

Thermal Hall conductance and a relative topological invariant of gapped two-dimensional systems

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Abstract

We show that derivatives of thermal Hall conductance of a 2d lattice quantum system with respect to parameters of the Hamiltonian are well-defined bulk quantities provided correlators of local observables are short-range. This is despite the fact that thermal Hall conductance itself has no meaning as a bulk transport coefficient. We use this to define a relative topological invariant for gapped 2d lattice quantum systems at zero temperature. Up to a numerical factor, it can be identified with the difference of chiral central charges for the corresponding edge modes. This establishes bulk-boundary correspondence for the chiral central charge. We also show that for Local Commuting Projector Hamiltonians relative thermal Hall conductance vanishes identically, while for free fermionic systems it is related to the electric Hall conductance via the Wiedemann-Franz law.

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I. INTRODUCTION AND OVERVIEW

There has been much theoretical as well as experimental interest in thermal Hall conductivity and conductance of various materials. Just to give a couple of recent examples: (1) thermal Hall conductance has been used to probe the non-Abelian nature of the $\nu = 5/2$ FQHE state [1]; (2) an unusual behavior of thermal Hall conductivity at low temperatures was observed in cuprate superconductors in the pseudogap region [2].

Despite many theoretical works on the thermal Hall effect (see e.g. [3–6]), there are still unresolved issues with the very definition of thermal Hall conductivity. In fact, all known approaches to defining thermal Hall conductivity as a bulk property are plagued with ambiguities. To see what the issue is in the simplest possible setting, consider a macroscopic system where the only locally conserved quantity is energy, such as an insulating crystal at positive temperature. One could expect that thermal Hall conductivity appears as a transport coefficient in the hydrodynamics of phonons, but this is not the case: the hydrodynamics of phonons does not allow for a time-reversal-odd transport coefficient at leading order in the derivative expansion. The conservation law for the energy density ϵ is

$$\frac{\partial \epsilon}{\partial t} = -\nabla \cdot \mathbf{j}^E. \quad (1)$$

In the hydrodynamic limit one can expand the energy current \mathbf{j}^E to first order in derivatives of ϵ , or equivalently in derivatives of the temperature T :

$$\mathbf{j}_m^E = -\kappa_{m\ell}(T)\partial_\ell T. \quad (2)$$

Hence the conservation law becomes

$$c(T)\frac{\partial T}{\partial t} = \kappa_{m\ell}\partial_m\partial_\ell T + \kappa'_{m\ell}\partial_m T \partial_\ell T, \quad (3)$$

where $c(T)$ is the heat capacity and the prime denotes derivative with respect to T . The r.h.s. of this equation depends only on the symmetric part $\kappa_{m\ell}^S$ of the tensor $\kappa_{m\ell}$, which by Onsager reciprocity is the same as its time-reversal-even part. The anti-symmetric part $\kappa_{m\ell}^A$ has no observable effect. While the energy current through a surface (or, in the 2d context, through a line) depends on the whole tensor $\kappa_{m\ell}$, the contribution of $\kappa_{m\ell}^A$ can be thought of as a boundary effect. Indeed, if we define

$$\beta_{m\ell}^A(T) = \int^T \kappa_{m\ell}^A(u) du, \quad (4)$$

then in 3d the Stokes' theorem gives

$$-\int_{\Sigma} d\Sigma_m \kappa_{m\ell}^A \partial_{\ell} T = -\frac{1}{2} \int_{\partial\Sigma} d\mathbf{l}_k \epsilon_{kpm} \beta_{pm}^A(T). \quad (5)$$

Similarly, in 2d the contribution of $\kappa_{m\ell}^A$ to the energy current through a line can be written as a boundary term. The conclusion seems to be that thermal Hall conductivity has no meaning as a bulk transport property, either in 3d or 2d.

In the 2d case the tensor $\kappa_{m\ell}^A$ reduces to a single quantity ¹, the thermal Hall conductance $\kappa^A = \frac{1}{2} \epsilon_{m\ell} \kappa_{m\ell}^A$, and there is an alternative line of reasoning which suggests that in certain circumstances κ^A can be defined in bulk terms. At low temperatures and for systems with a bulk energy gap, it seems natural to relate $\kappa^A(T)$ to the chiral central charge of the edge CFT:

$$\kappa^A(T) \simeq \frac{\pi T}{6} (c_R - c_L). \quad (6)$$

This follows from the fact that a chiral 1+1d CFT at temperature T carries an equilibrium energy current $I^E = \frac{\pi T^2}{12} (c_R - c_L)$ [10, 11]. Thus in a strip of a 2d material whose boundaries are kept at temperatures T and $T + \Delta T$ which are close to each other and much smaller than the bulk energy gap there is a net energy current

$$I^E \simeq \frac{\pi T}{6} (c_R - c_L) \Delta T. \quad (7)$$

If we define $\kappa^A = I^E/\Delta T$, we get (6). It has been shown in [12] that the chiral central charge of the edge modes (and more generally, the equilibrium energy current carried by the edge modes) is independent of the particular edge. Hence the low-temperature thermal Hall conductance of a gapped 2d material defined via (6) is a well-defined bulk property ². This approach does not say anything about thermal Hall conductance at general temperatures or about thermal Hall conductance of materials which are not gapped.

In this paper, we show that although thermal Hall conductance is not a well-defined bulk transport coefficient, the difference of thermal Hall conductances of two 2d materials

¹ We use the notation κ^A instead of the more standard κ_{xy} to avoid confusion with the off-diagonal component of κ^S which may be nonzero if rotational invariance is broken.

² For gapped 2d systems at low temperatures, one can try to define thermal Hall conductance as the coefficient of the gravitational Chern-Simons term in the low-energy effective action [7–9]. As explained in [5], the energy current corresponding to the gravitational Chern-Simons term is of higher order in derivatives, in agreement with the above discussion. However, there is no natural way to couple a typical condensed matter system to gravity, therefore this prescription is ambiguous.

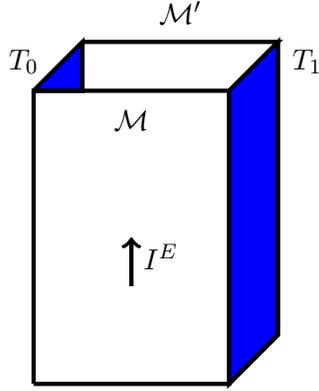


FIG. 1: Protocol for measuring the relative thermal Hall conductance of materials \mathcal{M} and \mathcal{M}' . If κ_{xy}^S is non-zero, one needs to repeat the measurement with the materials $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}'}$ where time-reversal symmetry is broken in the opposite way.

with suitable properties can be defined unambiguously provided the two materials can be continuously deformed into each other without encountering a bulk phase transition. The properties we require are exponential decay of correlations in both space and time, although this might be relaxed. Thus, properly speaking, one should talk about the relative thermal Hall conductance $\kappa_{\mathcal{M}\mathcal{M}'}^A$ of materials \mathcal{M} and \mathcal{M}' . To measure it, one needs to study energy flow along the surface of a cylinder made out of \mathcal{M} and \mathcal{M}' (see Fig. I) in the situation when the two junctions between \mathcal{M} and \mathcal{M}' are maintained at different temperatures. For a 2d crystal with a square symmetry or an isotropic disordered material where the off-diagonal component of κ^S is negligible, $\kappa_{\mathcal{M}\mathcal{M}'}^A$ is the energy flow perpendicular to the temperature gradient divided by the temperature difference. More generally, one would need to repeat the measurement for the materials $\bar{\mathcal{M}}$ and $\bar{\mathcal{M}'}$ where time-reversal invariance is broken in the opposite way in order to be able to separate κ^A from the off-diagonal part of κ^S . While the result of such measurement will depend on the way \mathcal{M} and \mathcal{M}' are glued together, we show that this dependence becomes vanishingly small when the diameter of the cylinder is much larger than the correlation length.

From a mathematical viewpoint, we construct a 1-form $d\kappa^A = g_\alpha(T, \lambda)d\lambda^\alpha$ on the parameter space of 2d lattice quantum systems with a finite correlation length and define the relative thermal Hall conductance $\Delta\kappa_{\mathcal{M}\mathcal{M}'}^A$ as the integral of this 1-form along a path connecting \mathcal{M} and \mathcal{M}' . We show that the 1-form is exact, so the integral does not depend on the choice of the path. The components of the 1-form $d\kappa^A$ have a transparent hydrodynamic

significance: they enter the conservation equation for energy density when the parameters λ^α are slowly varying in space:

$$c(T)\frac{\partial T}{\partial t} = \kappa_{m\ell}\partial_m\partial_\ell T + \kappa'_{m\ell}\partial_m T\partial_\ell T + \partial_\alpha\kappa_{m\ell}^S\partial_m\lambda^\alpha\partial_\ell T + g_\alpha\varepsilon_{m\ell}\partial_m\lambda^\alpha\partial_\ell T. \quad (8)$$

Since re-scaling temperature is equivalent to re-scaling the Hamiltonian, we can formally regard temperature as one of the parameters that one can vary. The corresponding component of $d\kappa^A$ drops out of the conservation equation.

To get the absolute thermal Hall conductance for a given material \mathcal{M} , one needs an additional assumption. Suppose \mathcal{M} can be connected to a trivial insulator without going through a bulk phase transition.³ Then one can choose \mathcal{M}' to be the trivial insulator \mathcal{M}_0 and identify $\kappa_{\mathcal{M}\mathcal{M}_0}^A$ with the thermal Hall conductance of \mathcal{M} . Equivalently, since the above assumption implies that \mathcal{M} can be deformed to the time-reversed material $\bar{\mathcal{M}}$, we can identify the thermal Hall conductance of \mathcal{M} with $\frac{1}{2}(\kappa_{\mathcal{M}\bar{\mathcal{M}}}^A - \kappa_{\bar{\mathcal{M}}\mathcal{M}}^A)$. The above protocol for measuring thermal Hall conductance can then be simplified: one would make a cylinder out of \mathcal{M} and $\bar{\mathcal{M}}$, measure the flow of energy in the presence of a small temperature difference between the junctions, and define the thermal Hall conductance of \mathcal{M} as half the perpendicular energy flow divided by the temperature difference.

