Proceedings of the 1998 IEEE
International Conference on Robotics & Automation
Leuven, Belgium • May 1998

Minimum-Deflection Grasps and Fixtures

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Abstract— **This paper presents an approach to planning compliant grasps and fixtures in which the object exhibits minimal deflection under external disturbances. The approach, which applies to general two- and three-dimensional grasps and fixtures represented by any quasi-rigid compliance model, employs a quality measure that characterizes the grasped or fixtured object's worst-case deflection caused by disturbing wrenches lying in the unit wrench ball. To ensure well-defined notions of deflection and wrench balls, frame-invariant rigid body velocity and wrench norms are for the first time used. As illustrated by its application to fixtures of polygonal objects, our minimum-deflection approach can be effectively applied to planning grasps and fixtures where deflection significantly influences performance.**

1 Introduction

Compliance plays a dominant role in workpiece fixturing, and can also significantly influence the performance of robotic grasps. This paper presents an approach to planning compliant grasps and fixtures in which the object exhibits *minimal deflection* under an applied work load. Loosely speaking, the deflection of a compliantly grasped or fixtured object is the typical (e.g. maximal) displacement of the object's individual particles. In applications such as part machining and assembly insertion, the magnitude of deflection will limit the accuracy of the overall process for which the fixture was designed. Our approach is based on a quality measure that characterizes the object's worst-case deflection caused by disturbing wrenches lying in a unit wrench ball. Valid for general 2D and 3D grasps and fixtures employing *any* number of fingers or fixels, this quality measure holds for *all* compliance models. In establishing the notions of deflection and wrench balls, we use rigid body velocity and wrench norms that are *invariant* to change of reference frame location. To our knowledge, this is the first time frame-invariant norms are applied to quantifying grasp or fixture effectiveness.

A *quality measure* is a scalar-valued function quantifying grasp or fixture effectiveness. Prior research on quality measures has mostly focused on rigid grasps where compliance is ignored. Li and Sastry [9] defined a quality measure as the smallest singular value of the matrix whose columns consist of the generating wrenches, i.e., wrenches due to unit finger forces. The quality measures suggested by Kirkpatrick, Mishra and Yap [7], Ferrari and Canny [3], and Teichmann [20] compute the maximal wrench ball inscribed in the convex hull of the generating wrenches. Markenscoff and Papadimitriou [14] minimized the worst-case finger forces needed to balance a family of pure forces, while Mirtich and Canny [15] treated pure forces and torques lexicographically. The quality measures proposed by Kerr and Roth [6], Trinkle [21], and Bicchi [1] characterize the margin by which grasp contact constraints are satisfied.

The effectiveness of compliant grasps and fixtures has been less studied. Prattichizzo, Salisbury and Bicchi [19] defined robustness measures that quantify a compliant grasp's sensitivity to perturbations of a given work load. In Ref. [13], we presented a quality measure that determines the characteristic stiffness of compliant grasps and fixtures. Compared with the prior works on rigid as well as compliant grasps, the quality measure presented in this paper is *invariant* to change of reference frame location, considers work loads of *all* directions in the wrench space. Moreover, the quality measure *directly* characterizes the object's deflection, and hence has great utility in applications where deflection is a major concern. This quality measure can be used with *any* compliance model using quasi-rigid bodies, such as those in Refs. [2, 4, 5, 12, 18]. For concreteness, however, we will use the compliance model [12] for illustration.

The quality measure is defined as the *norm* of the object's worst-case displacement due to an external wrench lying in the *unit wrench ball*—the set of wrenches whose norms are less than or equal to unity. This approach shares with the works of Refs. [3, 7, 20] in the use of wrench norms. However, while those works exclusively considered rigid grasps, this work concerns compliant grasps. More importantly, the wrench norms in those works depend on reference frame location, whereas we use *frame-invariant* wrench and rigid body velocity norms. The Euclidean norm of wrench components has been employed to define quality measures for rigid grasps [3, 7, 9, 20]. However, the Euclidean wrench and rigid body velocity norms are frame-dependent and hence ill-defined [8]. Lin and Burdick investigated the issue of frame-choice effects via objective kinematic metric functions [11], and for the first time developed frame-invariant, physically meaningful velocity and wrench norms [10]. The current work draws on those results in considering minimum-deflection grasps and fixtures.

