi G g_{\alpha\beta} T^{\alpha\beta}, \qquad C^{\alpha\beta}{}_{\mu\nu} = 0, \tag{6}$$

where $C^{\alpha\beta}{}_{\mu\nu}$ is a quantity known as the Weyl tensor. One can show that in (1 + 1) dimensions the Weyl tensor vanishes identically[4], so the Nordstrøm equations are equivalent to equation (5).

The paper is organized as follows. We first describe two different methods of deriving the toy model from basic physical principles: in section 2, we derive the toy model by extrapolating from Newtonian gravity, and in section 3, we derive the toy model by starting with a flat spacetime and constructing a relativistic field theory that obeys the principle of equivalence. We then consider two solutions to the field equation for the toy model: in section 4 we present a solution for a stationary point particle, and in section 5 we present a solution for a static source of uniform density.

2. Deriving the toy model from Newtonian gravity

One way to derive the toy model is to look for a relativistic theory that reduces to Newtonian gravity in the nonrelativistic limit. It is straightforward to construct the analog to Newtonian gravity in (1 + 1) dimensions: the field equation for the gravitational potential ϕ is given by

$$\partial_x^2 \phi(t, x) = 2G\rho(t, x), \tag{7}$$

where G is the gravitational constant and ρ is the mass density, and the equation of motion for a point particle moving in this potential is given by

$$\ddot{z}(t) = -\partial_x \phi(t, x)|_{x=z(t)}, \tag{8}$$

[2] See [3, p 232].
[3] See [12]; a discussion of the Nordstrøm theory that is accessible to students is given in [1, p 429].
[4] See [3, p 238].

where $z(t)$ is the position of the particle at time t. A key feature of this equation of motion is that it does not involve the particle mass, which means that the trajectory of the particle depends only on its initial position and velocity, and not on its constitution. This suggests that we might be able to explain gravitational effects in terms of a property intrinsic to space itself, rather than in terms of a particle–field interaction, and we will therefore look for a relativistic generalization of Newtonian gravity that explains gravitational effects in this way.

A further hint that can guide us towards the correct relativistic generalization is the observation that the effects of gravitational acceleration can be locally transformed away by a suitable choice of coordinates. We can demonstrate this by choosing a system of coordinates in which the position of the particle is specified in terms of its separation $\Delta(t) = z(t) - z_R(t)$ from a freely falling reference particle with trajectory $z_R(t)$. From equation (8), we find that the equation of motion for Δ is

$$\ddot{\Delta}(t) = \ddot{z}(t) - \ddot{z}_R(t) = -\Delta(t)\partial_x^2\phi(t, x)|_{x=z_R(t)}, \tag{9}$$

where we have assumed that Δ is small enough that we can expand in Δ and retain only the first-order term. Using equation (7), we can substitute for $\partial_x^2\phi(t, x)$ to obtain

$$\ddot{\Delta}(t) = -2G\rho(t, z_R(t))\Delta(t). \tag{10}$$

We can also write this equation as

$$\ddot{\Delta}(t) = \frac{1}{m}F_t(t), \tag{11}$$

where m is the mass of the particle and

$$F_t(t) \equiv -2mG\rho(t, z_R(t))\Delta(t) \tag{12}$$

is a quantity that we will call the tidal force[5]. Note that the tidal force is negligible if the particle is close to the reference trajectory, so in this limit equation (11) has the same form as the equation of motion for a free particle in the absence of a gravitational field.

Based on these observations, we are led to consider a geometric theory of gravity in which particle trajectories correspond to geodesics in a curved spacetime. We can then view the tidal acceleration described by equation (10) as geodesic deviation caused by the spacetime curvature. Thus, we would like to interpret equation (10) as the nonrelativistic limit of the equation of geodesic deviation[6]

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}s^2} = -\frac{R}{2}\Delta, \tag{13}$$

where R is the curvature scalar corresponding to the metric tensor $g_{\mu\nu}$ and s is the proper time defined relative to $g_{\mu\nu}$. By comparing equation (10) with equation (13), we can relate the curvature scalar to the mass density:

$$R = 4G\rho. \tag{14}$$

This should be the nonrelativistic limit of our new field equation. This cannot be the correct relativistic field equation, since R is a scalar under general coordinate transformations while ρ is the time–time component of the energy–momentum tensor $T^{\alpha\beta}$. Note, however, that the trace of the energy–momentum tensor is a scalar, and reduces to ρ in the nonrelativistic limit. Thus, we take

$$R = 4Gg_{\alpha\beta}T^{\alpha\beta} \tag{15}$$

as our gravitational field equation.

[5] The name 'tidal force' comes from the fact that one can define an analogous force for Newtonian gravity in $(3 + 1)$ dimensions, and it is the action of this force on the oceans of the Earth that generates the tides (see [13], section 1.6).
[6] This equation is discussed in [1, p 30].

3. Deriving the toy model from the principle of equivalence

A second way to derive the toy model is to start with a flat spacetime and construct a relativistic field theory that satisfies the principle of equivalence[7]. The principle of equivalence states that all forms of energy–momentum act as a source for gravity, including the energy–momentum of the gravitational field itself. This means that the gravitational field is self-interacting, so the gravitational field equation must be nonlinear. We can obtain the correct nonlinear field equation by using the following procedure. First, we construct a theory that describes a point particle interacting with a scalar field ϕ via an arbitrary coupling. Next, we calculate the total energy–momentum tensor for the particle–field system, and impose the principle of equivalence by choosing the form of the coupling such that the source for the field is given by the trace of this tensor. Finally, we show that the resulting theory can be interpreted geometrically, so we can either view the theory as describing a particle coupled to a scalar field in a flat spacetime, or as describing a free particle moving in a curved spacetime with metric tensor[8] $g_{\mu\nu} = \mathrm{e}^{2\phi}\eta_{\mu\nu}$.

