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Control of Mechanical Systems with Symmetries and Nonholonomic Constraints

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Abstract

This paper presents initial results on the control of mechanical systems for which group symmetries exist (i.e., the dynamics are invariant under the action of a Lie group) that are not fully annihilated by the addition of nonholonomic constraints. These types of systems are characterized by the persistence of momentum-like drift terms which are not directly controllable via the inputs to the system. We show that for systems with nonholonomic constraints (in direct contrast with unconstrained systems with symmetries or systems with holonomic constraints) there exists the possibility for controlling these momentum terms. The snakeboard is used as a motivating example, and some comment is given as to the utility of these equations for general robotic locomotion.

1. Introduction

Making use of modern advances in geometric mechanics, researchers have made great progress in analyzing the mechanics of locomotion. The problem of locomotion can be found in many different areas of study because it asks the fundamental question of how does a system use its control inputs to move from one place to another. By utilizing the inherent mathematical structure found in these types of problems, one can formulate the dynamics of a wide variety of locomotion problems in a very intuitively appealing and insightful manner. Doing so leads to a stronger comprehension of the *mechanics* of locomotion, but leaves open some very basic questions about the *control* of such types of systems.

An important by-product of the mechanics research in locomotion has been the development of a theoretical bridge between systems with two different types of nonholonomic constraints. On one hand, there are systems with *external* (often called kinematic) constraints which include wheeled vehicles [8], grasping with point-finger contacts, and some models of snakes [5], paramecia [10], and even legged locomotion [4]. On the other hand, unconstrained systems with Lie group symmetries possess *internal* (often called dynamic) nonholonomic constraints, which often take the form of momentum conservation laws. Examples include satellites in space [3] and the problem of the "falling cat" [7].

Naturally, there are problems for which both internal and external constraints may exist and interact in a nontrivial manner. Bloch et al. [1] have studied this case and formulated a generalized momentum to describe the interaction of the external constraints with the internal symmetries. There is strong evidence to suggest that many different modes of locomotion (such as undulatory, legged, etc.) are governed by equations of this form. To illustrate these ideas, we will examine the snakeboard model [1, 6], which has been an important motivating example behind the theoretical progress for this mixed kinematic and dynamic constraint case.

Along with the problem of mechanics comes a number of associated issues to investigate. For instance, extensive work has been done in the limiting cases of either purely kinematic *or* purely dynamic constraints, including controllability [2, 7] and trajectory generation [8, 4]. Along these lines, Bloch, Reyhanoglu, and McClamroch [2], studied the case of fully dynamic systems with nonholonomic constraints. Their paper, however, required that all of the unconstrained dynamics be controlled, and was not intended to address the special structure inherent in systems with Lie group symmetries.

2. Background and formulation

The use of Lie groups will be important for the analysis performed in this paper, motivated principally from our studies of robotic locomotion, where displacements occur in some subgroup of $SE(3)$, most often $SE(2)$ or $SO(3)$. The reader should keep in mind, however, that the results hold for general Lie groups. Formally, the displacement of a robot's body fixed frame is considered as a *left translation*. That is, if the initial position of a rigid body is denoted by g, and it is displaced by an amount h, then its final position is hg. Hence, we can describe the evolution of the *position* of the robot using a Lie group with respect to some inertial frame.

The remaining components of the system are assumed to be controllable, and these configuration variables will be represented by a shape manifold M. Thus, the configuration space will be the product manifold given by $Q = G \times M$. In the mechanics literature, the manifold Q defines a *trivial principal fiber bundle* that is said to have *fibers*, G, over a *base* space, M. We can then decompose the coordinates on Q into fiber and base coordinates, i.e., $q = (g,r) \in G \times M = Q$.

The group translation induces a left action, $\Phi : G \times Q \to Q$ that satisfies the two properties: (1) $\Phi(e,q) = q$ for all $q \in Q$, and e the identity element of G; and (2) $\Phi(h,\Phi(g,q)) = \Phi(hg,q)$ for every $g,h \in G$ and $q \in Q$. It will be useful to consider the left action as a map from Q into Q, with the element $h \in G$ held fixed. Notationally, $\Phi_h : Q \to Q$ is given by $(g,r) \mapsto (\Phi(h,g),r) = (hg,r)$. The *lifted action*, which describes the effect of Φ_h on velocity vectors in TQ, is the tangent map, $T_q\Phi_h : T_qQ \to T_{hq}Q$.