It is easy to see that for gapped 2d materials the absolute thermal Hall conductance thus defined behaves at low temperatures as $\frac{\pi T}{6}(c_R - c_L)$, where $c_R - c_L$ is the chiral central charge of the edge modes. Indeed, we can create a junction between \mathcal{M} and $\bar{\mathcal{M}}$ by first deforming \mathcal{M} to the trivial insulator and then deforming the trivial insulator to $\bar{\mathcal{M}}$. Such a junction carries the edge modes for both \mathcal{M} and $\bar{\mathcal{M}}$, with opposite orientation. For temperatures below the gap, energy transport will occur only on the edges, so the above protocol will give the energy flow

$$I^E \simeq \frac{\pi}{6}(T_1^2 - T_0^2)(c_R - c_L), \quad (9)$$

which corresponds to $\kappa_{\mathcal{M}}^A = \frac{\pi T}{6}(c_R - c_L)$. We stress however that our definition of thermal Hall conductance works in far greater generality, including the case of temperatures much

³ It seems plausible that for $T > 0$ the space of 2d materials with a finite correlation length is connected, in the sense that one can deform any \mathcal{M} to a trivial insulator without going through a phase transition. Phase transitions at nonzero temperatures are usually associated with spontaneous symmetry breaking and typically can be turned into cross-overs by adding suitable symmetry-breaking perturbations. However, to be on the safe side, we state this as an assumption.

higher than the bulk energy gap, or cases where the correlation length is finite for $T > 0$ but diverges at $T \rightarrow 0$. In such cases, energy transport occurs throughout the material, and one cannot meaningfully separate bulk and edge modes, nevertheless both relative and absolute thermal Hall conductances are well-defined bulk quantities. What matters is not the presence of the energy gap at low T , but the rapid decay of correlations in space and time.

Ambiguities also affect transport coefficients describing the thermoelectric effect. They can be dealt with in the same manner as in the case of thermal Hall conductance. The discussion can also be extended to systems of particles interacting via a potential interaction. These issues will be discussed elsewhere.

Since the definition of thermal Hall conductance is rather subtle, we begin with a discussion of electric Hall conductance. Some of the subtleties arise already in this context. Then we move on to the case of thermal Hall effect and define a 1-form on the space of parameters whose integral along a curve defines the relative thermal Hall conductance. We show that the resulting quantity is path-independent and thus can be regarded as a difference of absolute thermal Hall conductances of the two endpoints. Finally, we apply our results to gapped 2d materials at temperatures below the energy gap and show that the relative thermal Hall conductance does not change as one varies the parameters of the Hamiltonian. Thus for such systems the relative thermal Hall conductance is a relative topological invariant. In view of the above discussion, it can be identified, up to a factor $\pi/6$, with the difference of chiral central charges of the two gapped phases. We also show that for systems described by Local Commuting Projector Hamiltonians the thermal Hall conductance relative to the $T = \infty$ state vanishes. Therefore such systems cannot have edge modes with a nonzero chiral central charge. This is an energy counterpart of the recently proved result that in such systems the zero-temperature electric Hall conductance vanishes [13]. In the concluding section we compare our approach to thermal Hall conductance with the results in the literature.

In one of the Appendices, we show by a direct computation that for free fermionic systems the relative thermal Hall conductance of the $T = 0$ and $T = \infty$ states is related to the zero-temperature electric Hall conductance through a version of the Wiedemann-Franz law.

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II. ELECTRIC HALL CONDUCTANCE OF A LATTICE SYSTEM AS A TOPOLOGICAL INVARIANT

A. Electric currents on a lattice

A 2d lattice system has a Hilbert space $V = \otimes_{p \in \Lambda} V_p$, where Λ (“the lattice”) is a uniformly discrete subset of \mathbb{R}^2 , and all V_p are finite-dimensional. An observable is localized at a point $p \in \Lambda$ if it has the form $A \otimes 1_{\Lambda \setminus p}$ for some $A : V_p \rightarrow V_p$. An observable is localized on a subset $\Lambda' \subset \Lambda$ if it commutes with all observables localized at any $p \notin \Lambda'$. A local observable A is an observable localized on a compact set $\Lambda' \subset \Lambda$, which will be called the support of A .

Hamiltonian of a lattice system has the form

$$H = \sum_{p \in \Lambda} H_p, \quad (10)$$

where the operators $H_p : V \rightarrow V$ are Hermitian and local. We will assume that the Hamiltonian has a finite range δ , which means that each H_p is a local observable supported in a ball of radius δ centered at x . This implies that $[H_p, H_q] = 0$ whenever $|p - q| > 2\delta$. We will also assume that the operators H_p are uniformly bounded, i.e. there exists $C > 0$ such that $\|H_p\| < C$ for all $p \in \Lambda$.

To define electric currents, we assume that the system has an on-site $U(1)$ symmetry. Thus we are given a $U(1)$ action on each V_p , with the generator Q_p (a Hermitian operator on V_p with integral eigenvalues). The total $U(1)$ charge is $Q_{tot} = \sum_{p \in \Lambda} Q_p$. Further, we assume that $[Q_{tot}, H_p] = 0$ for any $p \in \Lambda$. Since the time derivative of Q_q is

$$\frac{dQ_q}{dt} = i \sum_{p \in \Lambda} [H_p, Q_q], \quad (11)$$

it appears natural to define the $U(1)$ current from q to p by $J_{pq} = -i[H_p, Q_q]$. However, this does not satisfy a physically desirable property $J_{qp} = -J_{pq}$. Instead we define

$$J_{pq} = i[H_q, Q_p] - i[H_p, Q_q]. \quad (12)$$

The lattice current thus defined satisfies $J_{qp} = -J_{pq}$ as well as

$$\frac{dQ_q}{dt} = - \sum_{p \in \Lambda} J_{pq}. \quad (13)$$

Suppose Λ is decomposed into a disjoint union of two sets, $\Lambda = A \cup B$, $A \cap B = \emptyset$. The current from B to A is defined as

$$J(A, B) = \sum_{p \in A} \sum_{q \in B} J_{pq}. \quad (14)$$

More generally, given a function $\eta : \Lambda \times \Lambda \rightarrow \mathbb{R}$, one can define

$$J(\eta) = \frac{1}{2} \sum_{p, q} \eta(p, q) J_{pq}. \quad (15)$$

In the case $\eta(p, q) = \chi_B(q) - \chi_B(p)$, where $\chi_B(p) = 1$ for $p \in B$ and $\chi_B(p) = 0$ otherwise, $J(\eta)$ reduces to $J(A, B)$.

B. Chains and cochains

So far, we have encountered local operators H_p and Q_p which depend on a point and operators J_{pq} which depend on a pair of points. Later we will also encounter objects which depend on three points. It is useful to introduce a suitable terminology for such objects. A quantity $A(p_0, \dots, p_n)$ which depends on $n + 1$ points of Λ , is skew-symmetric under the exchange of points, and decays rapidly (say, faster than any power of the distance) when the distance between any two points becomes large, will be called a chain of degree n , or an n -chain. (One can visualize $n + 1$ ordered points of Λ as the vertices of an abstract oriented n -dimensional simplex, and a formal linear combination of simplices is usually called a chain.) For example, the operators J_{pq} form an operator-valued 1-chain. The decay condition is satisfied here because $[H_p, Q_q] = 0$ for $|p - q| > \delta$.

There is an operation ∂ on chains which lowers the degree by 1:

$$(\partial A)(p_1, \dots, p_n) = \sum_{q \in \Lambda} A(q, p_1, \dots, p_n). \quad (16)$$

Although the sum is infinite, the operation is well-defined for $n > 0$ since we assumed rapid decay when q is far away from any of the points p_1, \dots, p_n . This operation satisfies $\partial \circ \partial = 0$. The chain ∂A is called the boundary of the chain A . Using this notation, the conservation equation (13) can be written as

$$\frac{dQ}{dt} = -\partial J. \quad (17)$$

Applying ∂ to both sides, we get the conservation of the total electric charge $Q_{tot} = \partial Q$. Some care is needed here, since Q_{tot} is an infinite sum and thus is an unbounded operator.

Dually, an n -cochain is a function of $n + 1$ points of Λ which is skew-symmetric, but need not decay when one of the points is far from the rest. We will only consider real-valued cochains. A natural operation on cochains is:

$$(\delta\alpha)(p_0, \dots, p_{n+1}) = \sum_{j=0}^{n+1} (-1)^j \alpha(p_0, \dots, p_{j-1}, p_{j+1}, \dots, p_{n+1}). \quad (18)$$

It increases the degree by 1 and satisfies $\delta \circ \delta = 0$. The cochain $\delta\alpha$ is called the coboundary of the cochain α . The evaluation of an n -chain A on an n -cochain α is

$$A(\alpha) = \frac{1}{(n+1)!} \sum_{p_0, \dots, p_n} A(p_0, \dots, p_n) \alpha(p_0, \dots, p_n). \quad (19)$$

An example of a 1-cochain is a function $\eta(p, q)$ which appears in (15), then the operator $J(\eta)$ is simply the evaluation of the operator-valued 1-chain J on a 1-cochain η . This operator is, in general, unbounded. Note that the evaluation of a real-valued or complex-valued chain on a cochain does not necessarily make sense, because of convergence issues. To make it well-defined, one can require the chain to decay rapidly or vanish when any of the points is far from a particular point of Λ . This, however, does not hold for most chains we naturally encounter here. Alternatively, we can impose an additional condition on cochains: if we truncate the cochain by declaring that it vanishes whenever the distance between any two points is larger than some δ , then we may require the truncated cochain to vanish when any of the points is far from some fixed point. We may call such cochains "compactly supported". For example, if we regard χ_B as a 0-cochain, then $\eta = \delta\chi_B$ is compactly-supported if either A or B are compact. One can evaluate an arbitrary complex-valued chain on this cochain and get a well-defined number. Or, when one evaluates an operator-valued chain like J on this cochain, one gets an operator $J(A, B) = J(\eta)$ which is bounded.

With this being said, we can state a kind of "Stokes' theorem"

$$A(\delta\beta) = \partial A(\beta) \quad (20)$$

which applies to any n -chain A and any $(n-1)$ -cochain β for which both sides of the equation are well-defined. In the special case $A = J$ and $\beta = \chi_B$ for some finite set B , combining (20) and the conservation equation (17) we get that the current through the boundary of B (represented by the 1-cocycle $\delta\chi_B$) is equal to the rate of change of the total charge in B .

Finally we note that given an n -cochain α and an m -cochain γ one can define an $(n+m)$ -cochain $\alpha \cup \gamma$ by

$$(\alpha \cup \gamma)(p_0, \dots, p_{n+m}) = \frac{1}{(n+m+1)!} \sum_{\sigma \in \mathcal{S}_{n+m+1}} (-1)^\sigma \alpha(p_{\sigma(0)}, \dots, p_{\sigma(n)}) \gamma(p_{\sigma(n)}, \dots, p_{\sigma(n+m)}). \quad (21)$$

where \mathcal{S}_{n+m+1} is the permutation group on $n+m+1$ objects. This operation is associative and satisfies

$$\alpha \cup \gamma = (-1)^{nm} \gamma \cup \alpha, \quad \delta(\alpha \cup \gamma) = \delta\alpha \cup \gamma + (-1)^n \alpha \cup \delta\gamma. \quad (22)$$

The operations δ and \cup on cochains are analogous to operations d and \wedge on differential forms and have the same formal properties.