2 Modelling Compliance in C-Space

As a convention, we will use the term *fixture* to refer to both grasps and fixtures. A *fixture* consists of an object, denoted $\mathcal{B}$, contacted by k fixture elements (or fixels), denoted $\mathcal{A}_1, \ldots, \mathcal{A}_k$. We assume that the bodies $\mathcal{B}$ and $\mathcal{A}_i$ are *quasi-rigid*, i.e., elastic deformations are restricted to the vicinity of the contacts so that the *overall* motions of the bodies can be considered *rigid*. Hence, by further assuming $\mathcal{A}_i$ to be *stationary*, we can focus on the configuration space of $\mathcal{B}$ (regarded as a rigid body), reviewed as follows.

Let $\mathcal{F}_W$ be a stationary world reference frame, and $\mathcal{F}_B$ a frame fixed to $\mathcal{B}$. A *configuration* of $\mathcal{B}$ is a pair $q = (p, R)$, where $p \in \mathbb{R}^3$ is the position, and $R \in SO(3)$ the orientation of $\mathcal{F}_B$ relative to $\mathcal{F}_W$. The set of all configurations, denoted $\mathcal{C}$, is $\mathcal{B}$'s *configuration space* (c-space). The *tangent space* to $\mathcal{C}$ at configuration q is the set of all *tangent vectors* (*velocities*) of $\mathcal{B}$ at q. Tangent vectors, viewed as instantaneous displacements of $\mathcal{B}$, can be used to approximate *small displacements*. The *wrench space* at q is the set of all *wrenches*, or *covectors*, acting on $\mathcal{B}$ at configuration q.

Tangent vectors and covectors are given coordinates as follows [16]. A tangent vector can be specified by $\dot{q} = (v, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3$, where v is the velocity of $\mathcal{F}_B$'s origin and ω is $\mathcal{F}_B$'s angular velocity relative to the stationary frame $\mathcal{F}_W$. A covector can be written as $\boldsymbol{w} = (f, \tau) \in \mathbb{R}^3 \times \mathbb{R}^3$, where $f \in \mathbb{R}^3$ is a force acting at $\mathcal{F}_B$'s origin and τ is a torque. As vectors in 3D space, v, ω, f and τ can be given coordinates in either of the frames $\mathcal{F}_B$ and $\mathcal{F}_W$. When these vectors are given coordinates in $\mathcal{F}_B$, $\dot{q}$ is called a *body velocity*, and $\boldsymbol{w}$ a *body wrench*. If $\mathcal{F}_W$ is used instead, $\dot{q}$ is a *hybrid velocity* and $\boldsymbol{w}$ a *hybrid wrench*. The body and hybrid coordinates, indexed using b and h, respectively, are related by $\dot{q}_h = \mathcal{R}\dot{q}_b$ and $\boldsymbol{w}_h = \mathcal{R}\boldsymbol{w}_b$, where $\mathcal{R} = \mathrm{diag}(R, R)$ with R the orientation of $\mathcal{F}_B$ relative to $\mathcal{F}_W$. Since these simple rules can be used to transform any expression between body and hybrid coordinates, we will use body velocities and wrenches for convenience. Body coordinates are *not* affected by change of world frame, but depend on the choice of body frame. Let $\bar{\mathcal{F}}_B$ denote a new body frame which is displaced from $\mathcal{F}_B$ by a rigid displacement (R, d). Then the body coordinates with respect to $\bar{\mathcal{F}}_B$, denoted using overbars, are related to those with respect to $\mathcal{F}_B$ by [16]

$$\dot{\bar{q}} = T^{-1}\dot{q} \quad \text{and} \quad \bar{\boldsymbol{w}} = T^T\boldsymbol{w}, \tag{1}$$

The 6×6 matrix T takes the form $T = \left(\begin{smallmatrix} R & \hat{d}R \\ 0 & R \end{smallmatrix}\right)$, where for any $x \in \mathbb{R}^3$, $\hat{x}$ is a skew-symmetric matrix such that $\hat{x}y = x \times y$ for all $y \in \mathbb{R}^3$.