In working with mechanical systems, we assume the existence of a Lagrangian function, $L(q,\dot{q})$. We are interested in systems with nonholonomic constraints, which may take the form of no-slip wheel conditions or viscous friction. Given k linear velocity constraints, we can write them as a vector-valued set of k equations:

$$\omega^i_j(q)\dot{q}^j = 0, \qquad \text{for} \quad i = 1 \ldots k. \tag{1}$$

This class of constraints includes most commonly investigated nonholonomic constraints.

Linear constraints can be incorporated into the dynamics through the use of Lagrange multipliers. That is, the dynamic equations become

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \lambda_j \omega^j_i - \tau_i = 0, \tag{2}$$

where λ is an unknown multiplier representing a force of constraint, and τ is an external forcing function. Solving for Lagrange multipliers can be an involved procedure, and may hide much of the intrinsic geometric structure of the problem.

For systems in which the Lagrangian and the constraints are left-invariant, i.e., for which $\omega^i(hq) = T^*_{hq}\Phi_{h^{-1}}\omega(q)$ and $L(\Phi_h q, T_q\Phi_h \dot{q}) = L(q, \dot{q})$, it was shown in [1, 9] that the equations of motion can be transformed into the following form:

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \tilde{I}^{-1}(r)p, \tag{3}$$

$$\dot{p} = \frac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T\sigma_{p\dot{r}}(r)\dot{r} + \frac{1}{2}p^T\sigma_{pp}(r)p, \tag{4}$$

$$\ddot{r} = u. \tag{5}$$

These equations, of course, deserve a good deal of comment (to gain a much better insight into these equations, refer to [1]). Eqs. 3 and 5 are the fiber and base equations, respectively. They will define velocity vectors for the configuration variables. Eq. 4 is called the *generalized momentum equation*, where p is a momentum vector associated with the momentum along each of the kinematically unconstrained fiber directions. Notice that in Eq. 5 we have assumed the base (shape) space to be fully controllable, with acceleration inputs, u.

Of particular importance, however, is the term $\mathbb{A}(r)$ in Eq. 3. In the geometric mechanics nomenclature, $\mathbb{A}$ is said to define a *connection* on the fiber bundle Q. The connection will satisfy certain geometric properties. Most importantly, it defines the relationship between control velocities on TM and group velocities on TG. As might be expected, derivatives of $\mathbb{A}$ will have a direct correspondence to Lie brackets of the control and drift vector fields.

The $\tilde{I}^{-1}p$ term determines the effect of the momentum on the fiber equations. For the terms, $\sigma_{\dot{r}\dot{r}}$, $\sigma_{p\dot{r}}$, and σ_{pp} of the generalized momentum equation, we mention only that they are strictly functions of the base variables, r, and so the generalized momentum equation can be written *solely* as a function r and p [9]. With the inputs as accelerations, Eqs. 3–5 can be written in a standard form for nonlinear control systems with drift:

$$\dot{z} = f(z) + h_i(z)u^i, \tag{6}$$

where $z = (g, p, r, \dot{r}) \in N = G \times \mathbb{R}^p \times M \times T_r M$.

Example 1 Now let us turn to a formulation of the snakeboard problem in terms of the relationships derived above. The *Snakeboard* is a commercial variant of the skateboard, in which the wheel trucks are allowed to rotate independently. The simplified model of the *Snakeboard* (referred to as the snakeboard model) is shown in Figure 1. We briefly recall here the description of the snakeboard as developed in [6].

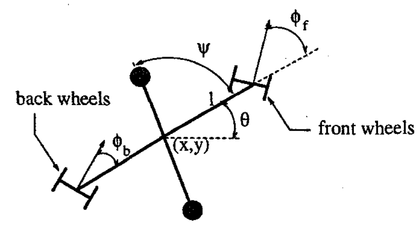

Fig. 1: The simplified model of the *Snakeboard*.

We begin by following two assumptions made in [1, 9] First, we assume that the wheels are controlled to move out of phase with each other, in opposite directions. In other words, using the symbols given in Figure 1, let $\phi = \phi_b = -\phi_f$. Second, along the lines of Bloch et al., we will assume that $J + J_r + 2J_w = ml^2$.