C. Electric Hall conductance

In order to define electric Hall conductance, let us pick two real-valued functions α and γ on the lattice Λ and consider the expression

$$\sigma(\alpha, \gamma) = \beta \int_0^\infty e^{-st} \langle \langle J(\delta\alpha, t); J(\delta\gamma) \rangle \rangle dt. \quad (23)$$

Here $J(\delta\alpha)$ is the value of the 1-chain $J = \{J_{pq}\}$ on a 1-cochain $\delta\alpha$, s is an infinitesimal parameter to be sent to $+0$ in the end, Heisenberg-picture operators are defined as usual, $A(t) = e^{iHt} A e^{-iHt}$, and double brackets $\langle \langle \dots \rangle \rangle$ denote Kubo's canonical correlation function [14]

$$\langle \langle A; B \rangle \rangle = \frac{1}{\beta} \int_0^\beta du \langle e^{uH} A e^{-uH} B \rangle - \langle A \rangle \langle B \rangle, \quad (24)$$

while single brackets $\langle \dots \rangle$ here and below denote thermal average. We note for future use three properties of the Kubo pairing $\langle \langle A; B \rangle \rangle$. First, it is symmetric:

$$\langle \langle A; B \rangle \rangle = \langle \langle B; A \rangle \rangle. \quad (25)$$

Second, it satisfies

$$\left\langle \left\langle \frac{dA(t)}{dt}; B \right\rangle \right\rangle = \langle \langle i[H, A(t)]; B \rangle \rangle = \frac{1}{\beta} \langle i[B, A(t)] \rangle. \quad (26)$$

Third, it measures the static linear response, in the sense that an infinitesimal variation δH of the Hamiltonian leads to a variation of the equilibrium expectation value of an observable B by

$$\delta \langle B \rangle = -\beta \langle \langle \delta H; B \rangle \rangle. \quad (27)$$

Here it is assumed that the observable B itself does not depend on H and thus does not vary when one varies H . More generally, if the observable B depends on H , one has

$$\delta\langle B \rangle = -\beta\langle\langle\delta H; B\rangle\rangle + \langle\delta B\rangle. \quad (28)$$

For a suitable choice of functions α and γ , $\sigma(\alpha, \gamma)$ is related to the Hall conductance. To see this, let us take $\alpha(p)$ to depend only on the x -coordinate of p :

$$\alpha(p) = \begin{cases} 1, & x(p) < 0, \\ \frac{R-x(p)}{R}, & 0 \leq x(p) \leq R, \\ 0, & x(p) > R. \end{cases} \quad (29)$$

Let $\gamma(p)$ depend only on the y -coordinate of p and be a step-function, $\gamma(y) = \theta(b - y(p))$. Then $J(\delta\gamma)$ is the net current across the line $y = b$, while

$$J(\delta\alpha) = -\frac{d}{dt}Q(\alpha), \quad (30)$$

where $Q(\alpha) = \sum_p \alpha(p)Q_p$ is the perturbation of the Hamiltonian which creates a constant electric field in the x -direction for $0 < x < R$. Then $\sigma(\alpha, \gamma)$ is the standard Kubo formula for the xy component of the electric conductance tensor. To get the electric Hall conductance, one needs to anti-symmetrize in α and γ . We will denote the anti-symmetric part $\sigma^A(\alpha, \gamma)$.

The expression for $\sigma(\alpha, \gamma)$ contains an improper integration over time and a quadruple summation over an infinite set Λ . Therefore, before proceeding, we should discuss its convergence. The behavior of correlators and susceptibilities for large time separations is a measure of the dissipative properties of the system. In “mixing” systems, both quantum and classical, the decay is exponential, with a finite correlation time τ_0 . In integrable systems, such as translationally-invariant systems of free fermions, some correlators do not decay with time, resulting in the divergence of the symmetric part of the conductivity tensor, while the anti-symmetric part stays finite. Since typical systems of interest are sufficiently chaotic, either because of disorder or because of interactions, we assume that the correlation time for local operators is finite, and will deal with the case of translationally-invariant free fermions separately. Finiteness of the relaxation time is a common assumption in transport theory, but it is difficult to prove. Note that one expects it to hold only in the thermodynamic limit, since otherwise one encounters the problem of Poincaré recurrence.

The spatial decay of susceptibilities, on the other hand, is expected to be exponential, with a finite correlation length ξ_0 , away from phase transitions. We will assume this is

the case. For gapped systems at $T = 0$ the spatial decay of both ordinary correlators and susceptibilities can be proved rigorously [15, 16].

With these assumptions, the integral over time and the summation over the points of Λ are rapidly convergent. This is somewhat non-trivial, since the operator $J_{pq}(t)$ is not a local operator for $t \neq 0$, even if though J_{pq} is. However, the Lieb-Robinson bound implies that it is approximately local, with the support growing linearly with t , and exponentially small “tails” outside this region. The corresponding velocity is known as the Lieb-Robinson velocity v_{LR} . Then one can easily see that restricting the summation to points which are within distance L from the region where both $\delta\alpha$ and $\delta\gamma$ are nonzero introduces only an exponentially small error if $L \gg \max(\xi_0, v_{LR}\tau_0)$. More precisely, it is easy to show that the error is no more than

$$\text{const} \times \exp\left(-\min\left(\frac{L}{2\xi_0}, \frac{L}{2v_{LR}\tau_0}\right)\right). \quad (31)$$

An important property of $\sigma(\alpha, \gamma)$ is that it is independent of the choice of functions α and γ as long as their asymptotic behavior is fixed. Indeed, let us deform the 0-cochain α into α' such that $\Delta\alpha = \alpha' - \alpha$ is supported on a strip $-L' < x(p) < L'$. The conductivity will change by

$$\Delta\sigma(\alpha, \gamma) = \beta \int_0^\infty dt e^{-st} \langle\langle J(\delta(\Delta\alpha), t); J(\delta\gamma) \rangle\rangle. \quad (32)$$

Using the Stokes' theorem (20) and the conservation law (11) we find

$$J(\delta(\Delta\alpha)) = (\partial J)(\Delta\alpha) = -\frac{dQ(\Delta\alpha)}{dt}. \quad (33)$$

Therefore

$$\Delta\sigma(\alpha, \gamma) = -\beta \int_0^\infty dt e^{-st} \langle\langle \frac{dQ(\Delta\alpha, t)}{dt}; J(\delta\gamma) \rangle\rangle = \beta \langle\langle Q(\Delta\alpha); J(\delta\gamma) \rangle\rangle. \quad (34)$$

This expression is well-defined because of our assumptions about the decay of Kubo pairings and because $Q(\Delta\alpha)$ is supported on a vertical strip of finite width, while $J(\delta\gamma)$ is supported on a horizontal strip of finite width. In particular, truncating $\Delta\alpha$ to zero for $|y| > L$ introduces a change of order $L^{-\infty}$ for sufficiently large L . Now we use the operator identity $J(\delta\gamma) = i[Q(\gamma), H]$ and get

$$\Delta\sigma(\alpha, \gamma) = -i \langle\langle [Q(\Delta\alpha), Q(\gamma)] \rangle\rangle = 0, \quad (35)$$

where we used the ultra-locality of the charge density $[Q_p, Q_q] = 0$. An identical argument shows that $\sigma(\alpha, \gamma)$ does not change if we modify γ without changing its asymptotic behavior.

Since $\sigma(\alpha, \gamma)$ does not depend on the choice of cochains α, γ , from now on we will make a particular choice which depends on two real numbers a, b . We will take $\alpha(p) = \theta(a - x(p))$ and $\gamma(p) = \theta(b - y(p))$. Then $J(\delta\alpha)$ is a current from the half-plane $x < a$ to the half-plane $x > a$, and $J(\delta\gamma)$ is the current from the half-plane $y < b$ to the half-plane $y > b$. Accordingly, we will write $\sigma(a, b)$ instead of $\sigma(\alpha, \gamma)$.

D. Electric Hall conductance at zero temperature as a topological invariant

The independence of $\sigma(\alpha, \gamma)$ on the choice of cochains is a strong property. For a gapped system at $T = 0$, it can be used to show that $\sigma(a, b)$ is unaffected by the variations of the Hamiltonian which preserve the above decay assumptions. To show this, we essentially follow the approach of Niu and Thouless [17] which was made rigorous by Watanabe [16].

To make things well-defined, we put the system on a torus of a finite size $L \times L$. We pick two numbers a and b (now defined modulo L) and define operators J_a and J_b as follows:

$$J_a = \sum_{x(p) > a, x(q) < a} J_{pq}, \quad J_b = \sum_{y(p) > b, y(q) < b} J_{pq}. \quad (36)$$

This might seem ambiguous, since $x(p)$ and $y(p)$ are now defined only modulo L . However, since $J_{pq} = 0$ for $|p - q| > \delta$, to define J_a we only need to consider p and q which are close to the line $x = a$, and then there is no ambiguity. The situation for J_b is similar. The operators J_a and J_b have a clear physical meaning: they represent currents through the lines $x = a$ and $y = b$, respectively. Now we consider the following expression:

$$\sigma(a, b; L) = \beta \int_0^\infty dt \langle \langle J_a(t); J_b \rangle \rangle. \quad (37)$$

By our assumptions, the integral is absolutely convergent, and for $L \gg \max(\xi_0, v_{LR}\tau_0)$ only a neighborhood of the point (a, b) contributes appreciably. More precisely, if one truncates the sum over lattice points outside of a square which contains the intersection of the lines and has size $L' \gg \max(\xi_0, v_{LR}\tau_0)$, then $\sigma(a, b; L)$ will change by an amount of order $(L')^{-\infty}$. Thus if the state $\langle \dots \rangle$ on \mathbb{R}^2 is the limit of a state on T^2 , then

$$\sigma(a, b) = \lim_{L \rightarrow \infty} \sigma(a, b; L), \quad (38)$$

and the convergence is very rapid: $\sigma(a, b) - \sigma(a, b; L) = O(L^{-\infty})$.

Our strategy will be to show that for $T = 0$ variation of $\sigma(a, b; L)$ with respect to the parameters of the Hamiltonian is of order $L^{-\infty}$. Taking the limit $L \rightarrow \infty$ we then conclude that $\sigma(a, b)$ is a topological invariant for $T = 0$. Note that in order to be able to set $T = 0$ from the start, we need to assume that the correlation time τ_0 is finite at $T = 0$. If we take the limit $T \rightarrow 0$ instead, then in order to follow the above strategy we need to be able to exchange the order of limits $T \rightarrow 0$ and $L \rightarrow \infty$. As a quick look at eq. (31) shows, the limit $L \rightarrow \infty$ is not uniform in T unless we require τ_0 to remain finite at $T = 0$. The easiest way out is to assume that at least a small amount of disorder is present, so that τ_0 does not diverge as $T \rightarrow 0$.

