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A Quality Measure for Compliant Grasps

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Abstract— **This paper presents a systematic approach for quantifying the quality of compliant grasps. Appropriate tangent and cotangent subspaces to the object's configuration space are studied, from which frame-invariant characteristic compliance parameters are defined. Physical and geometric interpretations are given to these parameters, and a practically meaningful method is proposed to make the parameters comparable. A frame-invariant quality measure is then defined, and grasp optimization using this quality measure is discussed with examples.**

1 Introduction

This paper presents a quality measure for compliant grasps. Compliance plays a dominant role in passive grasps such as workpiece fixturing, and can also be used to model the finger forces in active grasps. To our knowledge, this is the first systematic approach to quantifying the quality of compliant grasps. The approach is frame-invariant and physically appealing. It applies to the grasping of 2D and 3D objects by any number of fingers, and can be used to determine the optimal finger placement. For the sake of convenience, the term *grasping* will also apply to *fixturing.*

Compliant grasps have received much attention. Hanafusa and Asada [3] used a linear spring model to find stable 3-fingered planar grasps. Nguyen [11] used a linear spring model to compute the stiffness matrix of more general grasps. Howard and Kumar [4] also used a linear spring compliance model to study grasp stability, but included the effects of contact geometry. In studying compliance due to friction, Cutkosky and Wright [1] noted that stability is influenced by initial loading as well as local curvature. While the linear spring compliance model has been widely used by roboticists, it is not supported by experiments or results from elasticity theory. Rimon and Burdick [14] used *overlap functions* to model nonlinear compliance effects. Lin, Burdick and Rimon [7] use these overlap functions to compute and analyze the grasp stiffness matrix for various contact models, including the widely verified and theoretically justified Hertz model. While the overlap model is used for illustration, our grasp quality measure can be used with any compliance model.

Nearly all prior work on quantifying grasp effectiveness has assumed rigid body mechanics. Let the wrench (i.e. force and torque) due to a unit force applied by a contacting finger be termed a *generating wrench.* Li and Sastry [6] suggests a quality measure that is the smallest singular value of the *grasp matrix*, whose columns consist of the generating wrenches. Kirkpatrick, Mishra and Yap [5] define the radius of the maximal ball inscribed in the convex hull of the generating wrenches as a quality measure. This idea is also followed by Ferrari and Canny [2]. However, these quality criteria are flawed by their dependence on the choice of coordinate frame; a grasp which is optimal under one choice of reference frame may fail to be optimal under another. Several authors have devised schemes to avoid this problem. Markenscoff and Papadimitriou [9] minimize the worst-case finger forces needed to balance any external unit force acting on the object. Mirtich and Canny [10] first compute the grasps that best counteract pure forces. Among these grasps, the one that best resists torques is chosen to be optimal. Teichmann [15] finds the largest inscribed ball (as defined in Ref. [5]) for all choices of coordinate frames, but does not discuss the computation of the optimal grasp.

This paper concerns the systematic development of quality measures for compliant grasps. Frame invariance is one of the main attributes of our approach. We consider frame-invariant subspaces of the object's tangent and cotangent spaces, from which frame-invariant characteristic compliance parameters are defined. We give novel geometric interpretations to these parameters, which are also defined by Patterson and Lipkin [12] in a different manner. We also propose a practically meaningful method for making these parameters comparable, and define a frame-invariant quality measure. Examples demonstrate these ideas.

2 Background

A grasp or fixturing arrangement consists of an object $\mathcal{B}$ contacted by k fingers $\mathcal{A}_1, \ldots, \mathcal{A}_k$. We assume that the contacts are *frictionless*, and that the bodies have a *smooth* boundary near the contact points. The bodies are assumed to be *quasi-rigid*, and the fingers $\mathcal{A}_i$ stationary. In the quasi-rigid assumption, deformations due to compliance effects are assumed to be localized to the vicinity of the contact points, so that the overall motion of $\mathcal{B}$ relative to $\mathcal{A}_i$ can be described using rigid body kinematics. This is an excellent assumption for

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fixturing of mechanical parts.

Since the fingers $\mathcal{A}_i$ are stationary, we can focus on the following *configuration space* (c-space) of $\mathcal{B}$, which is denoted $\mathcal{C}$. Choose a fixed world reference frame, $\mathcal{F}_W$, and a frame $\mathcal{F}_B$ fixed to $\mathcal{B}$. A *configuration* of $\mathcal{B}$ is specified by the position, $d \in \mathbb{R}^3$, and orientation, $R \in SO(3)$, of $\mathcal{F}_B$ relative to $\mathcal{F}_W$. C-space is given *hybrid coordinates* $q = (d, \theta) \in \mathbb{R}^3 \times \mathbb{R}^3$, which map to $(d, R(\theta))$. The mapping $R(\theta)$ is given by $R(\theta) = \exp(\widehat{\theta})$, where $\widehat{\theta}$ is a skew-symmetric matrix such that $\widehat{\theta}x = \theta \times x$ for $x \in \mathbb{R}^3$. The *tangent space* to $\mathcal{C}$ at a configuration q, denoted by $T_q\mathcal{C}$, is the set of all tangent vectors, or velocities of $\mathcal{B}$, at q. In hybrid coordinates, tangent vectors can be written as vectors $\dot{q} = (v, \omega)$, where $v \in \mathbb{R}^3$ is the velocity of the origin of $\mathcal{F}_B$, and $\omega \in \mathbb{R}^3$ is the angular velocity of $\mathcal{F}_B$. The *wrench space* at q, denoted by $T_q^*\mathcal{C}$, is the set of all wrenches (or covectors) acting on $\mathcal{B}$. A wrench takes the form $w = (f, \tau)$ in hybrid coordinates, where $f \in \mathbb{R}^3$ is a force acting at $\mathcal{F}_B$'s origin and $\tau \in \mathbb{R}^3$ is a torque. In the planar case, letting the z-axis be perpendicular to the plane and dropping the identically zero components, we have $v, f \in \mathbb{R}^2$ and $\omega, \tau \in \mathbb{R}$.