With these assumptions, the snakeboard has a configuration manifold given by $Q = SE(2) \times \mathbb{S}^1 \times \mathbb{S}^1$. $SE(2)$ describes the position and orientation of the board with respect to some inertial reference frame. As coordinates for Q we shall use $(x, y, \theta, \psi, \phi)$ where (x, y, θ) describes the position and orientation, ψ is the angle of the rotor with respect to the board, and ϕ and $-\phi$ are, respectively, the angles of the back and front wheels with respect to the board. The configuration space easily splits into a trivial fiber bundle structure, with $q = (g, r)$ given by $g = (x, y, \theta) \in G = SE(2)$ and $r = (\psi, \phi) \in M = \mathbb{S}^1 \times \mathbb{S}^1$. The left action for a group element, $h = (a^1, a^2, \alpha) \in G$, is given by the map:

$$\Phi_h(x, y, \theta, \psi, \phi) = (x\cos\alpha - y\sin\alpha + a^1, \quad x\sin\alpha + y\cos\alpha + a^2, \theta + \alpha, \psi, \phi).$$

For the snakeboard, we denote by m the mass of the board, l the length from the board's center of mass to the wheels, and J, J_r, and J_w the inertias of the board, rotor and wheels, respectively. For the snakeboard, the unconstrained Lagrangian is given simply by kinetic energy terms as

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) + \frac{J}{2}\dot{\theta}^2 + \frac{J_r}{2}(\dot{\psi} + \dot{\theta})^2 + \frac{J_w}{2}(\dot{\phi}^2 + \dot{\theta}^2).$$

The control torques are applied at the wheels and the rotor, and the wheels of the snakeboard are assumed to roll without lateral sliding. The nonholonomic wheel constraints for the back and front wheels are, respectively,

$$-\sin(\theta + \phi)\dot{x} + \cos(\theta + \phi)\dot{y} - l\cos(\phi)\dot{\theta} = 0, \tag{7}$$

$$-\sin(\theta - \phi)\dot{x} + \cos(\theta - \phi)\dot{y} + l\cos(\phi)\dot{\theta} = 0 \tag{8}$$

A quick set of calculations shows that both the Lagrangian and the constraint one-forms are invariant with respect to the lifted group action. The momentum is defined along unconstrained directions tangent to the fiber. Thus, we define the *constrained fiber distribution* for this problem to be the one-dimensional subspace,

$$\mathcal{S}_q = \mathrm{sp}\{a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta}\}$$

where $a = -l\cos^2\phi\cos\theta$, $b = -l\cos^2\phi\sin\theta$, and $c = \sin(2\phi)$. All vectors in $\mathcal{S}_q$ are tangent to the group G and satisfy the constraints in Eqs. 7 and 8. The generalized momentum, p, is thus one-dimensional. Let $\langle\!\langle , \rangle\!\rangle$ denote the inner product defined by the kinetic energy metric for our mechanical system. Then

$$\begin{aligned} p &= \langle\!\langle \dot{q}, X(q) \rangle\!\rangle \\ &= \langle\!\langle (\dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}, \dot{\phi}_b, \dot{\phi}_f), (a, b, c, 0, 0, 0) \rangle\!\rangle \\ &= ma\dot{x} + mb\dot{y} + ml^2 c\dot{\theta} + J_r c\dot{\psi} + J_w c(\dot{\phi}_b + \dot{\phi}_f), \end{aligned}$$

where $X(q) \in \mathcal{S}_q$.

Writing the equations in the form of Eqs. 3–5 gives

$$\mathbb{A} = \begin{pmatrix} -\frac{J_r}{2ml}\sin 2\phi & 0 \\ 0 & 0 \\ \frac{J_r}{2ml^2}\sin^2\phi & 0 \end{pmatrix} \quad \text{and} \quad \tilde{I}^{-1} = \begin{pmatrix} -\frac{1}{2l} \\ 0 \\ \frac{1}{2ml^2}\tan\phi \end{pmatrix}.$$

The generalized momentum equation is then just

$$\dot{p} = 2J_r\cos^2\phi\,\dot{\phi}\dot{\psi} - \tan\phi\,\dot{\phi}p.$$

Finally, the dynamics on the base space reduce to

$$\begin{aligned} (1 - \frac{J_r}{ml^2}\sin^2\phi)\ddot{\psi} &= \frac{J_r}{2ml^2}\sin 2\phi\,\dot{\phi}\dot{\psi} - \frac{1}{2ml^2}\dot{\phi}p + \frac{1}{J_r}\tau_\psi \\ \ddot{\phi} &= \frac{1}{2J_w}\tau_\phi, \end{aligned}$$

which is easily inverted to show controllability of the base variables, which we will write as $\ddot{\psi} = u_\psi$ and $\ddot{\phi} = u_\phi$.