Let us say that a quantity depending on L is negligible if it is of order $L^{-\infty}$ for large L . The independence of $\sigma(a, b)$ of a, b implies that $\sigma(a, b; L)$ changes by a negligible amount when we change a and b . We are now going to use this property to show that at $T = 0$ the derivative of $\sigma(a, b; L)$ with respect to the parameters of the Hamiltonian is negligible. It is sufficient to show this for families of Hamiltonians of the form $H(\lambda) = H + \lambda V$, where V is a local operator supported on a region of a fixed diameter $D \ll L$. The general case is an immediate consequence, since we can write an arbitrary deformation as a sum of such deformations. The number of terms in this sum scales like L^2 , hence we still get a negligible derivative.

On a torus the energy spectrum is discrete. For simplicity, we assume that the ground state is unique in the limit $L \rightarrow \infty$. However, as remarked in [16], this is not an essential requirement, and the analysis can be extended to the case of multiple ground states, if they are locally indistinguishable, i.e. if the correlators of all local observables in any two ground states become the same in the limit $L \rightarrow \infty$. In the limit $T \rightarrow 0$ the Kubo pairing of two operators becomes

$$\beta \langle \langle A; B \rangle \rangle \xrightarrow{T \rightarrow 0} \sum_{n>0} \frac{\langle 0|A|n\rangle \langle n|B|0\rangle + \langle 0|B|n\rangle \langle n|A|0\rangle}{E_n - E_0}, \quad (39)$$

where the summation is over energy levels of the Hamiltonian H , and $n = 0$ is the ground state. Time integration can be done explicitly at zero temperature:

$$\begin{aligned} \int_0^\infty dt e^{-st} \sum_{n>0} \frac{\langle 0|J_a|n\rangle \langle n|J_b|0\rangle e^{i(E_0 - E_n)t} + \langle 0|J_b|n\rangle \langle n|J_a|0\rangle e^{-i(E_0 - E_n)t}}{E_n - E_0} \\ = -i \sum_{n>0} \frac{\langle 0|J_a|n\rangle \langle n|J_b|0\rangle - \langle 0|J_b|n\rangle \langle n|J_a|0\rangle}{(E_0 - E_n)^2}, \end{aligned} \quad (40)$$

where we set s to zero in the end since the system is gapped. This expression is already anti-symmetric in α and γ , since the dissipative part of the conductivity tensor vanishes at zero temperature if the energy gap is present.

In order to compute the variation with respect to a local change of the Hamiltonian it is convenient to introduce the resolvent $G = (z - H)^{-1}$ and to rewrite the Hall conductance as a contour integral in the z -plane:

$$\sigma^A(a, b) = -i \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a G^2 J_b), \quad (41)$$

where the contour of integration encloses the point $z = E_0$ counterclock-wise and the trace is over all energy eigenstates.

Since changing a and b affects $\sigma(a, b; L)$ only by a negligible amount, we can choose them so that the distance between the support of the perturbation V and the lines $x = a$ and $y = b$ is of order L . The variation of (41) is proportional to

$$\begin{aligned} \frac{\partial}{\partial \lambda} \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a G^2 J_b) &= \oint_{z=E_0} \frac{dz}{2\pi i} \{ \text{Tr} (G V G J_a G G J_b) \\ &+ \text{Tr} (G J_a G V G G J_b) + \text{Tr} (G J_a G G V G J_b) \}. \end{aligned} \quad (42)$$

where we have used the fact that variations of J_a, J_b are zero since the supports of J_a and J_b are more than a distance 2δ away from the support of V . We also used

$$\frac{\partial G}{\partial \lambda} = G \frac{\partial H}{\partial \lambda} G = G V G. \quad (43)$$

Subtracting a total derivative

$$\begin{aligned} 0 &= \oint \frac{dz}{2\pi i} \text{Tr} \frac{\partial}{\partial z} (G J_a G V G J_b) = \oint \frac{dz}{2\pi i} \{ \text{Tr} (G G J_a G V G J_b) \\ &+ \text{Tr} (G J_a G G V G J_b) + \text{Tr} (G J_a G V G G J_b) \}, \end{aligned} \quad (44)$$

from the above expression, we get

$$\frac{\partial}{\partial \lambda} \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a G^2 J_b) = - \oint \frac{dz}{2\pi i} \text{Tr} ([V, G J_a G] G J_b G). \quad (45)$$

In Appendix B we show that correlators of the form

$$\oint \frac{dz}{2\pi i} \text{Tr} ([A, G B G] G C G), \quad (46)$$

where A, B, C are local operators and the support of A is away from the support of B , are exponentially suppressed for gapped systems. Therefore we have

$$\frac{\partial \sigma^A(a, b; L)}{\partial \lambda} = -i \frac{\partial}{\partial \lambda} \oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} (G J_a(x_0) G^2 J_b(y_0)) = O(L^{-\infty}), \quad (47)$$

as claimed.

III. THERMAL HALL CONDUCTANCE OF A LATTICE SYSTEM

A. Energy currents and energy magnetization on a lattice

The energy current from site q to site p is defined to be

$$J_{pq}^E = -i[H_p, H_q], \quad (48)$$

so that

$$\frac{dH_q}{dt} = - \sum_{p \in \Lambda} J_{pq}^E. \quad (49)$$

Since $[H_p, H_q] = 0$ whenever $|p - q| > \delta$, J_{pq}^E is an operator-valued 1-chain. The energy flux from B to $A = \Lambda \setminus B$ is defined to be

$$J^E(A, B) = \sum_{p \in A} \sum_{q \in B} J_{pq}^E = J^E(\delta\chi_B), \quad (50)$$

where χ_B is the same as before.

In an equilibrium state we have

$$\partial \langle J^E \rangle = 0. \quad (51)$$

This suggests that there might exist a real-valued 2-chain M such that

$$\langle J^E \rangle = \partial M^E. \quad (52)$$

In fact, such a chain necessarily exists because the cohomology of ∂ in degree 1 is trivial. M^E is not unique: it is defined up to

$$M^E \mapsto M^E + \partial P, \quad (53)$$

where P is an arbitrary real-valued 3-chain. The equation (52) is a lattice analog of the continuum equation

$$\langle J_k^E(p) \rangle = -\epsilon_{kj} \partial_j M^E(p), \quad (54)$$

which defines “energy magnetization” M^E [4]. Note that in the continuum energy magnetization is a function of spatial coordinates, while on the lattice it is a 2-chain. Unfortunately, there is no canonical choice of M^E , either in the continuum or on the lattice. However, as noted in [18], if we let the Hamiltonian depend on some parameters λ_ℓ , then there is

a canonical expression for the 2-chain $\mu^E = dM^E$ valued in the space of 1-forms on the parameter space which solves the equation

$$d\langle J^E \rangle = \partial\mu^E. \quad (55)$$

Namely:

$$\mu^E(p, q, r) = -\beta\langle\langle dH_p; J_{qr}^E \rangle\rangle - \beta\langle\langle dH_r; J_{pq}^E \rangle\rangle - \beta\langle\langle dH_q; J_{rp}^E \rangle\rangle, \quad (56)$$

where $d = \sum_l d\lambda_l \frac{\partial}{\partial\lambda_l}$ is the differential on space of local Hamiltonians. The identity (55) is easily verified using (26), (28) and the definition of the energy current (48). Note that μ_E is a 2-chain thanks to the assumed decay properties of the Kubo pairing.

B. Thermal Hall conductance

A naive application of the Kubo formula gives the following expression for the off-diagonal components of the thermal conductance tensor:

$$\kappa_{\text{Kubo}}(\alpha, \gamma) = \beta^2 \int_0^\infty dt e^{-st} \langle\langle J^E(\delta\alpha, t); J^E(\delta\gamma) \rangle\rangle, \quad (57)$$

where α and γ are as above and the extra factor of β arises from the definition of the thermodynamic force as ∇T rather than $\frac{1}{T}\nabla T$. However, this expression is incorrect both for physical and mathematical reasons. As explained in [4], it is physically incorrect because it includes non-transport circulating energy currents which should not be part of the definition of the thermal Hall conductance. Also, κ_{Kubo} typically diverges as $1/T$ for low T . It is mathematically incorrect because it depends on the choice of the cochains α and γ . Namely, using the same manipulations as in Section II C it is easy to check that under $\alpha \mapsto \alpha + \Delta\alpha$, where $\Delta\alpha$ is compactly supported, κ_{Kubo} changes by

$$\Delta\kappa_{\text{Kubo}}(\alpha, \gamma) = -\beta\langle i[H(\Delta\alpha), H(\gamma)] \rangle. \quad (58)$$

This is well-defined, but in general nonzero. Note that only the anti-symmetric part $\kappa_{\text{Kubo}}^A(\alpha, \gamma)$ suffers from this problem. The symmetric part does not depend on the choice of α and γ , in agreement with the fact that it is an actual transport coefficient.

As discussed in the introduction, the root cause of these problems is that only the derivative of the thermal Hall conductance with respect to parameters is a physical quantity.

Accordingly, we define a 1-form on the space of parameters

$$d\kappa^A(\alpha, \gamma) = d\kappa_{\text{Kubo}}^A(\alpha, \gamma) - 2\beta\mu^E(\delta\alpha \cup \delta\gamma). \quad (59)$$

The additional term both subtracts the microscopic circulating current and makes the formula independent of the choice of cochains. The latter can be shown as follows. Let us fix γ and deform cochain α into α' such that $\Delta\alpha = \alpha' - \alpha$ is supported on a strip $-L' < x(p) < L'$. Using the conservation law we get

$$d\kappa_{\text{Kubo}}^A(\Delta\alpha, \gamma) = -\beta^2 \int_0^\infty dt e^{-st} d\langle\langle \frac{dH(\Delta\alpha, t)}{dt}; J^E(\delta\gamma) \rangle\rangle = \beta^2 d\langle\langle H(\Delta\alpha); J^E(\delta\gamma) \rangle\rangle. \quad (60)$$

Due to the decay of the Kubo pairings, we can replace the cochain $\Delta\alpha$ with a truncated cochain $\widetilde{\Delta\alpha}$ which vanishes for $|y| > L$ while introducing an error of order $L^{-\infty}$ for a sufficiently large L . Since the cochain $\widetilde{\Delta\alpha}$ is compactly supported, the following manipulation is legitimate

$$\beta^2 d\langle\langle H(\widetilde{\Delta\alpha}); J^E(\delta\gamma) \rangle\rangle = \beta^2 d\langle\langle H(\widetilde{\Delta\alpha}); -i[H, H(\gamma)] \rangle\rangle = -\beta d\langle i[H(\widetilde{\Delta\alpha}), H(\gamma)] \rangle, \quad (61)$$

Similarly, replacing $\Delta\alpha$ with $\widetilde{\Delta\alpha}$ introduces an error of order $L^{-\infty}$ in the energy magnetization term. After this truncation we get

$$\begin{aligned} \mu^E(\delta\widetilde{\Delta\alpha} \cup \delta\gamma) &= \partial\mu^E(\widetilde{\Delta\alpha} \cup \delta\gamma) = d\langle J^E(\widetilde{\Delta\alpha} \cup \delta\gamma) \rangle \\ &= -\frac{1}{2} \sum_{p, q \in \Lambda} d\langle i[H_p, H_q] \rangle \widetilde{\Delta\alpha}(p)(\gamma(q) - \gamma(p)) \\ &= -\frac{1}{2} \sum_{p, q \in \Lambda} d\langle i[H_p, H_q] \rangle \widetilde{\Delta\alpha}(p)\gamma(q) = -\frac{1}{2} d\langle i[H(\widetilde{\Delta\alpha}), H(\gamma)] \rangle, \end{aligned} \quad (62)$$

where we have truncated $\Delta\alpha$ for $|y| > L$, introducing an error of order $L^{-\infty}$, in the first step we have used Stokes' theorem, in the second step we have used eq. (55), and in the last steps we have used the definition of J^E and the cup product. Therefore $-2\beta\mu^E$ is exactly the right term to cancel the change in $d\kappa_{\text{Kubo}}^A$.