The hybrid parametrization of c-space depends on the choice of frames. Consider a new world frame, $\bar{\mathcal{F}}_W$, displaced from $\mathcal{F}_W$ by (d_w, R_w), and a new object frame, $\bar{\mathcal{F}}_B$, displaced from $\mathcal{F}_B$ by (d_b, R_b). A configuration with coordinates q would now have different coordinates $\bar{q}$. The tangent and cotangent vectors transform as follows:

$$\dot{\bar{q}} = T^{-1}\dot{q}, \qquad \bar{w} = T^T w, \tag{1}$$

where the transformation matrix for the 3D and 2D cases are given by

$$T_{6\times 6} = \begin{pmatrix} R_w & \widehat{R_0 d_b} R_w \\ 0 & R_w \end{pmatrix} \text{ and } T_{3\times 3} = \begin{pmatrix} R_w & JR_0 d_b \\ 0 & 1 \end{pmatrix}, \tag{2}$$

respectively. Here $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, and $R_0 = R(\theta_0)$. Since d_w and R_b do not appear in T, a translation of $\mathcal{F}_W$ or a rotation of $\mathcal{F}_B$ do not affect the transformation.

Rimon and Burdick [14] proposed a model for contact compliance which use *overlap functions.* These functions allow one to ignore the specific details of deformations when $\mathcal{B}$ and $\mathcal{A}_i$ are quasi-rigid. Rather, the net contact force is modelled as a function of the overlap of the two undeformed rigid body volumes that results from a relative displacement. The overlap approach is briefly reviewed here. In the absence of deformation, the two bodies $\mathcal{B}$ and $\mathcal{A}_i$ contact at a single point, and after deformation occurs the bodies inter-penetrate. The *overlap* between $\mathcal{B}$ and $\mathcal{A}_i$, denoted δ_i, is *the minimum amount of translation separating $\mathcal{B}$ from $\mathcal{A}_i$.* Clearly, δ_i depends on $\mathcal{B}$'s configuration: $\delta_i = \delta_i(q)$. We define $\delta_i = 0$ when $\mathcal{B}$ and $\mathcal{A}_i$ are disjoint or maintain surface contact. The net contact force is assumed to act on $\mathcal{B}$ at the initial contact point, in the direction of the separating translation. The force's magnitude, denoted f_i, is assumed to depend on the overlap $\delta_i(q)$: $f_i = f_i(\delta_i(q))$. The simplest model assumes that f_i is a linear function of the overlap:

$$f_i(\delta_i) = k_i\delta_i, \tag{3}$$

where k_i is determined by the material and surface properties of $\mathcal{B}$ and $\mathcal{A}_i$. While this model is linear in δ_i, it is typically *not* linear, since δ_i is nonlinear in q. More sophisticated contact models can be formulated by choosing appropriate functions $f_i(\delta_i)$. See [7] for details.

Consider a grasp of $\mathcal{B}$ at a configuration q_0. The arrangement of fingers forms an *equilibrium grasp* if (in the absence of any external wrench) the finger forces produce a zero net wrench on $\mathcal{B}$. When subjected to an arbitrary external disturbance, $\mathcal{B}$ may be displaced from q_0. The grasp is *stable* if $\mathcal{B}$ returns to q_0 after the external disturbance is removed. A more formal discussion of stability can be found in Ref. [14].

The elastic potential energy of the system consisting of the object $\mathcal{B}$ and fingers $\mathcal{A}_1, \ldots, \mathcal{A}_k$ is:

$$\Pi(q) = \sum_{i=1}^{k} \int_0^{\delta_i(q)} f_i(\delta) d\delta. \tag{4}$$

It can be verified that $\delta_i(q)$ is differentiable almost everywhere, hence $\Pi(q)$ is differentiable. In the absence of a disturbing wrench, an equilibrium grasp is characterized by:

$$\nabla\Pi(q_0) = \sum_{i=1}^{k} f_i(\delta_{i0})\nabla\delta_{i0} = \vec{0}, \tag{5}$$

where $\delta_{i0} = \delta_i(q_0)$ and $\nabla\delta_{i0} = \nabla\delta_i(q_0)$. A sufficiently small displacement of $\mathcal{B}$ can be approximated by a tangent vector. For this reason we will interchangeably use the terms tangent vector and local displacement. The *stiffness matrix* is defined as the Hessian $K = D^2\Pi(q_0)$ of the potential Π. Denoting $f_i' = \frac{df_i}{d\delta_i}$ and $D^2\delta_{i0} = D^2\delta_i(q_0)$, it follows from (5) that

$$K = \sum_{i=1}^{k} \left\{ f_i'(\delta_{i0})\nabla\delta_{i0}\nabla\delta_{i0}^T + f_i(\delta_{i0})D^2\delta_{i0} \right\}. \tag{6}$$

Therefore, the stiffness matrix can be computed from the overlaps δ_i and their derivatives. The reader is referred to Ref. [7] for the computation of K. As is well known, at points $q = q_0 + \dot{q}$ in the vicinity of q_0 the stiffness matrix gives the wrench acting on $\mathcal{B}$, according to the formula $w = K\dot{q}$.