Now consider a fixture, whose elastic behavior can be characterized by a scalar-valued function, called the *elastic potential* and denoted Π, of $\mathcal{B}$'s configuration q. We call a configuration q_0 an *equilibrium configuration* if $\mathcal{B}$ is in equilibrium under *nonzero* fixel forces, in the absence of external disturbances. The nonzero fixel forces are called *preloading* forces and the fixture is called a preloaded *equilibrium fixture*. An equilibrium configuration q_0 is a *critical point* of Π, that is, $\nabla\Pi(q_0) = 0$. The 6×6 Hessian matrix, $K = D^2\Pi(q_0)$, is the equilibrium fixture *stiffness matrix*. As an elastic system, the fixture at q_0 is (quasi-statically) *stable* if q_0 is a local minimum of Π. The fixture is stable if K is positive definite. In this case, if $\dot{q}$ is a body velocity approximating a small displacement of $\mathcal{B}$ due to a disturbing body wrench $\boldsymbol{w}$, we have the following linear relationship: $\boldsymbol{w} = K\dot{q}$. When body velocities and wrenches are used, the stiffness matrix is not influenced by change of world frame. With respect to a new body frame $\bar{\mathcal{F}}_B$, Eq. (1) yields the following transformation rule:

$$\bar{K} = T^T K T. \tag{2}$$

A specific formula must be used to actually compute the stiffness matrix. In this paper, we use the formula in Ref. [12] that employs the overlap compliance model. However, the quality measure presented in this paper is valid for stiffness matrix formulas based on other compliance models.

3 Tangent Vector and Covector Norms

This section discusses a few frame-invariant norms, which may or may not be induced from inner products, of rigid body velocities and wrenches [10, 11]. We briefly review the notions of norms and inner products [17]. Let V be a vector space. A *norm* on V is a positive definite function $\|\cdot\|\colon V \to \mathbb{R}$ such that $\|\alpha x\| = |\alpha|\|x\|$, and $\|x + y\| \leqslant \|x\| + \|y\|$ for all $x, y \in V$ and $\alpha \in \mathbb{R}$. An *inner product* on V is a positive definite, symmetric bilinear function $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$. An inner product *induces* a norm: $\|x\| = \langle x, x\rangle^{1/2}$, but a given norm is in general *not* inducible from an inner product unless it satisfies the parallelogram law.

3.1 Tangent Vector Norms

We consider tangent vector norms, which allow us to assess the *size*, or *length*, of rigid body velocities (or instantaneous displacements). Given a body velocity $\dot{q} \in \mathbb{R}^6$, one might define its norm using the *Euclidean norm* of $\mathbb{R}^6$: $\|\dot{q}\| = (\dot{q}^T\dot{q})^{1/2}$. While widely used, this norm assumes different values as $\dot{q}$ transforms according to (1) with respect to different choices of body frame. Moreover, a length scale, which *cannot* be naturally chosen, is needed to unify the dimensions of the translation and rotation components of $\dot{q}$. The lack of a natural length scale is an additional drawback of the Euclidean approach. In the following we review two tangent vector norms that are free of these undesirable features.

We first discuss a velocity norm that is induced from an inner product. Let $\dot{q}_i = (v_i, \omega_i)$ $(i = 1, 2)$ be body velocities. Define a bilinear function by

$$\langle \dot{q}_1, \dot{q}_2 \rangle = \int_{\mathcal{B}} \nu(r)(\omega_1 \times r + v_1)^T(\omega_2 \times r + v_2)dr, \quad (3)$$

where $\mathcal{B}$ denotes the region of $\mathbb{R}^3$ occupied by $\mathcal{B}$ with respect to $\mathcal{F}_B$, r represents the location of points in $\mathcal{B}$, and $\nu\colon \mathcal{B} \to \mathbb{R}$ is a non-negative scalar-valued function. Note that it is easy to obtain an equivalent expression in terms of hybrid velocities. It can be shown [10] that the bilinear function $\langle \cdot, \cdot \rangle$ is an *inner product*, and satisfies the following *frame-invariance* condition. Suppose that when a new body frame $\bar{\mathcal{F}}_B$ is used, r, $\mathcal{B}$ and $\dot{q}_i$ in Eq. (3) transform to $\bar{r}$, $\bar{\mathcal{B}}$ and $\dot{\bar{q}}_i = (\bar{v}_i, \bar{\omega}_i)$. In addition, the function ν transforms to $\bar{\nu}$ such that $\bar{\nu}(\bar{r}) = \nu(r)$. Then $\langle \dot{q}_1, \dot{q}_2 \rangle = \langle \dot{\bar{q}}_1, \dot{\bar{q}}_2 \rangle$, where $\langle \dot{\bar{q}}_1, \dot{\bar{q}}_2 \rangle$ is computed by replacing the quantities in the right-hand side of Eq. (3) with their counterparts corresponding to $\bar{\mathcal{F}}_B$.