3.1. Free point particle in flat spacetime

Let us begin by considering a free point particle of mass m moving in a flat spacetime. We can describe the motion of the particle by specifying its trajectory $z^\mu(\lambda)$, where λ is an arbitrary parameter that labels points along the particle's worldline. It is convenient to parameterize the trajectory in terms of the proper time τ, which is defined such that

$$\mathrm{d}\tau = (\eta_{\mu\nu} v^\mu v^\nu)^{1/2}\,\mathrm{d}\lambda, \tag{16}$$

where

$$v^\mu \equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}\lambda}. \tag{17}$$

Given a trajectory expressed in terms of an arbitrary parameter λ, we can use equation (16) to re-parameterize the trajectory in terms of the proper time. We then define the velocity of the particle to be

$$w^\mu \equiv \frac{\mathrm{d}z^\mu}{\mathrm{d}\tau}. \tag{18}$$

The mass density in the rest frame of the particle is

$$\bar{\rho}_p(x) = m \int \delta^{(2)}(x - z(\tau))\,\mathrm{d}\tau, \tag{19}$$

and in appendix A we show that the energy–momentum tensor for the particle is[9]

$$\bar{T}_p^{\mu\nu}(x) = m \int w^\mu w^\nu \delta^{(2)}(x - z(\tau))\,\mathrm{d}\tau. \tag{20}$$

We can describe the dynamics of the particle in terms of the action

$$S_p = \int L_p\,\mathrm{d}\lambda = \int \mathcal{L}_p\,\mathrm{d}^2x, \tag{21}$$

[7] This method of deriving the toy model is based on a similar method for deriving general relativity by considering a tensor field in a flat spacetime (see [14] and the references within; a treatment that is accessible to beginning students is given in [13], chapter 3).

[8] Here $\eta_{\mu\nu}$ is the Minkowski tensor, defined such that $\eta_{00} = -\eta_{11} = 1$, $\eta_{01} = \eta_{10} = 0$.

[9] The bars on these quantities indicate that they are defined from the point of view in which the spacetime is flat; later, when we reinterpret the theory, we will need to distinguish these quantities from the corresponding quantities defined from the point of view in which the spacetime is curved. Here $\delta^{(2)}(x - z(\tau)) \equiv \delta(t - z^0(\tau))\delta(x - z^1(\tau))$.

where L_p, the relativistic Lagrangian, is given by

$$L_p = -m(\eta_{\alpha\beta} v^\alpha v^\beta)^{1/2}, \tag{22}$$

and $\mathcal{L}_p$, the Lagrangian density, is given by

$$\mathcal{L}_p = -m \int (\eta_{\alpha\beta} v^\alpha v^\beta)^{1/2} \delta^{(2)}(x - z(\lambda))\, \mathrm{d}\lambda = -\bar{\rho}_p(x). \tag{23}$$

The equation of motion for the particle can be obtained from L_p via the Euler–Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \frac{\partial L}{\partial v^\gamma} - \frac{\partial L}{\partial z^\gamma} = 0. \tag{24}$$

Note that

$$\frac{\partial L_p}{\partial v^\gamma} = -m(\eta_{\alpha\beta} v^\alpha v^\beta)^{-1/2} \eta_{\gamma\mu} v^\mu, \qquad \frac{\partial L_p}{\partial z^\gamma} = 0. \tag{25}$$

Thus, from equations (16)–(18) and (24), it follows that the equation of motion for the particle is

$$\frac{\mathrm{d}w^\mu}{\mathrm{d}\tau} = 0. \tag{26}$$

As a consistency check, we can use this equation of motion to show that the energy–momentum tensor is conserved:

$$\partial_\mu \bar{T}_p^{\mu\nu}(x) = m \int \frac{\mathrm{d}w^\nu}{\mathrm{d}\tau} \delta^{(2)}(x - z(\tau))\, \mathrm{d}\tau = 0, \tag{27}$$

where we have used that

$$w^\mu \partial_\mu \delta^{(2)}(x - z(\tau)) = -\frac{\mathrm{d}}{\mathrm{d}\tau} \delta^{(2)}(x - z(\tau)) \tag{28}$$

and integrated by parts.

3.2. Point particle coupled to a scalar field in flat spacetime

Suppose we now couple the point particle to a scalar field ϕ. We will take the action for the system to be

$$S = \int (\mathcal{L}_f + \mathcal{L}_p + \mathcal{L}_i)\, \mathrm{d}^2 x, \tag{29}$$

where

$$\mathcal{L}_f = \frac{1}{4G} (\partial_\mu \phi)(\partial^\mu \phi) \tag{30}$$

describes a free massless scalar field,

$$\mathcal{L}_p = -\bar{\rho}_p \tag{31}$$

describes a free particle, and

$$\mathcal{L}_i = -f(\phi)\bar{\rho}_p \tag{32}$$

describes an arbitrary coupling between the particle and field. We can use this action to obtain equations of motion for the particle and field, and to construct the energy–momentum tensor for the system.