We observe that the two summands in Eq. (6) generally depend on the initial deformations δ_{i0}. It is shown in [7] that the second term depends on the surface curvatures at the contacts, while the first term does not. We say that the first term accounts for first order geometrical effects, while the second term accounts for second order (curvature) effects. If the first term alone is positive definite, the grasp is *stable to first order*. Otherwise, if the entire K is positive definite, the grasp is *stable to the second order*. The relative contributions of first and second order effects on grasp stability and stiffness are analyzed in Ref. [7].

We conclude this section with the following change-of-frame formula for the stiffness matrix:

$$\bar{K} = T^T K T, \tag{7}$$

where $\bar{K}$ is the stiffness matrix associated with the new frames $\bar{\mathcal{F}}_W$ and $\bar{\mathcal{F}}_B$. This formula can be derived from (1) and the fact that $\nabla\Pi(q_0) = 0$.

3 Principal Stiffness Parameters

This section defines the characteristic compliance parameters of a grasp, based on the stiffness matrix K, and the *compliance matrix* $C \triangleq K^{-1}$. For clarity, we note that $\boldsymbol{w} = K\dot{q}$ while $\dot{q} = C\boldsymbol{w}$. We use the following partition of K and C into 3×3 matrices:

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix}, \qquad C = \begin{pmatrix} C_{11} & C_{12} \\ C_{12}^T & C_{22} \end{pmatrix}. \tag{8}$$

Note that the diagonal blocks are positive definite, since K and C are. We use $\dot{q}_1$ and $\dot{q}_2$ for the translational and rotational components of $\dot{q} = (v, \omega)$, and use $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ for the force and torque components of $\boldsymbol{w} = (f, \tau)$.

3.1 Formal Development

The eigenvalues of K, which could provide important insight into the stiffness matrix, are not frame invariant. To circumvent this difficulty, we look at the tangent subspace defined by

$$V = \{\dot{q} \in T_{q_0}\mathcal{C} \;:\; f = (K\dot{q})_1 = 0\}.$$

That is, V consists of the small displacements that induce a pure reaction torque on $\mathcal{B}$. Using the partition of K, we obtain $V = \{(v, \omega) : v = -K_{11}^{-1}K_{12}\omega\}$, from which it follows that V can be parametrized as

$$\dot{q} = P\omega \quad \text{where} \quad P = \begin{pmatrix} -K_{11}^{-1}K_{12} \\ I \end{pmatrix}. \tag{9}$$

Let K_V denote the restriction of K to V. Recalling that the stiffness matrix represents the symmetric bilinear operator $D^2\Pi(q_0)$, we have that $\omega^T K_V \omega = \omega^T P^T K P \omega$ for arbitrary ω. Thus under our parametrization of V, K_V has the representation

$$K_V = P^T K P = K_{22} - K_{12}^T K_{11}^{-1} K_{12}.$$

Since K maps $\dot{q} \in V$ to pure-torque wrenches, we have that

$$(K\dot{q})_2 = K_V\,\omega. \tag{10}$$

Consider now two new frames $\bar{\mathcal{F}}_W$ and $\bar{\mathcal{F}}_B$, with overbars denoting objects associated with these frames. The linear operator K_V has the following invariance property.

Proposition 3.1 ([8]). *Let V and $\bar{V}$ be the subspaces parametrized by (9) in the q and $\bar{q}$ coordinates. Let $\bar{K}_V$ be the restriction of $\bar{K}$ to $\bar{V}$. Then $\bar{K}$ obeys the orthogonal transformation*

$$\bar{K}_V = {R_w}^T K_V R_w.$$

Hence the eigenvalues of K_V are frame-invariant.

Dually, consider the following wrench subspace:

$$W = \{\boldsymbol{w} \in T^*_{q_0}\mathcal{C} \;:\; \omega = (C\boldsymbol{w})_2 = 0\}.$$

In words, W is the subspace of wrenches that induce pure translation, and this subspace can be parametrized as

$$\boldsymbol{w} = Qf \qquad Q = \begin{pmatrix} I \\ -C_{22}^{-1}C_{12}^T \end{pmatrix}. \tag{11}$$

Using this parametrization, the restriction of C to W, denoted C_W, is $C_W = Q^T C Q = C_{11} - C_{12}C_{22}^{-1}C_{12}^T = K_{11}^{-1}$. Moreover, the resulting pure-translation is given by $v = (C\boldsymbol{w})_1 = C_W f$, where $\boldsymbol{w} \in W$.