When the function ν is the *mass density* of $\mathcal{B}$, we can readily recognize that $\frac{1}{2}\langle \dot{q}, \dot{q} \rangle$ is $\mathcal{B}$'s *kinetic energy*. However, an alternative interpretation can be given from a purely kinematic point of view. Let ν be chosen such that $\int_{\mathcal{B}} \nu(r)dV = 1$. Then the function ν can be interpreted as a *weighting function* for the contributions of the *velocities of $\mathcal{B}$'s individual points.* Thus, we may view this inner product as a *weighted average* over $\mathcal{B}$'s points, and call it the *weighted-average inner product.* This kinematically motivated interpretation allows the inner product to be used for applications that do *not* involve dynamics. From now on we will use this interpretation and assume that the condition $\int_{\mathcal{B}} \nu(r)dV = 1$ is always satisfied.

It can be shown from (3) that the weighted-average inner product can be computed by

$$\langle \dot{q}_1, \dot{q}_2 \rangle = \dot{q}_1^T M \dot{q}_2, \quad (4)$$

where the *inertia matrix* with respect to ν is partitioned as $M = \begin{pmatrix} M_{11} & M_{12} \\ M_{12}^T & M_{22} \end{pmatrix}$, with $M_{11} = I$, $M_{12} = -\int_{\mathcal{B}} \nu(r)\hat{r}dV$ and $M_{22} = -\int_{\mathcal{B}} \nu(r)\hat{r}^2 dV$. While named after inertia, M is a *kinematic* quantity if ν is a weighting function.

The weighted-average inner product induces the frame-invariant velocity norm

$$\|\dot{q}\|_2 = \langle \dot{q}, \dot{q} \rangle^{\frac{1}{2}} = (\dot{q}^T M \dot{q})^{\frac{1}{2}}, \quad (5)$$

which we call the *velocity 2-norm* because it involves a positive definite quadratic form. From the kinematic interpretation of the weighted-average inner product, this norm gives the *root mean square (RMS)* of the velocities of $\mathcal{B}$'s points with respect to the weighting function ν. For example, let $\nu(r) = \sum_{i=1}^{n} \nu_i \delta(r - r_i)$, where r_i are the coordinates of $\mathcal{B}$'s feature points, δ denotes the Dirac delta function, and $\sum_{i=1}^{k} \nu_i = 1$. Then, the 2-norm gives the RMS of the velocities of $\mathcal{B}$'s feature points.

The 2-norm is induced from the weighted-average inner product. However, *norms are a more basic notion than inner products, and do not always have to be defined from inner products.* We now consider a velocity norm that is not induced from an inner product. Let $\dot{q} = (v, \omega)$ be a body velocity. Then

$$\|\dot{q}\|_\infty = \max_{r \in \mathcal{B}} |v + \omega \times r|, \quad (6)$$

where $|x| = (x^T x)^{1/2}$ for all $x \in \mathbb{R}^3$, is a *frame-invariant* norm [10]. As can be shown, this norm, called the *maximum velocity norm*, is *not* inducible from an inner product. However, it has an attractive physical interpretation: $\|\dot{q}\|_\infty$ gives the *maximal velocity*, or if $\dot{q}$ is used to approximate a displacement, the *maximal displacement*, of $\mathcal{B}$'s points as $\mathcal{B}$ moves at velocity $\dot{q}$.

The computation of the maximum velocity norm is discussed in Ref. [10] for general objects, and is considered below for an object $\mathcal{B}$ whose convex hull is a polyhedron. Let I_V be an index set for the polyhedron's vertices. For a body velocity $\dot{q} = (v, \omega)$, the velocity of a vertex $i \in I_V$ with body coordinates r_i is $u_i = v - \hat{r}_i \omega$. Hence, $|u_i|^2 = \dot{q}^T A_i \dot{q}$ where $A_i = [I_3 \; -\hat{r}_i]^T [I_3 \; -\hat{r}_i]$, and

$$\|\dot{q}\|_\infty^2 = \max_{i \in I_V} \dot{q}^T A_i \dot{q}. \quad (7)$$

In most applications, the displacements of the points in the fixtured object $\mathcal{B}$ are very small. Thus, a displacement of $\mathcal{B}$ can be approximated by a tangent vector $\dot{q}$. The norm $\|\dot{q}\|$ then indicates the size, or length, of the displacement, and measures how far $\mathcal{B}$ is displaced from its original location. Motivated by this observation, we define $\|\dot{q}\|$ as the *deflection* of $\mathcal{B}$ corresponding to the displacement $\dot{q}$. In particular, we call $\|\dot{q}\|_2$ the *RMS-deflection*, and $\|\dot{q}\|_\infty$ the *∞-deflection* of $\mathcal{B}$.