Let $z = (x, y, \theta, p, \psi, \phi, \dot{\psi}, \dot{\phi}) \in N$, and then we can write the snakeboard equations in control form with

$$\begin{aligned} f = \Big(&\frac{\cos\theta}{2ml}(J_r\sin 2\phi\,\dot{\psi} - p), \frac{\sin\theta}{2ml}(J_r\sin 2\phi\dot{\psi} - p), \\ &-\frac{\tan\phi}{2ml^2}(J_r\sin 2\phi\dot{\psi} - p), \\ &2J_r\cos^2\phi\dot{\phi}\dot{\psi} - p\tan\phi\dot{\phi}, \dot{\psi}, \dot{\phi}, 0, 0\Big)^T. \end{aligned}$$

3. Local accessibility

First, we must give notions of accessibility and controllability. Let $\mathcal{R}^V(z_0, T)$ denote the set of reachable points in N from z_0 at time $T > 0$, using admissible controls, $u(t)$, and such that the trajectories remain in the neighborhood V of z_0 for all $t \leq T$. Furthermore, let

$$\mathcal{R}^V_T(z_0) = \cup_{t \leq T}\mathcal{R}^V(z_0, t)$$

be the set of all reachable points from z_0 within time T.

Definition 1 The system given by Eq. 6 is locally accessible *if for all $z \in N$, $\mathcal{R}^V_T(z)$ contains a non-empty open set of N for all neighborhoods V of z and all $T > 0$. The system is called* small-time locally controllable *(STLC) if z is an interior point of $\mathcal{R}^V_T(z)$ for all $T > 0$.*

For systems of the form of Eq. 6, we can check accessibility using the Lie algebra rank condition (LARC). To do so, let $\Delta_0 = \mathrm{span}\{f, h_1, \ldots, h_m\}$ (spanning over $\mathcal{C}^\infty$ functions of N), and iteratively define

$$\Delta_k = \Delta_{k-1} + \mathrm{span}\{[X, Y] \,|\, X, Y \in \Delta_{k-1}\}.$$

This is a nondecreasing sequence of distributions on N which ill terminate under certain regularity conditions. We will call $C = \Delta_\infty$ the *accessibility distribution.* The LARC states that the system will be globally (and hence locally) accessible if $C = TN$.

In their paper [4], Kelly and Murray show, similar to [7], that the controllability of a kinematic system can be determined solely from the connection, $\mathbb{A}$, its curvature, and higher derivatives. Using this as motivation, we define a sequence of subgroups of the Lie algebra, $\mathfrak{g}$, of G.

$$\begin{aligned} \mathfrak{h}_1 &= \mathrm{span}\{\mathbb{A}(X) : X \in T_rM\} \\ \mathfrak{h}_2 &= \mathrm{span}\{D\mathbb{A}(X, Y) : X, Y \in T_rM\} \\ &\vdots \\ \mathfrak{h}_k &= \mathrm{span}\{L_X\xi - [\mathbb{A}(Z), \xi], [\eta, \xi] : X \in T_rM, \\ &\qquad \xi \in \mathfrak{h}_{k-1}, \eta \in \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{k-1}\}, \end{aligned} \tag{9}$$

where the curvature (exterior derivative) of the connection is defined with respect to the structure equations as

$$D\mathbb{A}(X, Y) = d\mathbb{A}(X, Y) + [\mathbb{A}(X), \mathbb{A}(Y)],$$

with $[\mathbb{A}(X), \mathbb{A}(Y)]$ the Lie bracket on $\mathfrak{g}$ and d denoting exterior differentiation.