Proposition 3.2. *Let $\bar{W}$ be the subspace parametrized by (11) in the $\bar{q}$ coordinates, and let $\bar{C}_W$ be the restriction of the compliance matrix $\bar{C}$ to $\bar{W}$. Then $\bar{C}_W$ obeys the orthogonal transformation $\bar{C}_W = {R_w}^T C_W R_w$. Hence, the eigenvalues of $C_W = K_{11}^{-1}$ are frame-invariant.*

Propositions 3.1 and 3.2 lead to the following observations. The behavior of K on V characterizes the *rotational stiffness* of the grasp. Regardless of frame location, the same pure-torque is elicited in response to an instantaneous displacement in V. Similarly, the behavior of C on W characterizes the *translational compliance* of the grasp. A wrench in W generates the same pure-translation when using different frames. Since the tangent subspace V and the image of W under C_W span $T_{q_0}\mathcal{C}$, the two subspaces characterize the grasp compliance completely. Summarizing these observations and using the fact that $C_W = K_{11}^{-1}$, we call the eigenvalues μ_i $(i = 1, 2, 3)$ of K_V the *principal rotational stiffnesses*, and the eigenvalues σ_i $(i = 1, 2, 3)$ of $K_{11} = C_W^{-1}$ the *principal translational stiffnesses* of the grasp. In particular, $\sigma_{min} = \min\{\sigma_i\}$ is the *smallest* principal translational stiffness. The associated eigenvectors are called *principal rotational and translational stiffness directions*, respectively.

For planar grasps it can be shown that there is a *unique* location of the origin of $\bar{\mathcal{F}}_B$, given by

$$d_b = R_0^T J K_{11}^{-1} K_{12}, \tag{12}$$

such that $\bar{K}_{3\times3}$ takes the *block-diagonal* form $\bar{K} = \text{diag}({R_w}^T K_{11} R_w,\ \mu)$. That is, the translation and rotational effects are *decoupled* about this special point, called the *center of compliance* [11]. The principal translational and rotational stiffnesses of the grasp are physically the translational and rotational stiffnesses about the center of compliance.

3.2 Screw Coordinates Interpretation

While searching for a 3D analog of the center of compliance, Patterson and Lipkin [12] were the first to recognize the existence of the principal stiffness directions. They used screw coordinates, and now we show that our principal parameters are equivalent to the ones derived by Patterson and Lipkin. First we briefly review the notion of screw coordinates.

A one-dimensional tangent subspace of the form $\{\dot{q} = \theta(v,\omega) : \theta \in \mathbb{R}\}$ with $\|\omega\| = 1$, is given screw coordinates as follows. The *instantaneous screw axis* is parallel to ω and passes through the point $v \times \omega$. The *pitch*, h, is equal to $v \cdot \omega$. For a one-dimensional wrench subspace $\{w = \alpha(f,\tau) : \alpha \in \mathbb{R}\}$ with $\|f\| = 1$, the *screw axis* is parallel to f and passes through the point $f \times \tau$. The *pitch* is $h = f \cdot \tau$.

Consider now a tangent vector $\dot{q}_i \in V$, where $\dot{q}_i$ is an eigenvector of K_V associated with the eigenvalue μ_i. Using (9), there exists a unit vector ω_i such that $\dot{q}_i = P\omega_i$. Then (10) gives $\tau = (K\dot{q}_i)_2 = \mu_i\omega_i$. That is, the displacement along $\dot{q}$ causes a pure-torque about the screw axis associated with $\dot{q}_i$. On the other hand, for $w = Qf_i \in W$ where f_i is an eigenvector of C_W associated with the eigenvalue $1/\sigma_i$, we have that $v = (Cw)_1 = (1/\sigma_i)f_i$. Hence, the wrench w generates a pure-translation along its screw axis. Patterson and Lipkin [12] call the screw axis associated with these eigenvectors the twist- and wrench-compliant axes, respectively.

3.3 Geometric Interpretation

We now present a novel interpretation of the principal stiffnesses. Consider the quadratic form $\Phi(\dot{q}) = \frac{1}{2}\dot{q}^T K \dot{q}$, where $\dot{q} \in T_{q_0}\mathcal{C}$. The level set $\mathcal{S}$ defined by $\Phi(\dot{q}) = 1$ is a 5-dimensional elliptical surface, and a point on $\mathcal{S}$ corresponds to a displacement that produces unit elastic energy. Consider the intersection, denoted $\mathcal{S}_\omega$, of $\mathcal{S}$ with the subset of $T_{q_0}\mathcal{C}$ determined by the equation $\omega = \text{const}$. Letting $\Phi_\omega(v) \triangleq \Phi(v,\omega)$, the points $v \in \mathcal{S}_\omega$ satisfy

$$\Phi_\omega(v) = \frac{1}{2}u^T K_{11} u + \frac{1}{2}\omega^T K_V \omega = 1,$$

where $u = v + K_{11}^{-1}K_{12}\omega$. Hence for each fixed ω, $\mathcal{S}_\omega$ is an ellipsoid with principal semi-axes of length $\sqrt{(2 - \omega^T K_V \omega)/\sigma_i}$ $(i = 1,2,3)$. These lengths are frame-invariant, and when $\omega = 0$ the lengths are $\sqrt{2/\sigma_i}$ $(i = 1,2,3)$.