3.2 Covector Norms

Covector norms formalize the notion of *size* or *length* of wrenches. Similar to the case of velocities, wrench norms defined using the Euclidean norm of $\mathbb{R}^6$ is frame-dependent and involves unnatural comparison of torques with forces. We now present a frame-invariant wrench norm that is induced from an inner product. Other frame-invariant wrench norms are discussed in Ref. [10].

A wrench inner product can be defined using the weighted-average velocity inner product $\langle \cdot, \cdot \rangle$. Every covector $\boldsymbol{w}$ corresponds to a *unique* tangent vector $\dot{q}_w$ such that $\boldsymbol{w}(\dot{q}) = \langle \dot{q}_w, \dot{q} \rangle$ for all tangent vectors $\dot{q}$ [17]. In body coordinates, we have $\dot{q}_w = M^{-1}\boldsymbol{w}$. Thus, the weighted-average velocity inner product formula (4) leads to the following *frame-invariant wrench inner product*:

$$\langle\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle\rangle = \langle M^{-1}\boldsymbol{w}_1, M^{-1}\boldsymbol{w}_2 \rangle = \boldsymbol{w}_1^T M^{-1} \boldsymbol{w}_2, \quad (8)$$

where $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ are body wrenches. This inner product induces a *frame-invariant norm*, called the wrench 2-norm, given by

$$\|\boldsymbol{w}\|_2 = \langle\langle \boldsymbol{w}, \boldsymbol{w} \rangle\rangle^{1/2} = (\boldsymbol{w}^T M^{-1} \boldsymbol{w})^{1/2}. \quad (9)$$

As one may expect, the wrench 2-norm is closely related to the velocity 2-norm. Indeed, it can be shown [10] that

$$\|w\|_2 = \sup\{\frac{|w^T\dot{q}|}{\|\dot{q}\|_2} : \dot{q} \in \mathbb{R}^6\}.$$

The 2-norm of a body wrench $w = (f, \tau)$ acting on $\mathcal{B}$ has the following physical interpretation. Imagine that w is *generated* by a system of *distributed pure forces*, denoted $f(r)$ where $r \in \mathcal{B}$, with respect to the given weighting function ν. That is, $f = \int_{\mathcal{B}} \nu(r) f(r) dV$ and $\tau = \int_{\mathcal{B}} \nu(r)\, r \times f(r) dV$. Denote by $\mathcal{D}(w)$ the set of such force distributions such that for each f in $\mathcal{D}(w)$, the integral $\int_{\mathcal{B}} \nu(r)\, |f(r)|^2\, dV$ is finite. Then [10]

$$\|w\|_2 = \inf\Big\{\big(\int_{\mathcal{B}} \nu(r)\, |f(r)|^2\, dV\big)^{\frac{1}{2}} : f \in \mathcal{D}(w)\Big\}.$$

Therefore, $\|w\|_2$ is, with respect to the weighting function ν, the *greatest lower bound for the root mean square of the magnitudes of distributed forces that generate* w.

4 A Deflection-Based Quality Measure

Based on frame-invariant velocity and wrench norms as well as the notion of object deflection, this section presents a frame-invariant fixture quality measure that characterizes the worst-case deflection of the object in response to external disturbances. We focus on *stable fixtures with positive definite stiffness matrices*, since other fixtures are considered ineffective.

In practical applications, a fixture is often considered effective if the displacement of the fixtured object due to external disturbances is small. Since $\mathcal{B}$'s displacement due to a wrench w is approximately given by the velocity $\dot{q} = Cw$, where $C = K^{-1}$ is the *compliance matrix* of the fixture, the displacement scales linearly with the applied wrench. Thus, we can quantify this effectiveness requirement by defining the following quality measure:

$$Q_w = \sup\{\|Cw\| : w \in \mathbb{R}^6,\ \|w\| \leqslant 1\}. \qquad (10)$$

For the fixture to be effective, Q_w is desired to be small. This quality measure characterizes the *worst-case deflection of the object under the action of wrenches lying in the unit wrench ball*, and will hence be called the *worst-case deflection quality measure*. Recall that the unit wrench ball is the set of wrenches whose norms are less than or equal to unity. The notions of deflection and wrench balls depend on the choices of velocity and wrench norms, and the unit wrench ball is in general *not* a Euclidean sphere in $\mathbb{R}^6$. Provided frame-invariant norms are used, Q_w is *frame-invariant*.