Let us first consider the particle. Since the particle degrees of freedom only enter into the Lagrangian densities $\mathcal{L}_p$ and $\mathcal{L}_i$, we can obtain the equation of motion for the particle from the action

$$S_{pi} = \int (\mathcal{L}_p + \mathcal{L}_i)\, \mathrm{d}^2x = \int L_{pi}\, \mathrm{d}\lambda, \tag{33}$$

where the relativistic Lagrangian L_{pi} is given by

$$L_{pi} = -m(1+f)(\eta_{\mu\nu} v^\mu v^\nu)^{1/2}. \tag{34}$$

From the Euler–Lagrange equations (24), we find that the equation of motion for the particle is

$$\frac{\mathrm{d}}{\mathrm{d}\tau} w^\mu + (1+f)^{-1}(\partial_\nu f)(w^\mu w^\nu - \eta^{\mu\nu}) = 0. \tag{35}$$

Now let us consider the field. The field equation can be obtained from the Euler–Lagrange equations

$$\partial_\mu \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} - \frac{\delta \mathcal{L}}{\delta \phi} = 0, \tag{36}$$

where $\mathcal{L} = \mathcal{L}_f + \mathcal{L}_p + \mathcal{L}_i$. Note that

$$\frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)} = \frac{1}{2G} \partial^\mu \phi, \qquad \frac{\delta \mathcal{L}}{\delta \phi} = -\frac{\mathrm{d}f}{\mathrm{d}\phi} \bar{\rho}_p. \tag{37}$$

Thus, we find that the field equation is[10]

$$\Box \phi = -2G \frac{\mathrm{d}f}{\mathrm{d}\phi} \bar{\rho}_p. \tag{38}$$

We can calculate the total energy–momentum tensor $\bar{T}^{\alpha\beta}$ for the system by applying the procedure described in appendix A to the action given in equation (29). We find that

$$\bar{T}^{\alpha\beta} = \bar{T}_f^{\alpha\beta} + \bar{T}_{pi}^{\alpha\beta}, \tag{39}$$

where

$$\bar{T}_f^{\alpha\beta} = \frac{1}{2G}\left((\partial^\alpha \phi)(\partial^\beta \phi) - \frac{1}{2}\eta^{\alpha\beta}(\partial_\mu \phi)(\partial^\mu \phi) \right) \tag{40}$$

$$\bar{T}_{pi}^{\alpha\beta} = m(1+f) \int w^\alpha w^\beta \delta^{(2)}(x - z(\tau))\, \mathrm{d}\tau. \tag{41}$$

As a consistency check, we can use the particle equation of motion (35) and the field equation (38) to show that

$$\partial_\alpha \bar{T}_{pi}^{\alpha\beta} = -\partial_\alpha \bar{T}_f^{\alpha\beta} = \bar{\rho}_p \frac{\mathrm{d}f}{\mathrm{d}\phi} \partial^\beta \phi, \tag{42}$$

confirming that the total energy–momentum tensor is conserved:

$$\partial_\alpha \bar{T}^{\alpha\beta} = 0. \tag{43}$$

[10] Here $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \partial_t^2 - \partial_x^2$ is the d'Alembertian operator in $(1+1)$ dimensions.

3.3. Determining the coupling

We will impose the principle of equivalence by choosing the coupling $f(\phi)$ in such a way that the trace of the total energy–momentum tensor $\bar{T}^{\alpha\beta}$ acts as a source for the field ϕ. This means that the field equation must have the form

$$\Box\phi = -2G\eta_{\alpha\beta}\bar{T}^{\alpha\beta}. \tag{44}$$

If we substitute the field equation (38) and the expression for $\bar{T}^{\alpha\beta}$ given in equation (39), we see that this condition implies

$$\frac{\mathrm{d}f}{\mathrm{d}\phi} = 1 + f, \tag{45}$$

where we have used that $\eta_{\alpha\beta}\bar{T}^{\alpha\beta}_{f} = 0$ and $\eta_{\alpha\beta}\bar{T}^{\alpha\beta}_{pi} = (1+f)\bar{\rho}_p$. We can integrate this equation to obtain

$$f(\phi) = A\,\mathrm{e}^{\phi} - 1 \tag{46}$$

for some constant A. Since the particle should be free when the field vanishes, we require that $f(0) = 0$, which implies that $A = 1$:

$$f(\phi) = \mathrm{e}^{\phi} - 1. \tag{47}$$

Substituting $f(\phi)$ into the expressions we derived in the previous section, we find that the equation of motion for the particle is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}w^{\mu} + (\partial_{\nu}\phi)(w^{\mu}w^{\nu} - \eta^{\mu\nu}) = 0, \tag{48}$$

the field equation is

$$\Box\phi = -2G\,\mathrm{e}^{\phi}\bar{\rho}_p, \tag{49}$$

and the total energy–momentum tensor for the system is $\bar{T}^{\alpha\beta} = \bar{T}^{\alpha\beta}_{f} + \bar{T}^{\alpha\beta}_{pi}$, where $\bar{T}^{\alpha\beta}_{f}$ is given by equation (40), and

$$\bar{T}^{\alpha\beta}_{pi}(x) = m\,\mathrm{e}^{\phi}\int w^{\alpha}w^{\beta}\delta^{(2)}(x - z(\tau))\,\mathrm{d}\tau. \tag{50}$$

3.4. Geometric interpretation

We will now show that we can reinterpret the theory, so we can view it as describing a free point particle moving in a curved spacetime with metric tensor $g_{\mu\nu} = \mathrm{e}^{2\phi}\eta_{\mu\nu}$. To accomplish this we need to reinterpret both the particle equation of motion (48) and the field equation (49) in geometric terms.