Next, we examine a few of the lower order brackets in the accessibility distribution, C, which play an important role in the accessibility and controllability analyses to follow. The only nonzero first order brackets (those in Δ_1) are those which bracket the control vector field with the drift vector field. A quick calculation shows that

$$\alpha_i := [f, h_i] = \begin{pmatrix} -\mathbb{A}_i(r) \\ (\sigma_{\dot{r}\dot{r}})_{ij}\dot{r}^j + (\sigma_{p\dot{r}})^j_i p_j \\ e_i \\ 0 \end{pmatrix}.$$

Moving to the second order brackets, an interesting thing happens when we bracket h_i with α_j:

$$\beta_{ij} := [h_i, \alpha_j] = [h_i, [f, h_j]] = \begin{pmatrix} 0 \\ (\sigma_{\dot{r}\dot{r}})_{ij} \\ 0 \\ 0 \end{pmatrix}.$$

Thus, the $\sigma_{\dot{r}\dot{r}}$ term (which is a cross-coupling term for the base variables that drives the momenta) directly affects the momentum variables via the β_{ij} brackets. Viewing this coupling as a map, $\sigma_{\dot{r}\dot{r}} : TM \times TM \to \mathbb{R}^p$, surjectivity of $\sigma_{\dot{r}\dot{r}}$ implies that all of the momentum directions can be generated via this second order bracket. This leads us directly to a test for local accessibility. Detailed proofs of the results here can be found in [9].

Proposition 2 Assume that $\sigma_{\dot{r}\dot{r}}$ is onto and that

$$\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \cdots,$$

where the $\mathfrak{h}_k$'s are defined as above using the local form of the connection in Eqs. 3–5. Then the system given by these equations is locally accessible.

4. Local Controllability

Unfortunately, for nonlinear systems with drift, local accessibility may be quite different from local controllability. In examining small-time local controllability, we use sufficient conditions given bySussman [11]. For reasons of clarity and brevity, the treatment here will not be completely mathematically rigorous. For details on how this can be done more rigorously, please refer to [2, 9, 11].

Let $h_0 := f$ so that $\Delta_0 = \text{span}\{h_0, h_1, \ldots, h_m\}$, and let $\delta^i(X)$ be the number of times that h_i appears in a given set of brackets, X. Similar to Sussman, define the *degree*, δ, of a bracket X to be $\delta = \sum_{i=0}^{m} \delta^i(X)$. Then we have the following theorem due to Sussman:

Theorem 3 [11] Given the system of Eq. 6, with $h_0(z_0) = f(z_0) = 0$ at an equilibrium point $z_0 \in N$, assume that $(h_0, \ldots, h_m)$ satisfy the LARC at z_0. Further, assume that whenever X is a bracket for which $\delta^0(X)$ is odd and $\delta^1(X), \ldots, \delta^m(X)$ are all even, then there exist brackets $Y_1, \ldots, Y_k$ such that $X = \xi^i Y_i$, for some $\xi^1, \ldots, \xi^k \in \mathbb{R}$, and

$$\delta(Y_i) < \delta(X), \quad \text{for} \quad i = 1, \ldots, m.$$

Then the system defined by Eq. 6 is STLC from z_0.

In other words, we will define a "bad" bracket to be those for which the drift term appears an odd number of times and for which the control vector fields each appear an even number of times (including zero times). The requirement for small-time local controllability, then, will be that all "bad" brackets can be written in terms of brackets of lower degree.

Proposition 4 [9] Assume that the system defined by Eqs. 3–5 is locally accessible, that $\sigma_{\dot{r}\dot{r}}$ is surjective, and that $(\sigma_{\dot{r}\dot{r}})_{ii} \equiv 0$ for $i = 1, \ldots, m$ (no summation over i). Then this system is small-time locally controllable (STLC) from all equilibrium points, $z_0 \in N$.

Example 1: (cont'd) We return to the snakeboard example to investigate controllability. Obviously, the bracket of the control inputs, $[h_\psi, h_\phi]$, is identically zero. Similarly, the brackets $\alpha_\psi = [f, h_\psi]$ and $\alpha_\phi = [f, h_\phi]$ yield the respective velocity directions. Along with the control inputs, this will imply control of the base (controlled dynamics). In order to show accessibility and controllability (STLC), one of the first criteria to be satisfied is the conditions on $\sigma_{\dot{r}\dot{r}}$, given by the following third order brackets. First, we need the diagonal elements of $\sigma_{\dot{r}\dot{r}}$ to be zero. This is given by $\beta_{\phi\phi} = \beta_{\psi\psi} = 0$. Then, we look at off diagonal terms to show that $\sigma_{\dot{r}\dot{r}}$ is surjective. To see this, we simply give the necessary bracket:

$$\beta_{\phi\psi} = \begin{pmatrix}0, & 0, & 0, & 2J_r\cos^2\phi, & 0, & 0, & 0, & 0\end{pmatrix}^T,$$

The bracket $\beta_{\phi\psi} = [\alpha_\phi, h_\psi]$ is nonzero for all $\phi \neq \frac{\pi}{2}$ and so $\sigma_{\dot{r}\dot{r}}$ is surjective.