The worst-case deflection quality measure can also be defined over the unit displacement ball. Given a displacement $\dot{q}$ of $\mathcal{B}$, the fixels apply to $\mathcal{B}$ a restoring wrench, $w = K\dot{q}$. We define a quality measure by

$$Q_{\dot{q}} = \inf\{\|K\dot{q}\| : \dot{q} \in \mathbb{R}^6,\ \|\dot{q}\| \leqslant 1\}. \qquad (11)$$

For a fixture to be effective, the value of $Q_{\dot{q}}$ is preferred to be large. Since $C = K^{-1}$, we can show that $Q_{\dot{q}} = 1/Q_w$. Thus, Q_w defined in (10) also characterizes the *worst-case magnitude of the restoring wrench corresponding to all displacements lying in the unit displacement ball.* Note again that the unit displacement ball, consisting of displacements with associated deflections bounded by unity, is in general *not* a Euclidean sphere in $\mathbb{R}^6$.

In practice, one first computes the stiffness matrix K, and then obtains the compliance matrix C by inverting K. Therefore, the quality measure $Q_{\dot{q}}$ is more convenient. We will hereafter focus on $Q_{\dot{q}}$, and refer to it as the worst-case deflection quality measure as well.

Now let us compute the worst-case deflection quality measure with respect to different velocity and wrench norms. We first consider the computation with the velocity and wrench 2-norms, in which case $Q_{\dot{q}}$ characterizes the worst-case RMS-deflection of $\mathcal{B}$ over the 2-norm unit wrench ball. Denote the smallest eigenvalue of a symmetric matrix A by $\lambda_{\min}(A)$. From (5) and (9) we have $\|\dot{q}\|_2 = \dot{q}^T M \dot{q}$, and $\|K\dot{q}\|_2 = \dot{q}^T K M^{-1} K \dot{q}$. Letting $x = M^{1/2}\dot{q}$, we can rewrite (11) as

$$Q_{\dot{q}}^2 = \inf_{\dot{q} \in \mathbb{R}^6} \frac{(K\dot{q})^T M^{-1}(K\dot{q})}{\dot{q}^T M \dot{q}} = \inf_{x \in \mathbb{R}^6} \frac{x^T \widetilde{K}^2 x}{x^T x},$$

where $\widetilde{K} = M^{-1/2} K M^{-1/2}$ is called the *scaled stiffness matrix*. Thus, for the velocity and wrench 2-norms,

$$Q_{\dot{q}} = \lambda_{\min}(\widetilde{K}). \qquad (12)$$

That is, the quality measure is given by the *smallest eigenvalue of the scaled stiffness matrix* $\widetilde{K}$.

We next compute $Q_{\dot{q}}$ with respect to the maximum velocity norm and wrench 2-norm. The use of the maximum velocity norm allows the quality measure to indicate $\mathcal{B}$'s worst-case ∞-deflection, which is just the maximal displacement of $\mathcal{B}$'s body points. While computing $Q_{\dot{q}}$ with the maximum velocity norm is complicated for general objects, the formula (7) allows efficient computation for objects with polyhedral convex hulls. Using this formula and the wrench 2-norm formula (9) we have

$$Q_{\dot{q}}^2 = \inf_{\dot{q} \in \mathbb{R}^6} \frac{\dot{q}^T K M^{-1} K \dot{q}}{\max_{i \in I_V} \dot{q}^T A_i \dot{q}} = \min_{i \in I_V} \inf_{\dot{q} \in \mathbb{R}^6} \frac{\dot{q}^T K M^{-1} K \dot{q}}{\dot{q}^T A_i \dot{q}}.$$

Introducing the change of variables $y = M^{-1/2} K \dot{q}$ yields

$$Q_{\dot{q}} = \big(\max_{i \in I_V} \lambda_{\max}(M^{\frac{1}{2}} K^{-1} A_i K^{-1} M^{\frac{1}{2}})\big)^{-\frac{1}{2}}.$$

Thus, $Q_{\dot{q}}$ can be efficiently computed from a collection of eigenvalue problems for 6×6 symmetric matrices.

5 Planar Minimum-Deflection Fixtures

To illustrate the utility of the worst-case deflection quality measure, this section considers the minimum-deflection fixturing of polygonal objects by 3 and 4 fixels. We first consider some general properties of planar compliant fixtures.