We would like to conclude this section with a precaution. One might be tempted to define a function instead of just a 1-form on the space of local Hamiltonians by dropping the differential from the Kubo term in (59) and replacing the energy magnetization term with $\beta\langle i[H(\alpha), H(\gamma)] \rangle$. However, the latter expression is a double sum over an infinite set which does not absolutely converge. Any attempt to regularize the sum will introduce ambiguities. For example, one can try to define $\langle [H(\alpha), H(\gamma)] \rangle$ by summing first over α and then over γ ,

but one encounters an evaluation of a non-compact chain on a non-compact cochain on the second step which is ill-behaved. Alternatively, one can try to cut-off the sum, but then a boundary term will arise.

C. Path-independence of the thermal Hall conductance

We have defined a 1-form $d\kappa^A$ on the space of parameters of a lattice system whose integral along a curve Γ can be identified with the difference of thermal Hall conductances of the initial and final points of Γ . The definition of the 1-form depended on the rapid spatial decay of the Kubo pairings of local operators, as well as on the finiteness of the correlation time. Thus when choosing a curve connecting two points \mathcal{M} and \mathcal{M}' in the parameter space, one needs to avoid loci where phase transitions occur, or where the correlation time becomes infinite because of integrability. Since we are allowed to enlarge the parameter space by adding arbitrary local terms to the Hamiltonian, it is very plausible that such a curve exists for any two points \mathcal{M} and \mathcal{M}' .

An important consistency requirement is that the difference of the thermal Hall conductances thus computed does not depend on the choice of Γ . To show this, consider an arbitrary closed loop Γ in the parameter space. By our assumptions, the Kubo part of conductivity is well-defined for each point of Γ . Therefore $d\kappa^{\text{Kubo}}$ is an exact 1-form and its integral over any closed curve vanishes. Now consider the energy magnetization term for different values of the parameters of the Hamiltonian λ but with fixed 1-cochains α and γ . As in the previous section, we can replace α and γ with truncated cochains $\tilde{\alpha}, \tilde{\gamma}$ which are zero far away from the intersection of $\delta\alpha$ and $\delta\gamma$ while introducing an error in the energy magnetization term of order $C(\lambda)e^{-L/\xi(\lambda)}$. Since the closed curve Γ is compact, there exist non-zero $\xi_\Gamma = \inf_{\lambda \in \Gamma} \xi(\lambda)$ and non-zero $C_\Gamma = \sup_{\lambda \in \Gamma} C(\lambda)$ such that the error introduced by truncation is smaller than $C_\Gamma e^{-L/\xi_\Gamma}$. Since the intersection of $\tilde{\alpha}$ and $\tilde{\gamma}$ is compact, one can do the same manipulations as in eq. (62) to show that

$$\mu^E(\delta\tilde{\alpha} \cup \delta\tilde{\gamma}) = -\frac{1}{2}d\langle i[H(\tilde{\alpha}), H(\tilde{\gamma})] \rangle. \quad (63)$$

Since the r.h.s. is an exact 1-form, the integral of the energy magnetization term over any closed loop is of order $L^{-\infty}$. Since L is arbitrary, we conclude that the integral over any closed Γ is zero.

An equivalent way to state this is to say that the 1-form $\mu^E(\delta\alpha \cup \delta\gamma)$ is exact. Since by definition it is the differential of energy magnetization, this means that the energy magnetization exists as a globally-defined function on the parameter space. This function is defined up to an additive constant.

Non-rigorously, this must be true in order to avoid contradiction with the theorem about the absence of net energy currents in equilibrium quasi-1d systems [12]. Indeed, imagine slowly varying the parameters of the system as a function of the y coordinate while following a closed curve Γ . Suppose the parameters return to the original values at $y = L$ where L is large compared to the correlation length. Then we can compactify the y direction with period L , and regard this as a quasi-1d system. For large L the energy current in the x direction can be computed using the continuum equation (54). Since the net energy current should vanish, we get

$$0 = \int \langle J_x^E \rangle dy = \int_0^L \partial_y M^E dy \simeq \int \partial_\lambda M^E d\lambda = \int_\Gamma \mu^E. \quad (64)$$

The error in this computation should become arbitrarily small for $L \rightarrow \infty$. Since the energy current must vanish in equilibrium, we get the desired result.

D. A relative invariant of gapped 2d lattice systems

In this section we use the 1-form $d\kappa^A$ to define a relative topological invariant of gapped 2d lattice systems at zero temperature. We anticipate that in the case when both lattice systems admit a conformally-invariant edge, the invariant will be equal to $\pi/6$ times the difference of the chiral central charges for the two systems. We cannot necessarily connect two such systems by a curve Γ in the space of Hamiltonians without encountering a bulk phase transition. If we could, this would mean that they are in the same zero-temperature phase, and then by the result of [12] they would have to have the same chiral central charge for the edge modes, and therefore the relative invariant would vanish. Rather, the idea is to treat the temperature T as yet another parameter, and connect the two systems in the enlarged parameter space (T, λ) . At positive temperatures quantum phase transitions are smoothed out into cross-overs, and the two systems can now be deformed into each other while maintaining a finite correlation length.

Formally, the temperature can be regarded as a parameter because re-scaling the temper-

ature by a positive factor is equivalent to re-scaling the Hamiltonian by the inverse factor. Therefore one can extend the form $\kappa^A(\alpha, \gamma)$ to the open subset of the enlarged parameter space given by $T > 0$. In detail, this is done as follows. Given a Hamiltonian H , we define a one-parameter family of Hamiltonians by $H(\lambda) = \lambda H$. Then the above mentioned scaling symmetry implies

$$T \frac{d}{dT} \frac{\kappa_{Kubo}^A(\alpha, \gamma)}{T} = - \lambda \frac{d}{d\lambda} \Big|_{\lambda=1} \frac{\kappa_{Kubo}^A(\alpha, \gamma; \lambda)}{T}, \quad (65)$$

where $\kappa_{Kubo}^A(\alpha, \gamma; \lambda)$ denotes the Kubo part of κ^A computed with the Hamiltonian $H(\lambda)$. We have to divide κ^A by T in order to get an observable which is invariant under the rescaling $H \mapsto \lambda H, T \mapsto \lambda T$. Under $\alpha \mapsto \alpha + \Delta\alpha$, where $\Delta\alpha$ is supported on a strip $-L' < x < L'$, the expression on the r.h.s. of (65) changes by

$$\frac{1}{T^2} \lambda \frac{d}{d\lambda} \Big|_{\lambda=1} \langle i\lambda^2 [H(\Delta\alpha), H(\gamma)] \rangle_\lambda, \quad (66)$$

where $\langle \dots \rangle_\lambda$ denotes average with respect to the Gibbs state with the Hamiltonian λH . On the other hand, let us define a 2-chain τ^E by

$$\tau^E(p, q, r) = \beta \langle \langle H_p; J_{qr}^E \rangle \rangle + \beta \langle \langle H_r; J_{pq}^E \rangle \rangle + \beta \langle \langle H_q; J_{rp}^E \rangle \rangle. \quad (67)$$

This is just $-\mu^E$ with dH_p replaced with H_p . The computation in (62) then implies

$$\tau^E(\delta\Delta\alpha \cup \delta\gamma) = \frac{1}{2} \lambda \frac{d}{d\lambda} \Big|_{\lambda=1} \langle \lambda^2 i [H(\Delta\alpha), H(\gamma)] \rangle_\lambda. \quad (68)$$

Therefore the following combination is independent of the choice of cochains α and γ :

$$T \frac{d}{dT} \frac{\kappa_{Kubo}^A(\alpha, \gamma)}{T} - \frac{2}{T^2} \tau^E(\delta\alpha \cup \delta\gamma). \quad (69)$$

We can now define a 1-form on the subset $T > 0$ of the enlarged parameter space:

$$\Psi = \frac{d\kappa_{Kubo}^A(\alpha, \gamma)}{T} - \frac{2}{T^2} \mu^E(\delta\alpha \cup \delta\gamma) + \frac{d}{dT} \left(\frac{\kappa_{Kubo}^A(\alpha, \gamma)}{T} \right) dT - 2\tau^E(\delta\alpha \cup \delta\gamma) \frac{dT}{T^3}. \quad (70)$$

By construction, it is independent of the cochains α and γ . Its integral around any closed curve in the (T, λ) space is zero by the same argument as before, therefore Ψ is exact.

Given any two gapped zero-temperature lattice systems \mathcal{M} and \mathcal{M}' , we would like to define a relative topological invariant by integrating Ψ along a curve in the enlarged parameter space which connects \mathcal{M} and \mathcal{M}' . We need to check two things: that the integral

converges, and that it does not change as one deforms \mathcal{M} and \mathcal{M}' while keeping $T = 0$ and finite correlation length. Neither of these is obvious. The T -component of the 1-form Ψ is

$$\begin{aligned} \Psi_n &= \frac{d}{dT} \left(\frac{\kappa_{Kubo}^A(\alpha, \gamma)}{T} \right) - \frac{2}{T^3} \tau^E(\delta\alpha \cup \delta\gamma) = \\ &= -\frac{1}{2T^3} \left[\frac{d}{d\lambda} \Big|_{\lambda=1} \int_0^\infty \beta \langle \langle J_\lambda^E(\delta\alpha, t); J_\lambda^E(\delta\gamma) \rangle \rangle_\lambda dt + 2\tau^E(\delta\alpha \cup \delta\gamma) - (\alpha \leftrightarrow \gamma) \right]. \end{aligned} \quad (71)$$

Here $\langle \langle \dots \rangle \rangle_\lambda$ denotes the Kubo pairing at temperature T with respect to the Hamiltonian $H(\lambda) = \lambda H$, and J_λ^E is the energy current for the Hamiltonian $H(\lambda)$. We denoted the T -component Ψ_n to emphasize that it is the normal component to the boundary $T = 0$ of the enlarged parameter space. The convergence of the integral of Ψ requires the expression in parentheses to vanish faster than T^2 as $T \rightarrow 0$. Similarly, the independence of the integral of Ψ on the deformation of the endpoints requires the tangential component of Ψ ,

$$\Psi_t = \frac{1}{2T^2} \left(d \int_0^\infty \beta \langle \langle J^E(\delta\alpha, t); J^E(\delta\gamma) \rangle \rangle dt - 2\mu^E(\delta\alpha \cup \delta\gamma) - (\alpha \leftrightarrow \gamma) \right). \quad (72)$$

to vanish at $T = 0$. Thus the expression in parentheses should vanish faster than T^2 as $T \rightarrow 0$.