Let us begin with the particle equation of motion. For a particle moving in a spacetime described by the metric tensor $g_{\mu\nu}$, the proper time s is defined such that

$$\mathrm{d}s = (g_{\mu\nu}v^{\mu}v^{\nu})^{1/2}\,\mathrm{d}\lambda = \mathrm{e}^{\phi}(\eta_{\mu\nu}v^{\mu}v^{\nu})^{1/2}\,\mathrm{d}\lambda = \mathrm{e}^{\phi}\,\mathrm{d}\tau, \tag{51}$$

and the particle velocity is given by

$$u^{\mu} = \frac{\mathrm{d}z^{\mu}}{\mathrm{d}s} = \frac{\mathrm{d}\tau}{\mathrm{d}s}\frac{\mathrm{d}z^{\mu}}{\mathrm{d}\tau} = \mathrm{e}^{-\phi}w^{\mu}. \tag{52}$$

Using these relations, we can rewrite the equation of motion (48) in terms of s and u^{μ}:

$$\frac{\mathrm{d}}{\mathrm{d}s}u^{\mu} + (\partial_{\nu}\phi)(2u^{\mu}u^{\nu} - g^{\mu\nu}) = 0. \tag{53}$$

In appendix B we calculate the Christoffel symbols for the metric $g_{\mu\nu}$, and using these symbols we can express this equation of motion as

$$\frac{\mathrm{d}}{\mathrm{d}s}u^{\gamma} + \Gamma^{\gamma}{}_{\alpha\beta}u^{\alpha}u^{\beta} = 0. \tag{54}$$

This is just the geodesic equation; thus, we find that the particle moves along geodesics of the curved spacetime described by $g_{\mu\nu}$.

Next, we will reinterpret the field equation (49). In appendix A, we show that for a particle of mass m moving in a curved spacetime with metric tensor $g_{\mu\nu}$, the energy–momentum tensor is given by

$$T_p^{\alpha\beta}(x) = mg^{-1/2}\int u^{\alpha}u^{\beta}\delta^{(2)}(x - z(s))\,\mathrm{d}s, \tag{55}$$

where $g \equiv -\det g_{\mu\nu}$. From equations (19), (51), (52) and (55), we find that

$$g_{\alpha\beta}T_p^{\alpha\beta}(x) = \mathrm{e}^{-\phi}\bar{\rho}_p(x). \tag{56}$$

In appendix B, we show that the curvature scalar corresponding to $g_{\mu\nu}$ is

$$R = -2\mathrm{e}^{-2\phi}\eta^{\mu\nu}\partial_{\mu}\partial_{\nu}\phi = -2\mathrm{e}^{-2\phi}\,\Box\,\phi. \tag{57}$$

Thus, we can express the field equation (49) as

$$R = 4Gg_{\alpha\beta}T_p^{\alpha\beta}. \tag{58}$$

This is just the field equation (5) for the special case of a point particle source.

In summary, from one point of view the spacetime is flat and the particle interacts with a scalar field ϕ. The equation of motion for the particle is given by equation (48), and the field equation is given by equation (49). From another point of view, the spacetime is curved and the particle falls freely. The equation of motion for the particle is given by the geodesic equation (54), and the spacetime geometry is described by a metric tensor that satisfies the field equation (58).

4. Solution for a stationary point particle

To illustrate the results of the previous section, we will solve the field equation for the case of a stationary point particle. Let us first consider the system from the flat-spacetime point of view. We will take the trajectory of the particle to be $z^0(\tau) = \tau, z^1(\tau) = 0$, where τ is the proper time defined relative to the metric tensor $\eta_{\mu\nu}$. Substituting this trajectory into equation (19), we find that the mass density in the rest frame of the particle is given by

$$\bar{\rho}_p(x) = m\int \delta^{(2)}(x - z(\tau))\,\mathrm{d}\tau = m\delta(x). \tag{59}$$

If we substitute this result into the field equation (49), we find that

$$\partial_x^2\phi(x) = 2Gm\,\mathrm{e}^{\phi_0}\delta(x), \tag{60}$$

where $\phi_0 \equiv \phi(0)$, and where we have assumed that the potential is static. The solution to this equation is[11]

$$\phi(x) = Gm\,\mathrm{e}^{\phi_0}|x| + \phi_0. \tag{61}$$

We can also consider the system from the curved-spacetime point of view. The metric that describes the spacetime curvature is

$$\mathrm{d}s^2 = \mathrm{e}^{2\phi(x)}(\mathrm{d}t^2 - \mathrm{d}x^2), \tag{62}$$

[11] The most general solution includes an additional term Ex that describes a constant background field, but for simplicity we will assume that $E = 0$.

with $\phi(x)$ given by equation (61), and the curvature scalar is

$$R = 2\mathrm{e}^{-2\phi(x)}\partial_x^2\phi(x) = 4Gm\,\mathrm{e}^{-\phi_0}\delta(x), \tag{63}$$

so the spacetime is flat everywhere except on the particle worldline itself. In the curved-spacetime interpretation the parameter ϕ_0 can be thought of as defining a constant scaling factor e^{ϕ_0}. This scaling factor can always be eliminated by transforming to new coordinates $\bar{t} = \mathrm{e}^{\phi_0}t$, $\bar{x} = \mathrm{e}^{\phi_0}x$ in which the metric takes the form

$$\mathrm{d}s^2 = \mathrm{e}^{2\bar{\phi}(\bar{x})}(\mathrm{d}\bar{t}^2 - \mathrm{d}\bar{x}^2), \tag{64}$$

where $\bar{\phi}(\bar{x}) = a|\bar{x}|$ and $a \equiv Gm$. Thus, without loss of generality we can take $\phi_0 = 0$.