5.1 Planar Compliant Fixtures

In a planar fixture, the object is restricted to move in a plane. The stiffness matrix K reduces to a 3×3 matrix, and can be partitioned in the form $K = \begin{pmatrix} K_{11} & K_{12} \\ K_{12}^T & K_{22} \end{pmatrix}$, with K_{11}, K_{12} and K_{22} having dimensions 2×2, 2×1 and 1×1, respectively. Using the transformation rule (2), one can show that there is a *unique* location of $\bar{\mathcal{F}}_B$'s origin, with coordinates $p_c = JK_{11}^{-1}K_{12}$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, such that the stiffness matrix becomes *block-diagonal*. That is, $\bar{K} = \mathrm{diag}(R^T K_{11} R, \mu)$, where R is the orientation of $\bar{\mathcal{F}}_B$ relative to $\mathcal{F}_B$, and $\mu = K_{22} - K_{12}^T K_{11}^{-1} K_{12}$. This point, about which the object's translational and rotational stiffnesses are *decoupled*, is called the fixture's *center of compliance*. The orientation matrix R can be chosen such that $\bar{K}$ becomes *diagonal*: $\bar{K} = \mathrm{diag}(\sigma_1, \sigma_2, \mu)$, where σ_i are the eigenvalues of K_{11}. It can be shown (e.g. [13]) that the stiffness parameters σ_i and μ are in fact *frame-invariant*.

We focus on the worst-case deflection quality measure using the velocity and wrench 2-norms. The quality measure then gives $\mathcal{B}$'s worst-case RMS-deflection due to a unit 2-norm ball of wrenches, and can be computed by $Q_{\dot{q}} = \lambda_{\min}(\widetilde{K})$, where $\widetilde{K} = M^{-1/2} K M^{-1/2}$ is the scaled stiffness matrix. Given any weighting function $\nu(r)$ for a planar object, there exists a unique point, called the *centroid* of the object, such that when the body frame is based at this point, the 3×3 inertia matrix is diagonal: $M = \mathrm{diag}(1, 1, \rho_c^2)$, where $\rho_c = (\int_{\mathcal{B}} \nu(r) \, |r|^2 \, dV)^{1/2}$, called $\mathcal{B}$'s *radius of gyration*, is a purely kinematic quantity.

Consider a body frame $\mathcal{F}_B$ whose origin is at $\mathcal{B}$'s centroid. Let $p_c = (\xi, \eta)$ be the coordinates in $\mathcal{F}_B$ of the fixture's center of compliance. With a proper choice of $\mathcal{F}_B$'s orientation, the scaled stiffness matrix can be cast in the following form [10]:

$$\widetilde{K} = \begin{pmatrix} \sigma_1 & 0 & -\sigma_1 \widetilde{\eta} \\ 0 & \sigma_2 & \sigma_2 \widetilde{\xi} \\ -\sigma_1 \widetilde{\eta} & \sigma_2 \widetilde{\xi} & \widetilde{\mu}^2 + \sigma_2 \widetilde{\xi}^2 + \sigma_1 \widetilde{\eta}^2 \end{pmatrix}, \qquad (13)$$

where $\widetilde{\xi} = \xi/\rho_c$, $\widetilde{\eta} = \eta/\rho_c$, and $\widetilde{\mu} = \mu/\rho_c^2$. This formulation will be used subsequently for optimal 3- and 4-fixel fixtures of polygonal objects.

In the remainder of this paper, we assume frictionless contacts and invoke Ref. [12] to compute stiffness matrices. However, it is important to note that this is for illustration purposes only. The worst-case deflection quality measure also applies to other compliance models, which may include friction effects.

5.2 Optimal Three-Fixel Fixtures

For a fixture of a polygonal object by 3 fixels to be in equilibrium, the contact normals must be *concurrent* (i.e., intersect at a common point) and positively span the plane. It can be shown that such an equilibrium fixture is also stable [12]. Since each fixel must be placed on a different edge, we can consider all triplets of edges.