Next we consider the collection of points in $\mathcal{S}$, denoted $\mathcal{S}_n$, at which the vectors normal to $\mathcal{S}$ have zero v-components. For any $\dot{q} = (v,\omega) \in \mathcal{S}_n$, the condition $(\nabla\Phi(\dot{q}))_1 = 0$ implies that $v = -K_{11}^{-1}K_{12}\omega$ and consequently $\Phi(\dot{q}) = \frac{1}{2}\omega^T K_V \omega = 1$. By setting the v-coordinates of the points in $\mathcal{S}_n$ to zero, we obtain the projection of $\mathcal{S}_n$ to the subspace $v = 0$ as follows.

$$(\mathcal{S}_n)_{v=0} = \{(v,\omega) : v = 0 \text{ and } \frac{1}{2}\omega^T K_V \omega = 1\}.$$

This is an ellipsoid with principal semi-axes of lengths $\sqrt{2/\mu_i}$, where μ_i for $i = 1,2,3$ are the eigenvalues of K_V.

For planar grasps $\mathcal{S}$ is 2-dimensional. Fig. 1 shows two such ellipsoids for a 4-fingered grasp of the quadrilateral given in Fig. 4 with fingers placed on the edges AB (at the vertices A and B), BC (at C) and DA (at D), respectively. The upright ellipsoid in the figure corresponds to the origin located at the center of compliance with coordinates $(6.15, 5.54)$, while the slanted ellipsoid corresponds to the origin located at $(0,0)$. The lengths of the principal semi-axes of each horizontal cross section of $\mathcal{S}$ are frame invariant. Similarly, the projection of $\mathcal{S}$ is bounded by two points, whose ω-coordinates are $\pm\sqrt{2/\mu}$. These two points are frame invariant, and $\mathcal{S}$ is always bounded by the two horizontal planes $\omega = \pm\sqrt{2/\mu}$.

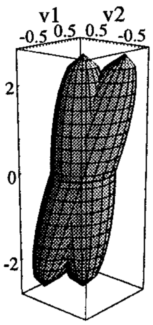

Fig. 1. The elastic energy ellipsoid in $T_{q_0}\mathcal{C}$

For the wrench space on which the quadratic form $\Psi(w) = \frac{1}{2}w^T C w$ is defined, we have the following analogous interpretation, shown in Fig. 2. The level set $\mathcal{T}$ given by $\Psi(w) = 1$ is a 5-dimensional elliptical surface corresponding to wrenches that induce unit elastic energy. The intersection $\mathcal{T}_f$ of $\mathcal{T}$ with the set $f = \text{const}$ is an ellipsoid whose principal semi-axes are equal to $\sqrt{\mu_i(2 - f^T K_{11}^{-1} f)}$ $(i = 1,2,3)$ and are frame-invariant. When $f = 0$, the principal semi-axes of $\mathcal{T}_f$ are given by $\sqrt{2\mu_i}$. Let $\mathcal{T}_h$ be the subset of $\mathcal{T}$ such that the normal vector to $\mathcal{T}$ at a point $w \in \mathcal{T}_h$ has zero τ-component. The projection of $\mathcal{T}_h$ to the subspace $\tau = 0$ is given by

$$(\mathcal{T}_h)_{\tau=0} = \{(f,\tau) : \tau = 0 \text{ and } \frac{1}{2}f^T K_{11}^{-1} f = 1\}.$$

Since $K_{11}^{-1} = C_W$ is frame invariant, the principal semi-axes of $(\mathcal{T}_h)_{\tau=0}$, given by $\sqrt{2\sigma_i}$, are frame invariant.

In the planar case, the elliptical surface $\mathcal{T}$ intersects the τ-axis at two points whose coordinates are $\pm\sqrt{2\mu}$ (Fig. 2). If $\mathcal{T}$ is vertically oriented, the horizontal projection of $\mathcal{T}$ is the planar ellipse $\frac{1}{2}f^T K_{11}^{-1} f = 1$. Any other $\mathcal{T}$ is inscribed in the vertical cylinder whose base set is this ellipse. These features can be observed in Fig. 2, for the same grasp as used for Fig. 1. The upright and slanted ellipsoids correspond to the same frames as their counterparts in Fig. 1.

4 A Frame-Invariant Quality Measure

Guaranteeing that the displacement of a grasped object will not exceed a specified tolerance is one of the

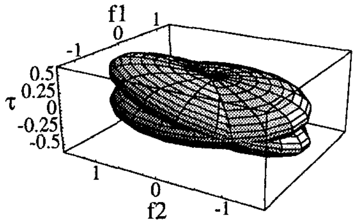

Fig. 2. The elastic energy ellipsoid in $T^*_{q_0}C$

most important concerns in fixture design [13]. Hence we wish to develop a grasp quality measure which is related to the deflection of the object under the action of disturbing forces. In particular, we wish to relate the principal translational and rotational stiffnesses to the object's deflection, and use this relation to evaluate alternative grasps.