4.1. Coordinate transformations

The (t, x) coordinates that we have been using up until now are convenient because they allow us to connect the toy model with the flat-spacetime interpretation, but we can gain further insight into our solution by considering an alternative system of coordinates that casts the metric into a simpler form. Since the spacetime is flat for $x > 0$ and $x < 0$, in each of these regions we can define coordinates in which the metric tensor is given by the Minkowski tensor $\eta_{\mu\nu}$. For the region $x \geqslant 0$ we define coordinates (u_+, v_+) by

$$u_+(t, x) = a^{-1}\,\mathrm{e}^{ax}\sinh at \tag{65}$$

$$v_+(t, x) = a^{-1}\,\mathrm{e}^{ax}\cosh at, \tag{66}$$

and for the region $x \leqslant 0$ we define coordinates (u_-, v_-) by

$$u_-(t, x) = a^{-1}\,\mathrm{e}^{-ax}\sinh at \tag{67}$$

$$v_-(t, x) = a^{-1}\,\mathrm{e}^{-ax}\cosh at. \tag{68}$$

It is straightforward to verify that if we apply these coordinate transformations to the metric given in equation (62), we obtain the Minkowski metric

$$\mathrm{d}s^2 = \mathrm{d}u_+^2 - \mathrm{d}v_+^2 = \mathrm{d}u_-^2 - \mathrm{d}v_-^2. \tag{69}$$

Note that events with $x > 0$ are described exclusively by the (u_+, v_+) coordinates, events with $x < 0$ are described exclusively by the (u_-, v_-) coordinates, and events with $x = 0$ (that is, events on the worldline of the particle) are described by both systems of coordinates. The two coordinate patches are therefore glued together along the particle worldline, which in the new coordinates corresponds to the hyperbolas $v_+^2 - u_+^2 = 1/a^2$ and $v_-^2 - u_-^2 = 1/a^2$. We will call these hyperbolas H_+ and H_-, respectively; note that points to the left of these hyperbolas do not correspond to physical events.

To describe the gluing, we need to relate the two coordinate descriptions of events $(t, 0)$ that lie on the particle worldline; from equations (65)–(68), we find that

$$u_+(t, 0) = u_-(t, 0), \qquad v_+(t, 0) = v_-(t, 0). \tag{70}$$

We also need to relate the two coordinate descriptions of vectors in the tangent space of the particle worldline. From equations (65) and (66), we find that as $x \to 0$ from above,

$$\begin{pmatrix}\mathrm{d}u_+\\ \mathrm{d}v_+\end{pmatrix} = \begin{pmatrix}\cosh at & \sinh at\\ \sinh at & \cosh at\end{pmatrix}\begin{pmatrix}\mathrm{d}t\\ \mathrm{d}x\end{pmatrix}, \tag{71}$$

and from equations (67) and (68), we find that as $x \to 0$ from below,

$$\begin{pmatrix}\mathrm{d}u_-\\ \mathrm{d}v_-\end{pmatrix} = \begin{pmatrix}\cosh at & -\sinh at\\ \sinh at & -\cosh at\end{pmatrix}\begin{pmatrix}\mathrm{d}t\\ \mathrm{d}x\end{pmatrix}. \tag{72}$$

Thus, at $x = 0$, we have that

$$\begin{pmatrix} \mathrm{d}u_- \\ \mathrm{d}v_- \end{pmatrix} = \begin{pmatrix} \cosh 2at & -\sinh 2at \\ \sinh 2at & -\cosh 2at \end{pmatrix} \begin{pmatrix} \mathrm{d}u_+ \\ \mathrm{d}v_+ \end{pmatrix}. \tag{73}$$

This is just a spatial reflection followed by a Lorentz transformation with $\beta = \tanh 2at$. An arbitrary vector in the tangent space of the particle worldline has two coordinate descriptions A_+^μ and A_-^μ, and these are related by the same transformation law:

$$\begin{pmatrix} A_-^0 \\ A_-^1 \end{pmatrix} = \begin{pmatrix} \cosh 2at & -\sinh 2at \\ \sinh 2at & -\cosh 2at \end{pmatrix} \begin{pmatrix} A_+^0 \\ A_+^1 \end{pmatrix}. \tag{74}$$

In summary, the new coordinates are convenient because they reduce the metric to Minkowski form, but the cost of this simplification is that instead of one global coordinate patch (t, x) we need two coordinate patches (u_+, v_+) and (u_-, v_-) that are glued together in a nontrivial way.

4.2. Trajectory of a freely falling test particle

Let us now consider the trajectory of a test particle[12] falling in the spacetime described by equation (62). Intuitively, we expect the test particle to be attracted by the gravitational field of the source particle and to undergo periodic oscillations about the source particle worldline. One can verify this by calculating the trajectory of the test particle in the (t, x) coordinate system, using either equation (48), the equation of motion for the flat-spacetime interpretation, or equation (54), the equation of motion for the curved-spacetime interpretation. We will consider an alternative approach, however, which makes use of the (u_+, v_+) and (u_-, v_-) coordinates, and helps clarify the way in which the two coordinate patches are glued together.