Finally, to demonstrate that the snakeboard is controllable, we need show that $\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \ldots$. We begin by computing $[\alpha_\phi, \alpha_\psi]$, which gives us the curvature of the connection, $D\mathbb{A}$. This yields terms of the form:

$$\begin{pmatrix}\frac{J_r}{ml}\cos 2\phi, & 0, & -\frac{J_r}{ml^2}\sin 2\phi\end{pmatrix}^T \in \mathfrak{h}_2.$$

Then, $[\alpha_\phi, [\alpha_\psi, \alpha_\phi]]$ yields

$$\begin{pmatrix}-\frac{2J_r}{ml}\sin 2\phi, & 0, & -\frac{2J_r}{ml^2}\cos 2\phi\end{pmatrix}^T \in \mathfrak{h}_3,$$

and $[\alpha_\phi, [\alpha_\psi, [\alpha_\psi, [\alpha_\phi, \alpha_\psi]]]]$ gives

$$\begin{pmatrix}0, & \frac{2J_r{}^2}{m^2l^3}\cos 2\phi, & 0\end{pmatrix}^T \in \mathfrak{h}_5.$$

Thus, $\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \mathfrak{h}_5$, and the conditions for Proposition 4 are satisfied.

5. Conclusion

This paper establishes easily computable accessibility and controllability results for systems on principal fiber bundles with external nonholonomic constraints. These types of systems are characterized by the existence of a connection, which relates the control inputs to the motion of the system. The connection has been discussed in its direct implications for accessibility and controllability. Furthermore, these types of systems will very often include drift vector fields, in the form of momentum terms. Research has shown that many problems of locomotion can be formulated in terms of this dynamical structure. Future work will be concerned with further exploiting the geometric structure of the problem, and in developing results governing trajectory generation and optimal control of locomotive gait patterns.

References

[1] A. M. Bloch, P. S. Krishnaprasad, J. E. Marsden, and R. M. Murray. Nonholonomic mechanical systems and symmetry. Technical report, California Institute of Technology, Pasadena, CA, 1994. To appear, *Archive for Rational Mechanics and Analysis*. Available electronically via http://avalon.caltech.edu/cds/reports/cds94-013.ps.

[2] A. M. Bloch, M. Reyhanoglu, and N. H. McClamroch. Control and stabilization of nonholonomic dynamic systems. *IEEE Transactions on Automatic Control*, 37(11):1746–1757, 1992.

[3] C. K. Chen and N. Sreenath. Control of coupled spatial multibody systems with nonholonomic constraints. In *Proc. of the 32nd IEEE Conf. on Decision and Control*, pages 949–954, December 1993.

[4] S. D. Kelly and R. M. Murray. Geometric phases and locomotion. *Journal of Robotic Systems*, 12(6):417–431, June 1995.

[5] P. S. Krishnaprasad and D. P. Tsakiris. *G*-snakes: Nonholonomic kinematic chains on Lie groups. In *Proc. 33rd IEEE Conf. on Decision and Control*, Lake Buena Vista, FL, December 1994.

[6] A. D. Lewis, J. P. Ostrowski, R. M. Murray, and J. W. Burdick. Nonholonomic mechanics and locomotion: The snakeboard example. In *Proc. IEEE Int. Conf. Robotics and Automation*, pages 2391–7, San Diego, CA, May 1994.

[7] R. Montgomery. Nonholonomic control and guage theory. In Z. Li and J. F. Canny, editors, *Nonholonomic Motion Planning*, pages 343–378. Kluwer, 1993.

[8] R. M. Murray and S. S. Sastry. Nonholonomic motion planning: Steering using sinusoids. *IEEE Transactions on Automatic Control*, 38(5):700–716, 1993.

[9] J. P. Ostrowski. *The Mechanics and Control of Undulatory Robotic Locomotion*. Ph.D. thesis, California Institute of Technology, Pasadena, CA, 1995.

[10] A. Shapere and F. Wilczek. Geometry of self-propulsion at low Reynolds number. *Journal of Fluid Mechanics*, 198:557–585, January 1989.

[11] H. J. Sussman. A general theorem on local controllability. *SIAM Journal on Control and Optimization*, 25(1):158–194, January 1987.