In Appendix B we give some non-rigorous arguments that in fact both expressions vanish exponentially fast as $T \rightarrow 0$. To see why this is plausible, consider eq. (72) for definiteness and denote the expression in parentheses $g(T)$. It is a 1-form on the space of parameters of the Hamiltonian. The first term in $g(T)$ is the exterior derivative of the same kind of current correlator which defines the electric Hall conductance, except that the electric current J is replaced with the energy current J^E . The same argument as in Section IID shows that at $T = 0$ this exterior derivative is of order $L^{-\infty}$ when evaluated on any perturbation of the Hamiltonian which a distance L away from the support of $\delta\alpha \cup \delta\gamma$. The same is true for the second term, because of the assumed decay of Kubo pairings. Since the sum of the two terms does not change as one varies α and γ , L can be arbitrarily large, and we conclude that $g(0) = 0$ when evaluated on any deformation of the Hamiltonian supported on a quadrant in \mathbb{R}^2 . Therefore $g(0) = 0$ identically. Further, in the presence of the energy gap one expects the low-temperature expansion to have a finite radius of convergence, therefore $g(T) - g(0)$ is exponentially suppressed for low T . Combining these statements, we get the desired result.

There is another limit where one can evaluate Ψ , namely $T \rightarrow \infty$. In this limit the expectation value $\langle A \rangle$ of a local operator A becomes the normalized trace over the local

Hilbert space, while the Kubo pairing becomes

$$\lim_{T \rightarrow \infty} \langle\langle A; B \rangle\rangle = \langle AB \rangle - \langle A \rangle \langle B \rangle. \quad (73)$$

Thus all components of Ψ are of order $1/T^3$ for large T , and therefore the relative thermal Hall conductance of any two high-temperature states is of order $1/T^2$. Hence another natural choice of a reference state (apart from the trivial insulator at $T = 0$) is the $T = \infty$ state. That is, one can define an absolute topological invariant of a gapped zero-temperature system \mathcal{M} by integrating the 1-form Ψ along any path connecting \mathcal{M} to the $T = \infty$ state.

The case of a Locally Commuting Projector Hamiltonian is particularly simple. In this case, since $J_{pq}^E = -i[H_p, H_q] = 0$ for all p, q , the T -component of the 1-form Ψ vanishes identically. Integrating Ψ along a path Γ along which only T changes, we find that $\kappa^A(T) - \kappa^A(\infty) = 0$. Thus the thermal Hall conductance relative to the $T = \infty$ state is zero for all temperatures.⁴ This implies that the chiral central charge of the edge modes must vanish for such a Hamiltonian. One can also show that the zero-temperature electric Hall conductance vanishes for such systems, but the proof is very different [13].

The case of gapped systems of free fermions is also fairly simple, since there are no phase transitions at any $T > 0$, and one can again integrate Ψ along a path with only T varying. Then one only needs to know the T -component of Ψ , which can be evaluated in complete generality. This computation is performed in Appendix C where it is shown that

$$\int_{T=0}^{T=\infty} \Psi = \left. \frac{\kappa^A}{T} \right|_{T=\infty} - \left. \frac{\kappa^A}{T} \right|_{T=0} = -\frac{\pi^2}{3} \sigma^A, \quad (74)$$

where σ^A is the electric Hall conductance at $T = 0$. If one defines κ^A/T to vanish at $T = \infty$, then this can be regarded as a form of the Wiedemann-Franz law. Note however that it cannot be interpreted too naively. For example, since Ψ is exponentially small for low T , most of the contribution to the integral (74) comes from T of order of the energy gap. Although one can define the absolute thermal Hall conductance at temperature T as

$$\kappa^A(T) = T \int_{\infty}^T \Psi, \quad (75)$$

⁴ Strictly speaking, to avoid potential phase transitions at $T > 0$, one needs to work with a finite-volume version of Ψ defined in Appendix B. Its T -component still vanishes for a system described by a Local Commuting Projector Hamiltonian, so the integral from any T to $T = \infty$ is still zero. Taking the infinite-volume limit we conclude that the relative thermal Hall conductance is identically zero.

and it will obey the Wiedemann-Franz law $\kappa^A \simeq \frac{\pi^2}{3} T \sigma^A$ at low T , $\kappa^A(T)$ is not determined by correlators measured at temperature T and a fixed Hamiltonian.

IV. CONCLUDING REMARKS

The problem of defining thermal Hall conductance as a bulk property has been previously discussed in [3, 4, 6]. Cooper, Halperin and Ruzin [3] explained the need to correct the Kubo formula for thermal Hall conductance with energy magnetization terms. Later Ref. [4] derived expressions for derivatives of energy magnetization M^E with respect to chemical potential and temperature. Similar formulas were obtained in [19] in the context of hydrodynamics of systems with Galilean invariance. From our point of view, these are evaluations of the 1-form μ^E on particular tangent vector fields to the enlarged parameter space.

One subtlety emphasized in this paper is that only derivatives of M^E with respect to parameters are unambiguously defined. The importance of this fact was previously stressed by Bradlyn and Read [6] who concluded that thermal Hall conductance is a purely edge effect. Our interpretation is somewhat different: we regard thermal Hall conductance as a well-defined relative transport coefficient. Yet another approach can be found in Ref. [4], where it was proposed to fix ambiguities in the definition of M^E utilizing the equation for the T -derivative of M^E . Specifically, the equation derived in [4] reads

$$\frac{d}{dT} \left(\frac{M^E}{T^2} \right) = \frac{f(T)}{T^3}, \quad (76)$$

where $f(T)$ is a certain linear combination of susceptibilities, and Ref. [4] proposed to pick the solution $M^E(T)$ such that both $M^E(T)$ and $dM^E(T)/dT$ are non-singular at $T = 0$. It is easy to see that there is no such solution if $f(0) \neq 0$, which is the case generically. In a gapped system, $f(T)$ differs from $f(0)$ by terms which are exponentially small for low T . In such a system, if $f(0)$ happens to vanish, the prescription of [4] is well-defined and leads to $M^E(T)$ which is exponentially small for low T . Even in such cases the prescription is unphysical, as can be seen by examining the computation of $M^E(T)$ for a simple one-parameter family of band Hamiltonians [20]. The Hamiltonian in that case depends on a real parameter λ so that for $\lambda < 0$ the system is a Chern insulator, for $\lambda > 0$ it is a trivial insulator, while for $\lambda = 0$ it is gapless. One finds that $M^E(T)$ computed following

the prescription in [4] has a discontinuity at $\lambda = 0$ for all $T > 0$, including $T = \infty$. The thermal Hall conductance has a similar discontinuity. This behavior is unphysical since for $T > 0$ there is no phase transition as one crosses from positive to negative λ , and all physical quantities remain smooth.

In our approach, if one normalizes $\kappa^A(T)$ to vanish for a trivial insulator at $T = 0$, one finds that in the Chern insulator $\kappa^A \sim T$ at low T , as expected from the relation between κ^A/T and the chiral central charge. The thermal Hall conductance is smooth for all $T > 0$ and λ and vanishes in the limit $T \rightarrow \infty$.

Another difference between [3, 4] and this work is that both [3, 4] discuss continuous systems and use Luttinger's approach to defining thermal responses via a fictitious gravitational potential [21]. This approach makes the computation of the thermal response formally analogous to that of the electric response. However, in order to derive differential equations for energy magnetization, the authors of [4] postulated that one can find an energy current \mathbf{j}^E which depends on the fictitious gravitational potential in a particularly simple way. It is possible to reformulate this postulate without making a reference to the gravitational potential. One finds the following condition:

$$i[h(\mathbf{r}), h(\mathbf{r}')] = (\mathbf{j}^E(\mathbf{r}) + \mathbf{j}^E(\mathbf{r}')) \cdot \nabla_{\mathbf{r}} \delta(\mathbf{r} - \mathbf{r}'), \quad (77)$$

where $h(\mathbf{r})$ is the energy density. This is a very restrictive condition: it is satisfied in a theory of free particles for a suitable choice of \mathbf{j}^E , but it cannot be satisfied once one allows for a potential interaction with a nonzero range. Clearly, one must relax eq. (77) in some way, but this leads to an energy current which does not satisfy the scaling relations assumed in [4], and then the differential equations for M^E derived there no longer apply. Similar remarks apply to [19], where it was assumed that the microscopic system can be coupled to Newton-Cartan geometry. This assumption is violated as soon as interactions with a non-zero range are allowed. Ref. [6] also assumes that a coupling to background geometry is possible, although the notion of background geometry is different from that of Ref. [19].

The case of lattice systems is simpler in this respect, in that there is a canonically-defined local energy current for any finite-range Hamiltonian, and this current tautologically satisfies a lattice analog of eq. (77). This allows one to derive lattice analogs of the differential equations in [4, 19] which hold in complete generality [18]. These differential equations are equivalent to the statement that the 1-form μ^E defined by eq. (56) satisfies the identity

$\partial\mu^E = d\langle J^E \rangle$. The price one has to pay is that μ^E is a 2-chain rather than a function, and the correction term for thermal Hall conductance must be expressed using the formalism of chains and cochains.

Appendix A: Exponential decay of certain correlators in a gapped phase

Let A , B , and C be local operators such that the supports of A and B are separated by at least L . Let $G = (z - H)^{-1}$ be the Green's function of a gapped Hamiltonian, and let E_0 be the energy of the ground state. For the time being we assume that the ground state is unique and comment on the more general case later. We are going to prove that the correlator

$$\oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} ([A, GBG] GCG), \quad (\text{A1})$$

is exponentially suppressed for large L . Note that the support of the operator C is not required to be separated from the supports of A and B . By performing the z integration we get

$$\begin{aligned} \oint \frac{dz}{2\pi i} \text{Tr} ([A, GBG] GCG) &= \langle AG_0BG_0^2C \rangle + \langle BG_0^2CG_0A \rangle - \langle CG_0^2AG_0B \rangle \\ &\quad - \langle CG_0AG_0^2B \rangle + \langle BG_0^2AG_0C \rangle + \langle BG_0AG_0^2C \rangle - \langle AG_0CG_0^2B \rangle \\ &\quad - \langle CG_0^2BG_0A \rangle + 2(\langle CG_0^3B \rangle - \langle BG_0^3C \rangle) \langle A \rangle \\ &\quad + (\langle AG_0^3B \rangle - \langle BG_0^3A \rangle) \langle C \rangle + (\langle CG_0^3A \rangle - \langle AG_0^3C \rangle) \langle B \rangle, \end{aligned} \quad (\text{A2})$$

where $\langle \dots \rangle$ denotes the average over the ground state and we have introduced the notation

$$G_0 = \sum_{n \neq 0} \frac{|n\rangle\langle n|}{E_0 - E_n}. \quad (\text{A3})$$

Now we use the following facts from [16] and other similar identities:

$$\begin{aligned} \langle O_1 G_0^n O_2 G_0^m O_3 \rangle &= \langle O_1 G_0^{n+m} O_3 \rangle \langle O_2 \rangle + O(e^{-L/\xi}), \\ \langle O_2 G_0^n O_1 G_0^m O_3 \rangle &= O(e^{-L/\xi}), \\ \langle O_1 G_0^n O_2 \rangle &= O(e^{-L/\xi}), \end{aligned} \quad (\text{A4})$$

if $n, m > 0$ and the support of operator O_2 is at least L distance away from the supports of O_1 and O_3 . Here $\xi > 0$ is a scale parameter which is finite for gapped systems. See [16] for the derivation of these identities.