Let $\dot{q} = \theta(v, \omega)$ be a displacement of $\mathcal{B}$, where $\|v\| = 1$ if $\omega = 0$, and $\|\omega\| = 1$ if $\omega \neq 0$. We define the *deflection* of $\mathcal{B}$ due to a displacement $\dot{q}$ as the *maximal* displacement of any point in $\mathcal{B}$. Since $\mathcal{B}$ is bounded, such a maximal displacement always exists and is independent of frame choice. If $\omega = 0$, the deflection is simply $|\theta|$. If $\omega \neq 0$, let $\rho_{max}(\dot{q})$ be the maximal distance from the screw axis of $\dot{q}$ to $\mathcal{B}$'s boundary points. The deflection is $|\theta|\sqrt{\rho_{max}(\dot{q})^2 + (v \cdot \omega)^2}$, where $v \cdot \omega$ is the pitch of $\dot{q}$. For planar grasps $v \cdot \omega = 0$ and $\mathcal{B}$'s deflection is $|\theta|\rho_{max}(\dot{q})$.

First we present our quality measure in the context of planar grasps. For planar grasps, we wish to compare the principal rotational stiffness μ with the smallest principal translational stiffness σ_{min}. As previously discussed, μ and σ_{min} are associated with pure rotation and translation of $\mathcal{B}$ with respect to the center of compliance. The deflection of the object can be used to compare these two parameters as follows. Consider a rotation of $\mathcal{B}$ of magnitude θ about the center of compliance. Then the deflection of $\mathcal{B}$ due to this rotation is $|\theta|\rho_{max}$, where ρ_{max} is the maximal distance from the center of compliance to $\mathcal{B}$'s boundary. The *equivalent stiffness* associated with μ, denoted μ_{eq}, is defined by the relationship:

$$\frac{1}{2}\mu_{eq}(\rho_{max}\theta)^2 = \frac{1}{2}\mu\theta^2,$$

where the right hand side is the elastic energy generated by the rotation θ. This relationship yields

$$\mu_{eq} = \frac{\mu}{(\rho_{max})^2}. \qquad (13)$$

The parameters μ_{eq} and σ_{min} are now comparable. We define the grasp *quality measure* as:

$$\mathcal{Q} = \min\{\sigma_{min}, \mu_{eq}\}. \qquad (14)$$

The scalar $\mathcal{Q}$ measures the *worst-case characteristic stiffness* based on $\mathcal{B}$'s deflection. Moreover, $\mathcal{Q}$ is frame invariant.

We now define the quality measure for a 3D grasps. For 3D objects, we must scale the principal rotational stiffnesses μ_i so that they become comparable with the translational stiffnesses. Let $\dot{q}_i = (v_i, \omega_i) \in V$ be the eigenvector of K_V associated with μ_i, such that $\|\omega_i\| = 1$. Then the elastic energy generated by the displacement $\theta\dot{q}_i$ is given by $\frac{1}{2}\mu_i\theta^2$, while the deflection of the object due to $\theta\dot{q}_i$ is $\theta\sqrt{(\rho_{max_i})^2 + (v_i \cdot \omega_i)^2}$, where $\rho_{max_i} = \rho_{max}(\dot{q}_i)$. Analogously to the 2D case, we define μ_{eq_i} by the following energy equivalence relationship

$$\frac{1}{2}\mu_{eq_i}\left(\theta\sqrt{(\rho_{max_i})^2 + (v_i \cdot \omega_i)^2}\right)^2 = \frac{1}{2}\mu_i\theta^2,$$

which yields

$$\mu_{eq_i} = \frac{\mu_i}{(\rho_{max_i})^2 + (v_i \cdot \omega_i)^2} \quad \text{for } i = 1, 2, 3. \qquad (15)$$

We define the following 3D grasp *quality measure*:

$$\mathcal{Q} = \min\{\sigma_{min}, \mu_{eq_1}, \mu_{eq_2}, \mu_{eq_3}\}. \qquad (16)$$

Again, $\mathcal{Q}$ is a frame-invariant scalar which measures the worst-case characteristic stiffness as determined by $\mathcal{B}$'s deflection.

5 Optimal Grasping of Polygons

To illustrate our methodology and its possible utility, we apply the quality measure (14) to the planar polygonal objects grasped by three or four disc fingers. For simplicity, we employ the overlap model of Eq. (3). Since each finger boundary has constant curvature and $\mathcal{B}$'s edges are straight, the stiffness coefficient k_i is assumed to be the same for all finger locations on a given edge. We exclude finger placements at vertices and choose coincident frames $\mathcal{F}_W$ and $\mathcal{F}_B$.

Let the *contact configuration space* (contact c-space) be the set of all possible contact arrangements (each contact can be parametrized by a scalar). For polygonal objects, the contact c-space can be decomposed into subspaces corresponding to different combinations of edges.

Consider the *computation of* ρ_{max} for polygons. If the center of compliance is at p, $\rho_{max}(p)$ is the distance from p to the farthest vertex of $\mathcal{B}$. For efficient computation, we may presort the plane into regions whose points correspond to the same farthest vertex. Let $\{v_1, \ldots, v_n\}$ be the vertices of $\mathcal{B}$. For vertex v_i, let H_j be the closed half plane that does not contain v_i and is bounded by the bisector between v_i and v_j ($j \neq i$). Let R_i be the intersection of these half planes. Then $\rho_{max}(p) = \|p - v_i\|$ for $p \in R_i$. Clearly, R_i is a convex polygonal region, and $\cup_{i=1}^n R_i = \mathbb{R}^2$.