Let us assume that the test particle starts out to the right of the source particle worldline, so initially we can describe its trajectory using the (u_+, v_+) coordinates. The metric tensor in this coordinate system is just the Minkowski metric, so the test particle moves at a constant velocity until it hits the hyperbola H_+. We then switch over to the (u_-, v_-) coordinate system, using equations (70) and (74) to obtain the initial position and velocity of the test particle in this system. The test particle once again moves at a constant velocity until it hits the hyperbola H_-, at which point we switch back to the (u_+, v_+) coordinate system and the whole process repeats itself.

We can illustrate this with a specific example. Consider a portion of a test particle trajectory that consists of two segments: for $-s_0 \leqslant s \leqslant 0$ the trajectory is given by

$$u_-(s) = s\cosh\theta, \qquad v_-(s) = a^{-1} - s\sinh\theta, \tag{75}$$

and for $0 \leqslant s \leqslant s_0$ the trajectory is given by

$$u_+(s) = s\cosh\theta, \qquad v_+(s) = a^{-1} + s\sinh\theta, \tag{76}$$

where $s_0 \equiv 2a^{-1}\sinh\theta$. Note that in each segment the particle moves at a constant velocity $\beta_\pm = \pm\tanh\theta$. We can verify that the two segments are glued together properly by checking that they satisfy equations (70) and (74) at $s = 0$:

$$u_+ = u_-, \qquad v_+ = v_-, \qquad \frac{\mathrm{d}u_+}{\mathrm{d}s} = \frac{\mathrm{d}u_-}{\mathrm{d}s}, \qquad \frac{\mathrm{d}v_+}{\mathrm{d}s} = -\frac{\mathrm{d}v_-}{\mathrm{d}s}. \tag{77}$$

where we have used that $t = 0$ for the point on the trajectory at $s = 0$.

It is straightforward to transform the trajectory into the (t, x) coordinate system by inverting equations (65)–(68). Figure 1 illustrates the trajectory for the case $\theta = 1/2$.

[12] We will assume the mass of the test particle is small compared to mass m of the source particle, so it does not significantly alter the spacetime geometry.

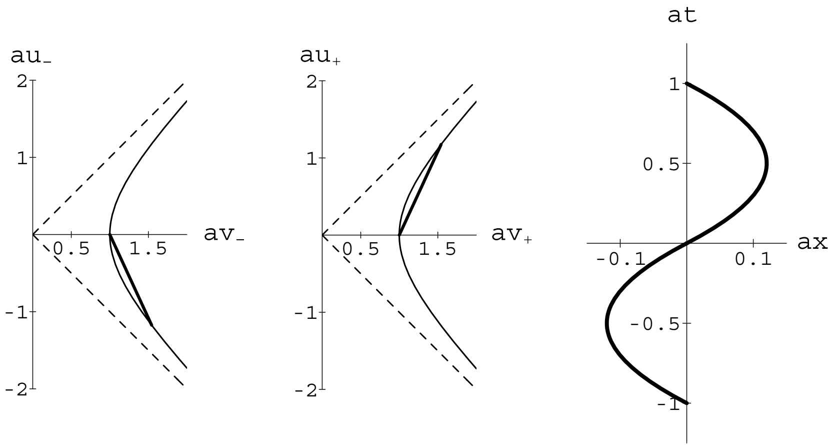

Figure 1. Test particle trajectory in the (u_-, v_-), (u_+, v_+) and (t, x) coordinate systems. The thick lines indicate the trajectory of the test particle; the thin lines in the (u_-, v_-) and (u_+, v_+) coordinate systems indicate the hyperbolas H_- and H_+.

5. Solution for a static, uniform density source

As a second example, we will present a solution for the case of a static, uniform density source[13]. This solution provides a simple example that illustrates some of the techniques involved in finding solutions for relativistic stars. We will assume that the source is a perfect fluid, so its energy–momentum tensor is given by[14]

$$T^{\alpha\beta}(x) = (\rho + p(x))u^\alpha(x)u^\beta(x) - p(x)g^{\alpha\beta}, \tag{78}$$

where ρ is the density, and $p(x)$ and $u^\alpha(x)$ are the pressure and velocity of the fluid at the point x. Let us choose a system of coordinates such that the metric tensor has the form $g_{\mu\nu} = \mathrm{e}^{2\phi}\eta_{\mu\nu}$. Since the fluid is static, in this coordinate system the components of the velocity vector u^α are

$$u^0 = \mathrm{e}^{-\phi} \qquad u^1 = 0. \tag{79}$$

Thus, the components of the energy–momentum tensor are

$$T^{00} = \mathrm{e}^{-2\phi}\rho, \qquad T^{11} = \mathrm{e}^{-2\phi}p, \qquad T^{01} = 0, \qquad T^{10} = 0. \tag{80}$$

The energy–momentum tensor must obey the conservation law

$$\nabla_\beta T^{\alpha\beta} = \partial_\beta T^{\alpha\beta} + \Gamma^\alpha{}_{\mu\beta}T^{\mu\beta} + \Gamma^\beta{}_{\mu\beta}T^{\alpha\mu} = 0, \tag{81}$$

where ∇_β indicates a covariant derivative. Substituting for the components of $T^{\alpha\beta}$ using equation (80), and for the Christoffel symbols using the results of appendix B, we find that

$$\nabla_\beta T^{1\beta} = \mathrm{e}^{-2\phi}(\partial_x p + (\rho + p)\partial_x\phi). \tag{82}$$

Thus, we obtain an equation of hydrostatic equilibrium:

$$\partial_x p = -(\rho + p)\partial_x\phi. \tag{83}$$

This equation describes how the pressure must vary inside the source in order to balance the gravitational attraction.