Using these we can simplify the first term in (A2). Separating C (which is by assumption a sum of local operators) into two parts $C = C_A + C_B$ where the support of C_A is far away from B and the support of C_B is far away from A , we get

$$\begin{aligned}\langle AG_0BG_0^2C \rangle &= \langle AG_0BG_0^2C_A \rangle + \langle AG_0BG_0^2C_B \rangle \\ &= \langle AG_0^3C_A \rangle \langle B \rangle + O(e^{-L/\xi}) = \langle AG_0^3C \rangle \langle B \rangle + O(e^{-L/\xi}).\end{aligned}\tag{A5}$$

Similarly, we have

$$\langle BG_0^2CG_0A \rangle = \langle BG_0^3A \rangle \langle C \rangle + O(e^{-L/\xi}),\tag{A6}$$

$$-\langle CG_0^2AG_0B \rangle = -\langle CG_0^3B \rangle \langle A \rangle + O(e^{-L/\xi}),\tag{A7}$$

$$-\langle CG_0AG_0^2B \rangle = -\langle CG_0^3B \rangle \langle A \rangle + O(e^{-L/\xi}),\tag{A8}$$

$$\langle BG_0^2AG_0C \rangle = \langle BG_0^3C \rangle \langle A \rangle + O(e^{-L/\xi}),\tag{A9}$$

$$\langle BG_0AG_0^2C \rangle = \langle BG_0^3C \rangle \langle A \rangle + O(e^{-L/\xi}),\tag{A10}$$

$$-\langle AG_0CG_0^2B \rangle = -\langle AG_0^3B \rangle \langle C \rangle + O(e^{-L/\xi}),\tag{A11}$$

$$-\langle CG_0^2BG_0A \rangle = -\langle CG_0^3A \rangle \langle B \rangle + O(e^{-L/\xi}).\tag{A12}$$

These eight terms exactly cancel the remaining six terms in (A2). Putting everything together, we get

$$\oint_{z=E_0} \frac{dz}{2\pi i} \text{Tr} ([A, GBG] GCG) = O(e^{-L/\xi}).\tag{A13}$$

We have assumed a single ground state in the above derivation. However, as noted in [16], exactly the same arguments work for a q -fold degenerate ground state assuming that they are indistinguishable by local operators, i.e. if

$$\langle p|O|q \rangle = \delta_{pq} \langle p|O|p \rangle + O(L^{-\infty})\tag{A14}$$

where $|p\rangle, |q\rangle$ are ground states, O is a local operator, and L is the size of the system.

Appendix B: The low-temperature behavior of the 1-form Ψ in a gapped system

Consider a 1-form $g(T)$ on the parameter space of the Hamiltonian given by

$$g(T) = \frac{1}{2}d \int_0^\infty \beta e^{-st} \langle \langle J^E(\delta\alpha, t); J^E(\delta\gamma) \rangle \rangle dt - (\alpha \leftrightarrow \gamma) - 2\mu^E(\delta\alpha \cup \delta\gamma).\tag{B1}$$

We will argue that $g(0) = \lim_{T \rightarrow 0} g(T) = 0$ for a gapped system and that $g(T) - g(0)$ is exponentially small for low T . Our arguments are not rigorous.

The idea is to put the system on a torus of size L and construct an approximation $g(T, L)$ such that

$$g(T) = \lim_{L \rightarrow \infty} g(T, L). \quad (\text{B2})$$

Assuming that the order of limits can be interchanged, we can learn about the behavior of $g(T)$ at low T by studying the behavior of $g(T, L)$ at low T .

As in the case of the electric Hall conductance, we replace the operators $J^E(\delta\alpha)$ and $J^E(\delta\gamma)$ with J_a^E and J_b^E , respectively, where

$$J_a^E = \sum_{x(p) > a, x(q) < a} J_{pq}^E, \quad J_b^E = \sum_{y(p) > b, y(q) < b} J_{pq}^E. \quad (\text{B3})$$

One can think of these operators as evaluations the operator-valued 1-chain J^E on 1-cochains

$$\phi(p, q) = \theta(a - x(q)) - \theta(a - x(p)), \quad \zeta(p, q) = \theta(b - y(q)) - \theta(b - y(p)). \quad (\text{B4})$$

These operators have natural analogues on a torus, and a natural finite-volume analog of the first term in (B1) is

$$g_1(T, L) = \frac{1}{2} d \int_0^\infty e^{-st} \beta \langle \langle J_a^E(t); J_b^E \rangle \rangle_L dt - (a \leftrightarrow b), \quad (\text{B5})$$

where we use subscript L to emphasize that the correlator is computed on a torus of size L . Note that on a torus the cochains ϕ, ζ are unambiguously defined only if the points p, q are sufficiently close to the lines $x = a$ and $y = b$, respectively. This does not cause a problem, however, since J_{pq}^E vanishes for $|p - q| > 2\delta$. That is, even though ϕ, ζ are only partially defined, the evaluations $J_a^E = J^E(\phi)$ and $J_b^E = J^E(\zeta)$ are well-defined operators for $L \gg \delta$.

Next let us construct a finite-volume analog of the second term in (B1). An apparent problem is that $\mu^E(p, q, r)$ does not vanish for large separations between p, q, r , they are just exponentially suppressed there. Since ϕ and ζ are only partially defined, the contraction $\mu^E(\phi \cup \zeta)$ is ambiguous. We deal with this by truncating $\mu^E(p, q, r)$ to zero whenever any of the pairwise distances exceeds $L/4$. Let $\tilde{\mu}^E$ be the truncated chain. We define the finite-volume analog of the second term in (B1) to be

$$g_2(T, L) = -2\tilde{\mu}^E(\phi \cup \zeta). \quad (\text{B6})$$

Next we can examine the dependence of $g(T, L) = g_1(T, L) + g_2(T, L)$ on the choice of a and b . The same manipulations as in Section III B show that the change in $g(T, L)$ is of order $L^{-\infty}$. Replacing μ^E with $\tilde{\mu}^E$ does not affect the argument.

At $T = 0$ this can be used to argue that $g(0) = 0$. It is sufficient to show this for a deformation of the Hamiltonian which vanishes in a large neighborhood of the point $x = a, y = b$. Then the same argument as in Section II D shows that $\lim_{T \rightarrow 0} g(T, L) = O(L^{-\infty})$, and therefore

$$\lim_{L \rightarrow \infty} \lim_{T \rightarrow 0} g(T, L) = 0. \quad (\text{B7})$$

Assuming that we can exchange the order of limits, we conclude that $\lim_{T \rightarrow 0} g(T) = 0$.

To determine the low- T behavior of $g(T, L)$ more precisely, we should examine $g_1(T, L)$ and $g_2(T, L)$ separately and show that both $g_1(T, L) - g_1(0, L)$ and $g_2(T, L) - g_2(0, L)$ are exponentially small at low T . We can replace Kubo pairings

$$\beta \langle \langle A; B \rangle \rangle_L = \int_0^\beta \langle A(-i\tau)B \rangle_L d\tau - \beta \langle A \rangle_L \langle B \rangle_L, \quad (\text{B8})$$

appearing in both $g_1(T, L)$ and $g_2(T, L)$ with the simplified expressions

$$\int_0^\beta \langle A(-i\tau)B \rangle_L d\tau. \quad (\text{B9})$$

For finite L this is legitimate because in a gapped system the expectation values are exponentially close to their $T = 0$ values. After this is done, we can write the simplified Kubo pairing in terms of the many-body Green's function $G(z) = (z - H)^{-1}$:

$$\int_0^\beta \langle A(-i\tau)B \rangle_L d\tau = Z^{-1} \oint e^{-\beta z} \frac{dz}{2\pi i} \text{Tr}(G(z)AG(z)B). \quad (\text{B10})$$

Here Z is the partition function, and the contour surrounds all the eigenvalues of H . Now if we deform the contour into a pair of contours, one surrounding $z = E_0$ and the other surrounding all other eigenvalues, we see that for low T the contribution of the first contour is exponentially close to its $T \rightarrow 0$ limit, while the contribution of the second one is exponentially small at low T . Thus $g_2(T, L) - g_2(0, L)$ is exponentially small at low T . For $g_1(T, L)$ there is an additional integration over time, and one cannot write a simple contour integral representation for it. Nevertheless, by inserting a complete set of states, it is easy to verify that $g_1(T, L) - g_1(0, L)$ is exponentially suppressed at low T . If we assume that the order of limits $T \rightarrow 0$ and $L \rightarrow \infty$ can be interchanged, we can conclude that $g(T)$ is exponentially small at low T . These arguments are at best heuristic, since it is far from clear when the interchange of the order of limits is legitimate.

Appendix C: Thermal Hall conductance for free fermions

Consider a free fermionic system on a lattice with a Hamiltonian

$$H = \sum_{p,q} a^\dagger(p)h(p,q)a(q). \quad (\text{C1})$$

The infinite matrix $h(p,q)$ is assumed Hermitian, $h(p,q)^* = h(q,p)$. The energy on site p is taken to be

$$H_p = \frac{1}{2} \sum_m (a^\dagger(p)h(p,m)a(m) + a^\dagger(m)h(m,p)a(p)). \quad (\text{C2})$$

Defining the charge operator as a 0-chain

$$Q_p = a^\dagger(p)a(p), \quad (\text{C3})$$

we find the electric current 1-chain

$$J_{pq} = i(a^\dagger(q)h(q,p)a(p) - a^\dagger(p)h(p,q)a(q)). \quad (\text{C4})$$

Contracting it with a 1-cochain $f(q) - f(p)$ for some function $f : \Lambda \rightarrow \mathbb{R}$, we get

$$J(\delta f) = -ia^\dagger[h, f]a, \quad (\text{C5})$$

where we now regard f as an operator in the one-particle Hilbert space.