5.1 Optimal Three-Finger Grasping

For 3-fingered planar equilibrium grasps, the stiffness matrix corresponding to a particular edge triplet (Fig. 3), can be computed according to (6). A formula is given in the following proposition. In the proposition, n_i are the unit contact normals pointing into $\mathcal{B}$. Also, the total initial finger force is $f_T = \sum_{i=1}^k f_i(\delta_{i0})$.

Proposition 5.1 ([8]). *If the origin of $\mathcal{F}_W$ coincides with the point of concurrency, the stiffness matrix for a grasp employing a given triplet of edges takes the form*

$$K = \begin{pmatrix} \sum_{i=1}^{3} k_i n_i n_i^T & 0 \\ 0 & \mu \end{pmatrix},$$

with the principal rotational stiffness given by

$$\mu = \frac{2 f_T\, a\, \sin\alpha_1 \sin\alpha_2 \sin\alpha_3}{\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3},$$

where α_i are the triangle's three angles and a is the radius of its circumscribed circle. Moreover, the center of compliance coincides with the concurrency point.

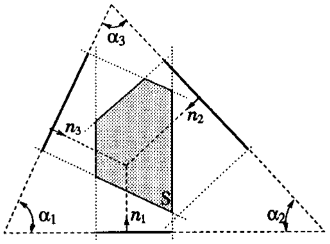

Fig. 3. 3-finger grasp on a particular edge triplet

It follows that σ_{min} is the smallest eigenvalue of the 2×2 matrix $\sum_{i=1}^{3} k_i n_i n_i^T$, and is *constant* for the given edge triplet. It is even more interesting to observe that μ, resulting from curvature effects and depending on the total finger force f_T, is also *constant* for all grasp arrangements on the same edge triplet. For the grasp to be stable, f_T must assume a positive value (i.e., initial deformations are nonzero). Since the first order effects are dominant, $\mu_{eq} \ll \sigma_{min}$ and therefore

$$\mathcal{Q} = \mu_{eq} = \frac{2 f_T\, a\, \sin\alpha_1 \sin\alpha_2 \sin\alpha_3}{\rho_{max}^2 (\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3)}. \quad (17)$$

In practice, f_T is fixed at a value which is the same for all edge triplets, and a threshold value ϵ can be chosen for σ_{min} such that a triplet with $\sigma_{min} < \epsilon$ is rejected.

For an edge triplet whose inward normals positively span $\mathbb{R}^2$, the collection of stable equilibrium grasps is parametrized by the location of the concurrency point. Consider the three strips in Fig. 3. The two lines bounding each strip are perpendicular to an edge, and pass through the edge's endpoints. For each point in the region S formed by intersecting the three strips, there exists a finger placement such that this point is the concurrency point of the contact normals.

For a given fixed preloading f_T, the quality measure (17) is maximized over a given edge triplet as ρ_{max}^2 is minimized. This agrees with the intuition that the deflection of $\mathcal{B}$ about the concurrency point due to a unit torque is minimized for the optimal grasp. For a given edge triplet, we maximize ρ_{max}^2, a positive definite quadratic function, over a collection of convex polygonal regions described in [8]. While these convex quadratic programming problems can be solved by many efficient algorithms, the optimal grasp arrangement is very intuitive when the geometric center[1] belongs to S. In this case the optimal concurrency point location coincides with the geometric center, and the optimal quality measure is given by (see footnote 1 for the radius r_0).

$$\mathcal{Q}_{opt} = \frac{2 f_T\, a\, \sin\alpha_1 \sin\alpha_2 \sin\alpha_3}{r_0^2 (\sin\alpha_1 + \sin\alpha_2 + \sin\alpha_3)}.$$

Example 5.1. Consider grasping a quadrilateral by three identical fingers (Fig 4). We take $k_i = 1$ $(i = 1,2,3)$ without loss of generality. The radius of the object is 6.7315 and its geometric center is at $(6.5, 1.75)$. For the edge triplets (AB, BC, CD) and (AB, BC, DA) stable grasps exist, with the optimal grasps given by arrangements I and II, respectively. We have $\sigma_{min} = 0.8609$ and $\mathcal{Q}/f_T = 0.081$ for grasp I, while $\sigma_{min} = 1.2764$ and $\mathcal{Q}/f_T = 0.0955$ for grasp II, which is globally optimal.

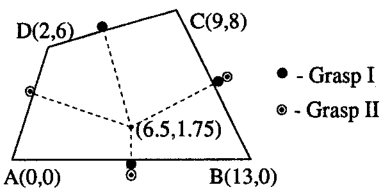

Fig. 4. 3-finger quadrilateral grasps

5.2 Optimal Four-Finger Grasping

A 4-fingered polygonal grasp involves three or four edges. Thus edge combinations of interest include all triplets and quadruplets of edges. For a given edge combination, let γ_i be the moment about the origin of the inward unit normal n_i to the i^{th} edge. Use $s = (s_1, s_2, s_3, s_4)^T$, where $s_i = \sqrt{k_i}\gamma_i$, to parametrize $\mathcal{G}$, the contact c-space. Since the geometric constraints on contact locations are all linear inequalities, $\mathcal{G}$ is a *convex polytope.* Moreover it is *bounded* since no finger can be placed at infinity.