[13] This is the (1+1)-dimensional analog of an exact solution due to Schwarzschild for the interior of a uniform-density star (see [1], pp 609–12).
[14] See [15], equation 2.10.7.

Using equation (80) for the components of $T^{\alpha\beta}$, and using the results of appendix B for the curvature scalar R, we find that the field equation is given by

$$R = -2\mathrm{e}^{-2\phi}\,\Box\phi = 4G g_{\alpha\beta}T^{\alpha\beta} = 4G(\rho - p). \tag{84}$$

Since the field is static, this reduces to

$$\partial_x^2\phi = 2G(\rho - p)\,\mathrm{e}^{2\phi}. \tag{85}$$

The equation of hydrostatic equilibrium (83) and the field equation (85) form a coupled set of equations that can be solved for the pressure $p(x)$ and the gravitational potential $\phi(x)$. We can simplify these equations by introducing a dimensionless spatial coordinate

$$u = (4Gp_0)^{1/2}\,\mathrm{e}^{\phi_0}x \tag{86}$$

and a dimensionless pressure

$$P(u) = p(u)/p_0, \tag{87}$$

where $p_0 \equiv p(0)$ and $\phi_0 \equiv \phi(0)$ are the pressure and gravitational potential at the center of the source. In terms of these new quantities, equations (83) and (85) become

$$P'(u) = -(\alpha + P(u))\phi'(u) \tag{88}$$

$$\phi''(u) = \tfrac{1}{2}(\alpha - P(u))\,\mathrm{e}^{2(\phi(u)-\phi_0)}, \tag{89}$$

where the primes denote derivatives with respect to u and where $\alpha \equiv \rho/p_0$ is the density-to-pressure ratio at the center of the source. We can integrate equation (88) to obtain

$$\phi(u) = \phi_0 - \log\left(\frac{\alpha + P(u)}{\alpha + 1}\right). \tag{90}$$

Or, solving for the pressure,

$$P(u) = (\alpha + 1)\,\mathrm{e}^{-(\phi(u)-\phi_0)} - \alpha. \tag{91}$$

Substituting this result into equation (89), we find

$$\phi''(u) = \alpha\,\mathrm{e}^{2(\phi(u)-\phi_0)} - \tfrac{1}{2}(\alpha + 1)\,\mathrm{e}^{(\phi(u)-\phi_0)}. \tag{92}$$

We will assume that the source is symmetric about $u = 0$, so $\phi'(0) = 0$. Using this boundary condition, we can write the solution to equation (92):

$$\phi(u) = \phi_0 - \log\left(\tfrac{1}{2}(\alpha + 1 - (\alpha - 1)\cosh u)\right). \tag{93}$$

If we substitute this result into equation (91), we find that

$$P(u) = \tfrac{1}{2}(\alpha^2 + 1 - (\alpha^2 - 1)\cosh u). \tag{94}$$

Thus, the pressure decreases as we move outwards from the center at $u = 0$, and eventually reaches zero at some point $u = r$ that defines the surface of the source. We can determine the location of this point by solving the equation $P(r) = 0$ for r; we find that

$$r = \log\left(\frac{\alpha + 1}{\alpha - 1}\right). \tag{95}$$

So far we have only considered the interior of the source, which corresponds to the region $|u| \leqslant r$. Outside the source the energy–momentum tensor vanishes, so the field equation is

$$\phi''(u) = 0, \tag{96}$$

and the potential is given by

$$\phi(u) = A|u| + B \tag{97}$$

for some constants A and B. We can determine these constants by matching the exterior solution given by equation (97) to the interior solution given by equation (93). From equations (93) and (95), we find that

$$\phi(\pm r) = \phi_0 + \log(1 + 1/\alpha) \tag{98}$$

$$\phi'(\pm r) = \pm 1. \tag{99}$$

Thus, the exterior solution, which applies to the region $|u| \geqslant r$, is given by

$$\phi(u) = |u| - r + \phi_0 + \log(1 + 1/\alpha). \tag{100}$$

It is instructive to compare this solution to the analogous solution for a constant-density source in Newtonian gravity. The field equation for Newtonian gravity is given by equation (7), so for a source with a constant density ρ we have that

$$\partial_x^2 \phi(x) = 2G\rho. \tag{101}$$

To determine how the pressure $p(x)$ varies inside the source, let us consider the forces acting on a small mass element of length δx. The mass element feels an inward force $-\rho\delta x \partial_x \phi(x)$ due to the gravitational field, and an outward force $\delta x \partial_x p(x)$ due to the pressure variation. If we require that these forces balance, we obtain an equation of hydrostatic equilibrium:

$$\partial_x p(x) = -\rho \partial_x \phi(x). \tag{102}$$

Equations (101) and (102) are the Newtonian analogs of equations (85) and (83). We can solve these equations for the gravitational potential and the pressure inside the source:

$$\phi(u) = \tfrac{1}{4}\alpha u^2 \tag{103}$$

$$P(u) = 1 - \tfrac{1}{4}\alpha^2 u^2, \tag{104}$$

where for simplicity we have assumed that $\phi_0 = 0$, and as before $\alpha \equiv \rho/p_0$, $u \equiv (4Gp_0)^{1/2}x$ and $P(u) = p(u)/p_0$. The surface of the source is located at the point $u = r$ where the pressure vanishes, so we find that $r = 2/\alpha$. Using these results, it is straightforward to show that the relativistic solution reduces to the Newtonian solution in the limit $\alpha \gg 1$ in which the density of the source is much larger than the pressure at the center of the source.