Similarly, the energy current operator is a 1-chain

$$\begin{aligned} J_{pq}^E = \frac{-i}{4} \sum_m & (a^\dagger(p)h(p,q)h(q,m)a(m) - a^\dagger(q)h(q,p)h(p,m)a(m) \\ & - a^\dagger(m)h(m,q)h(q,p)a(q) + a^\dagger(m)h(m,p)h(p,q)a(q) \\ & + a^\dagger(p)h(p,m)h(m,q)a(q) - a^\dagger(q)h(q,m)h(m,p)a(p)). \end{aligned} \quad (\text{C6})$$

Contracting it with a 1-cochain $f(q) - f(p)$, we get

$$J^E(\delta f) = -\frac{i}{2}a^\dagger[h^2, f]a. \quad (\text{C7})$$

The Gibbs state at temperature $T = 1/\beta$ is defined via

$$\langle a(p,t)a^\dagger(q,0) \rangle = \left\langle p \left| \frac{e^{-iht}}{1 + e^{-\beta h}} \right| q \right\rangle, \quad (\text{C8})$$

$$\langle a(p,t)^\dagger a(q,0) \rangle = \left\langle q \left| \frac{e^{iht}}{1 + e^{\beta h}} \right| p \right\rangle, \quad (\text{C9})$$

and Wick's theorem. Then

$$\langle J(\delta\alpha, t)J(\delta\gamma) \rangle = -\text{Tr} \left([h, \alpha] \frac{e^{-iht}}{1 + e^{-\beta h}} [h, \gamma] \frac{e^{iht}}{1 + e^{\beta h}} \right), \quad (\text{C10})$$

where the trace on the r.h.s. is taken over the 1-particle Hilbert space $L^2(\Lambda)$, and the functions $\alpha : \Lambda \rightarrow \mathbb{R}$ and $\gamma : \Lambda \rightarrow \mathbb{R}$ are regarded as Hermitian operators on this Hilbert space. The operators $[h, \alpha]$ and $[h, \gamma]$ are supported on a vertical and a horizontal strips, respectively.

Going to the energy basis, replacing $t \rightarrow t - i\tau$ and integrating over τ from 0 to β we get

$$\langle \langle J(\delta\alpha, t); J(\delta\gamma) \rangle \rangle = \frac{-1}{\beta} \sum_{n,m} \langle n|[h, \alpha]|m \rangle \langle m|[h, \gamma]|n \rangle e^{i(\varepsilon_n - \varepsilon_m)t} \frac{e^{\beta\varepsilon_n} - e^{\beta\varepsilon_m}}{(1 + e^{\beta\varepsilon_n})(1 + e^{\beta\varepsilon_m})(\varepsilon_n - \varepsilon_m)}, \quad (\text{C11})$$

where ε_n are 1-particle energy levels. Note that in the limit $T \rightarrow 0$, the fraction in this equation is equal to $\frac{\theta(\varepsilon_n) - \theta(\varepsilon_m)}{\varepsilon_n - \varepsilon_m}$ plus exponentially small terms. Thus at zero temperature ε_m and ε_n must have opposite signs. More generally, we can re-write the fraction as

$$\frac{f(\varepsilon_m) - f(\varepsilon_n)}{\varepsilon_n - \varepsilon_m} \quad (\text{C12})$$

where $f(\varepsilon) = \frac{1}{1 + e^{\beta\varepsilon}}$ is the Fermi-Dirac distribution.

Integrating over t , we get

$$\sigma(\alpha, \gamma) = i \sum_{n,m} \frac{\langle n|[h, \alpha]|m \rangle \langle m|[h, \gamma]|n \rangle}{\varepsilon_n - \varepsilon_m + i\delta} \frac{f(\varepsilon_n) - f(\varepsilon_m)}{\varepsilon_n - \varepsilon_m} \quad (\text{C13})$$

It is convenient to rewrite this expression using the 1-particle Green's functions $G_{\pm}(z) = 1/(z - h \pm i0)$. The following formulas are useful:

$$\langle a^\dagger A a \rangle = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left([G_+ - G_-] A \right), \quad (\text{C14})$$

$$\begin{aligned} -\beta \langle \langle a^\dagger A a; a^\dagger B a \rangle \rangle &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left([G_+ - G_-] A G_+ B + G_- A [G_+ - G_-] B \right) \\ &= -\frac{1}{2\pi i} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left(G_+ A G_+ B - G_- A G_- B \right), \end{aligned} \quad (\text{C15})$$

where we have suppressed the argument z for $G_{\pm}(z)$. Here A and B are operators on the 1-particle Hilbert space and we have assumed $\langle a^\dagger A a \rangle = \langle a^\dagger B a \rangle = 0$ in the last formula.

Also note that

$$hG_{\pm} = G_{\pm}h = zG_{\pm} - 1, \quad [G_{\pm}, A] = G_{\pm}[h, A]G_{\pm}. \quad (\text{C16})$$

Using the Green's functions, the electric conductance can be rewritten as

$$\sigma(\alpha, \gamma) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \{ [h, \alpha] G_+^2 [h, \gamma] (G_+ - G_-) - [h, \alpha] (G_+ - G_-) [h, \gamma] G_-^2 \}, \quad (\text{C17})$$

and the Kubo part of the thermal conductance as

$$\kappa_{\text{Kubo}}(\alpha, \gamma) = -\frac{\beta}{8\pi} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \{ [h^2, \alpha] G_+^2 [h^2, \gamma] (G_+ - G_-) - [h^2, \alpha] (G_+ - G_-) [h^2, \gamma] G_-^2 \}. \quad (\text{C18})$$

Note that we did not anti-symmetrize in α, γ for simplicity. This adds an exact piece to the 1-form Ψ , but does not affect the integrals we are interested in, because the symmetric part of the Kubo contribution happens not to contribute at endpoints.

The value of energy magnetization μ^E on a 2-cochain $\delta\alpha \cup \delta\gamma$ can be found to be

$$\begin{aligned} \mu^E(\delta\alpha \cup \delta\gamma) = \frac{1}{16\pi} \int_{-\infty}^{\infty} dz f(z) \text{Tr} \left(G_+ dh G_+ \left\{ [[h, \alpha], [h, \gamma]] + [h^2, \alpha] G_+ [h, \gamma] + [h, \alpha] G_+ [h^2, \gamma] \right. \right. \\ \left. \left. - [h^2, \gamma] G_+ [h, \alpha] - [h, \gamma] G_+ [h^2, \alpha] \right\} \right) - (G_+ \rightarrow G_-) \quad (\text{C19}) \end{aligned}$$

where dh is the variation of the 1-particle Hamiltonian. In the translationally invariant case, one can replace α and γ with momentum derivatives.

Using the above formulas, it is straightforward to compute the 1-form Ψ for any free system. Let us demonstrate this by computing the T -component of the 1-form Ψ .

For a global re-scaling of the Hamiltonian we have $dh = h$, and eq. (C19) can be simplified

$$\begin{aligned} \tau^E(\delta\alpha \cup \delta\gamma) = -\frac{1}{16\pi} \int_{-\infty}^{\infty} dz \text{Tr} \left\{ 2f(z) G_-^2 [h^2, \alpha] (G_+ - G_-) [h^2, \gamma] \right. \\ \left. - 2f(z) (G_+ - G_-) [h^2, \alpha] G_+^2 [h^2, \gamma] + 4f'(z) h^2 (G_+ - G_-) [h, \alpha] G_+ [h, \gamma] \right. \\ \left. - 4f'(z) G_- [h, \alpha] h^2 (G_+ - G_-) [h, \gamma] - f'(z) h (G_+ - G_-) [[h, \alpha], [h, \beta]] \right\} \quad (\text{C20}) \end{aligned}$$

Variation of $\kappa_{\text{Kubo}}(\alpha, \gamma)$ contains two pieces:

$$\begin{aligned} -\frac{\beta}{8\pi} d \left(\int_{-\infty}^{\infty} dz f(z) \text{Tr} \left\{ [h^2, \alpha] G_+^2 [h^2, \gamma] (G_+ - G_-) \right\} \right) \\ = \frac{\beta}{8\pi} \int_{-\infty}^{\infty} dz \text{Tr} \left\{ -2f(z) [h^2, \alpha] G_+^2 [h^2, \gamma] (G_+ - G_-) - 4f'(z) [h, \alpha] G_+^2 [h, \gamma] h^3 (G_+ - G_-) \right. \\ \left. + 4f'(z) [h, \alpha] G_+ [h, \gamma] h^2 (G_+ - G_-) - f'(z) [h, \alpha] [h, \gamma] h (G_+ - G_-) \right\} \quad (\text{C21}) \end{aligned}$$

and

$$\begin{aligned} & \frac{\beta}{8\pi} d \left(\int_{-\infty}^{\infty} dz f(z) \text{Tr} \left\{ [h^2, \gamma] G_-^2 [h^2, \alpha] (G_+ - G_-) \right\} \right) \\ &= \frac{\beta}{8\pi} \int_{-\infty}^{\infty} dz \text{Tr} \left\{ 2f(z) [h^2, \gamma] G_-^2 [h^2, \alpha] (G_+ - G_-) + 4f'(z) [h, \gamma] G_-^2 [h, \alpha] h^3 (G_+ - G_-) \right. \\ & \quad \left. - 4f'(z) [h, \gamma] G_- [h, \alpha] h^2 (G_+ - G_-) + f'(z) [h, \gamma] [h, \alpha] h (G_+ - G_-) \right\}. \quad (\text{C22}) \end{aligned}$$

Inserting these three contributions into eq. (71) we arrive at

$$\begin{aligned} \frac{d}{dT} \left(\frac{\kappa(\alpha, \gamma)}{T} \right) &= \frac{1}{2\pi T^3} \int_{-\infty}^{\infty} dz \text{Tr} \left\{ f'(z) [h, \alpha] G_+^2 [h, \gamma] z^3 (G_+ - G_-) \right. \\ & \quad \left. - f'(z) [h, \gamma] G_-^2 [h, \alpha] z^3 (G_+ - G_-) \right\}. \quad (\text{C23}) \end{aligned}$$

The right-hand side looks very similar to the electric conductance (C17). Indeed, integrating it over temperature from 0 to ∞ and using the formula

$$\int_0^{\infty} \frac{dT}{T^3} f'(z) = -\frac{\pi^2}{6|z|^3} = -\frac{\pi^2}{3z^3} (f(0) - f(\infty)) \quad (\text{C24})$$

gives

$$\frac{\kappa^A}{T} \Big|_{T=\infty} - \frac{\kappa^A}{T} \Big|_{T=0} = \frac{\pi^2}{3} \left(\sigma^A \Big|_{T=\infty} - \sigma^A \Big|_{T=0} \right). \quad (\text{C25})$$

Since at infinite temperature the electric Hall conductance vanishes, while the thermal Hall conductance can be defined to vanish, we arrive at the Wiedemann-Franz law.

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