Let $N_i = \sqrt{k_i} n_i$, and $h_i(s_i) = (N_i^T, s_i^T)^T$. Then the stiffness matrix is given by (see [7] for a proof)

$$K(s) = \begin{pmatrix} NN^T & Ns \\ (Ns)^T & s^T s \end{pmatrix}, \quad (18)$$

where $N = (N_1, N_2, N_3, N_4)$ and the second order effects have been neglected [7].

For a given edge combination, the contact normals do not change directions and the matrix N is constant. Hence the smallest principal translational stiffness is a constant. Using (18), we find the dependence of the principal rotational stiffness on contact configuration: $\mu(s) = s^T \Phi s$, where $\Phi = I - N^T (NN^T)^{-1} N$. From (12) and (18) the center of compliance as a function of s is given by $d_b(s) = \Gamma s$, where $\Gamma = J(NN^T)^{-1} N$. Thus in the polygonal region R_i we can write μ_{eq} as

$$\mu_{eq}(s) = \frac{s^T \Phi s}{(\Gamma s - v_i)^T (\Gamma s - v_i)}.$$

[1]Here the *geometric center* is the point p_0 at which $r_0 = \min_{p \in \mathbb{R}^2} \rho_{max}(p) = \rho_{max}(p_0)$, and r_0 is called the *radius*.

If there is some feasible s such that $\mu_{eq}(s) \geqslant \sigma_{min}$, then the corresponding grasp is optimal for this edge combination. Otherwise μ_{eq} can be maximized, which is considered below.

Define $d_i(s) = \det(h_{i+1}, h_{i+2}, h_{i+3}) \pmod 4$, for $i = 1,2,3,4$. Ref. [8] shows that the stiffness matrix $K(s)$ is positive definite if and only if $d_1(s)$, $-d_2(s)$, $d_3(s)$ and $-d_4(s)$ are all nonzero and have the same sign. Therefore the collection of stable grasps is $S_1 \cup S_2$, where S_1 and S_2 are *bounded convex polytopes* in $\mathbb{R}^4$.

$$S_1 = \mathcal{G} \cap \{s \in \mathbb{R}^4 : \ d_1(s), -d_2(s), d_3(s), -d_4(s) < 0\},$$

$$S_2 = \mathcal{G} \cap \{s \in \mathbb{R}^4 : \ d_1(s), -d_2(s), d_3(s), -d_4(s) > 0\}.$$

We can maximize μ_{eq} over convex polyhedral regions of the form $\mathcal{P} = S_i \cap \mathcal{D}_j$, where $\mathcal{D}_j = \{s \in \mathbb{R}^4 : \Gamma s \in R_j\}$. For $t \in \mathbb{R}$, define $\psi(t) = \max_{s \in \mathcal{P}} \phi(t,s)$ where

$$\phi(t,s) = s^T \Phi s - t\,(\Gamma s - v_i)^T(\Gamma s - v_i).$$

Proposition 5.2. *If the function ψ has a zero, it is unique. Moreover, $t^* = \mu_{eq}(s^*) = \max_{s \in \mathcal{P}} \mu_{eq}(s)$ if and only if $\psi(t^*) = \phi(t^*, s^*) = 0$.*

This proposition is proved in Ref. [8]. It follows that maximizing μ_{eq} over $\mathcal{P}$ is equivalent to solving the scalar equation $\psi(t) = 0$. The evaluation of the function ψ is an indefinite quadratic programming problem. While indefinite quadratic programming is NP-hard, there are many efficient approximate algorithms. In fact, with our 4-dimensional problems, an exhaustive search scheme is quite affordable. The remarkable fact is that global optimality is *guaranteed* at reasonable cost despite the nonconvex and strongly nonlinear nature of the quality measure.

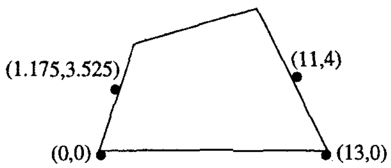

Fig. 5. Global optimal grasp of a quadrilateral

Example 5.2. Let us look at the quadrilateral used in Example 5.1 and assume $k_i = 1$. By considering all feasible edge combinations we can find the optimal grasp associated with each combination, and then determine the global optimal grasp arrangement. The global optimal grasp is the one in Fig. 5, with optimal quality measure equal to $\sigma_{min} = 1.684 < \mu_{eq} = 1.865$.

6 Conclusion

While compliance plays an important role in grasping and fixturing, systematic approaches to assessing the quality of compliant grasps have been lacking. In this paper we presented an effort along this direction. A frame-invariant quality measure was defined based on characteristic compliance parameters of the stiffness matrix. It applies to the grasping of 2D and 3D objects by any number of fingers, and can be used to determine the optimal finger placement. The promise of this quality measure is shown by examples applying it to polygonal grasps. We believe that this quality criterion will allow the development of more efficient and accurate algorithms for optimal planning of compliant grasps or fixtures.

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