Appendix A. Energy–momentum tensor

Here we show how to obtain a conserved energy–momentum tensor[15] for a system that is described in terms of an action S. First, we write the action in covariant form; that is, in a form that is invariant under general coordinate transformations. We can then obtain the energy–momentum tensor $T^{\alpha\beta}$ by varying this covariant action with respect to the metric tensor:

$$\delta S = -\frac{1}{2}\int T^{\alpha\beta}\delta g_{\alpha\beta} g^{1/2}\,\mathrm{d}^2 x, \tag{A.1}$$

where $g \equiv -\det g_{\mu\nu}$. For this method to be applicable, all the fields in the action must be dynamical; in other words, there can be no explicit spacetime dependence of the Lagrangian density. There are two results that are useful in performing the variations. First, the variation of the inverse metric tensor is given by

$$\delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}. \tag{A.2}$$

[15] See [15, pp 360–3] for a discussion of this method and a proof that the resulting energy–momentum tensor is conserved.

Second, the variation of g is given by[16]

$$\delta g = g g^{\mu\nu} \delta g_{\mu\nu}. \tag{A.3}$$

To illustrate this technique, we will calculate the energy–momentum tensor for a free point particle and for a free scalar field. For a free point particle, the action is

$$S_p = -m \int (\eta_{\alpha\beta} v^\alpha v^\beta)^{1/2} \, \mathrm{d}\lambda. \tag{A.4}$$

In covariant form, this action becomes

$$S_p = -m \int\int (g_{\alpha\beta} v^\alpha v^\beta)^{1/2} \delta^{(2)}(x - z(\lambda)) \, \mathrm{d}\lambda \, \mathrm{d}^2 x. \tag{A.5}$$

If we vary this action with respect to $g_{\alpha\beta}$, we find that the energy–momentum tensor is

$$T_p^{\alpha\beta} = m g^{-1/2} \int (g_{\alpha\beta} v^\alpha v^\beta)^{-1/2} v^\alpha v^\beta \delta^{(2)}(x - z(\lambda)) \, \mathrm{d}\lambda. \tag{A.6}$$

For a flat spacetime with $g_{\mu\nu} = \eta_{\mu\nu}$, the energy–momentum tensor is

$$T_p^{\alpha\beta} = m \int w^\alpha w^\beta \delta^{(2)}(x - z(\tau)) \, \mathrm{d}\tau. \tag{A.7}$$

For a free scalar field, the action is

$$S_f = \frac{1}{4G} \int (\partial_\mu \phi)(\partial^\mu \phi) \, \mathrm{d}x^2. \tag{A.8}$$

In covariant form, this action becomes

$$S_f = \frac{1}{4G} \int g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) g^{1/2} \, \mathrm{d}x^2. \tag{A.9}$$

If we vary this action with respect to $g_{\alpha\beta}$, we find that the energy–momentum tensor is

$$T_f^{\mu\nu} = \frac{1}{2G} \left(g^{\mu\alpha} g^{\nu\beta} (\partial_\alpha \phi)(\partial_\beta \phi) - \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} (\partial_\alpha \phi)(\partial_\beta \phi) \right). \tag{A.10}$$

For a flat spacetime with $g_{\mu\nu} = \eta_{\mu\nu}$, the energy–momentum tensor is

$$T_f^{\mu\nu} = \frac{1}{2G} \left((\partial^\mu \phi)(\partial^\nu \phi) - \frac{1}{2} \eta^{\mu\nu} (\partial_\alpha \phi)(\partial^\alpha \phi) \right). \tag{A.11}$$

Appendix B. Geometry in conformal coordinates

Here we calculate the Christoffel symbols and curvature scalar for the metric tensor $g_{\mu\nu} = \mathrm{e}^{2\phi} \eta_{\mu\nu}$. The Christoffel symbols are

$$\Gamma^\mu{}_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu} (\partial_\alpha g_{\nu\beta} + \partial_\beta g_{\alpha\nu} - \partial_\nu g_{\alpha\beta}) \tag{B.1}$$

$$= \eta^{\mu\nu} (\eta_{\nu\beta} \partial_\alpha \phi + \eta_{\alpha\nu} \partial_\beta \phi - \eta_{\alpha\beta} \partial_\nu \phi). \tag{B.2}$$

The Riemann curvature tensor is defined to be

$$R^\mu{}_{\alpha\nu\beta} = \partial_\nu \Gamma^\mu{}_{\alpha\beta} - \partial_\beta \Gamma^\mu{}_{\alpha\nu} + \Gamma^\gamma{}_{\alpha\beta} \Gamma^\mu{}_{\gamma\nu} - \Gamma^\gamma{}_{\alpha\nu} \Gamma^\mu{}_{\gamma\beta}, \tag{B.3}$$

and from this expression we can obtain the curvature scalar

$$R = g^{\alpha\beta} R^\mu{}_{\alpha\mu\beta} = -2 g^{\mu\nu} \partial_\mu \partial_\nu \phi = -2\mathrm{e}^{-2\phi} \eta^{\mu\nu} \partial_\mu \partial_\nu \phi = -2\mathrm{e}^{-2\phi} \Box \phi. \tag{B.4}$$

Note that we can also use equation (1) to express the Riemann tensor in terms of R.

[16] These results are derived in [3, p 258].

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