Given a fixture associated with an edge triplet, the stiffness matrix takes the following simple form in a body frame $\bar{\mathcal{F}}_B$ whose origin is at the concurrency point of the contact normals [12]: $\bar{K}_{11} = \sum_{i=1}^3 k_i n_i n_i^T$, $\bar{K}_{12} = 0$, and $\bar{K}_{22} = \mu = 2 f_T a \gamma$. In these formulas n_i are the unit inward contact normals, k_i are contact stiffness constants, f_T is the *total preloading fixel force* defined as the sum of the individual preloading fixel forces, a is the radius of the circumscribing circle of the triangle formed by the edges in the triplet, and $\gamma = (\prod_{i=1}^3 \sin \alpha_i)/(\sum_{i=1}^3 \sin \alpha_i)$ where α_i are the triangle's three interior angles (See Fig. 1). Since the stiffness matrix is block-diagonal in $\bar{\mathcal{F}}_B$, the fixture's center of compliance coincides with the concurrency point. For all concurrent fixel arrangements on the given edge triplet, the parameters σ_i, arranged such that $\sigma_1 \leqslant \sigma_2$, are *constant* since the contact normals are constant. The parameter μ is also *constant* for the edge triplet when the total preload f_T is specified.

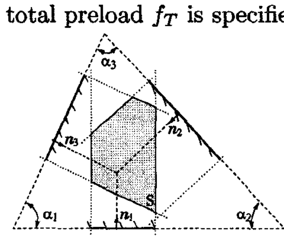

Fig. 1. An edge triplet

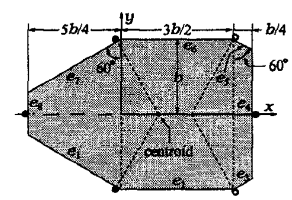

Fig. 2. 3-fixel octagon fixtures

Now let $\mathcal{F}_B$ be a body frame whose origin is at $\mathcal{B}$'s centroid. Given an edge triplet, since the contact normals have constant directions, the scaled stiffness matrix $\widetilde{K}$ of all fixtures on the edge triplet can be written as (13) in the *same* properly oriented frame $\mathcal{F}_B$. Moreover, we show in Ref. [10] that $\widetilde{\mu} = \mu/\rho_c^2 \ll \sigma_i$ for practical fixtures. This allows us to view $\widetilde{\mu}^2$ in the $(3,3)$ entry of $\widetilde{K}$ as a small perturbation. The quality measure, approximated as a perturbed eigenvalue, can then be computed as follows.

Lemma 5.1 ([10]). *For 3-fixel equilibrium fixtures, the worst-case deflection quality measure with respect to the velocity and wrench 2-norms is approximately given by* $Q_{\dot{q}} = \mu/(\rho_c^2 + \rho^2)$, *provided that* $\mu \ll \frac{1}{2}\sigma_1(\rho_c^2 + \rho^2)$, *where* $\rho = (\xi^2 + \eta^2)^{1/2}$ *is the distance between the concurrency point and* $\mathcal{B}$*'s centroid.*

We now consider optimal 3-fixel fixtures. For a given polygonal object, we assume that the optimal fixel placement is sought with respect to a specified value of f_T, the total preloading force. For a triplet of edges whose inward normals positively span $\mathbb{R}^2$, the set of stable equilibrium fixtures can be identified as follows. As shown in Fig. 1, construct three strips whose bounding lines are perpendicular to an edge and pass through the edge's endpoints. Denote by S the intersection of these three strips. For each point in S, there exists a fixel placement such that the contact normals intersect at this point. Therefore, the collection of stable equilibrium fixtures is parametrized by the location of the concurrency point, which belongs to the region S.

Note that for a given edge triplet, of all the terms appearing in Lemma 5.1, only ρ changes as the contact points vary along the edges. Thus, $Q_{\dot{q}}$ is maximized for the edge triplet when ρ, the distance between the concurrency point and $\mathcal{B}$'s centroid, is minimized. We can hence focus on minimizing $\rho(r)^2$ for all $r \in S$. Therefore, in the optimal fixel arrangement, the concurrency point of the contact normals is as close to the object's centroid as possible. It follows that the optimal fixel arrangement can be identified graphically. First, find the region S as shown in Fig. 1. Second, find the centroid of the object with respect to a given weighting function $\nu(r)$. If the centroid lies in S, then it is the optimal concurrency point. Otherwise the centroid lies outside S. Since S is a polygonal region, we can efficiently compute the closest point in S to the centroid. This point either lies on an edge of S or is a vertex of S.

Example 5.1. Consider the optimal fixturing of the octagon shown in Fig. 2 by three identical fixels. Assume that the total preload f_T is specified and that $k_i = 1$ for all contacts without loss of generality. Choosing a weighting function $\nu(r) = (1/8)\delta(r - r_j)$ where r_j are $\mathcal{B}$'s vertices, we find $\mathcal{B}$'s centroid at $(b/2, 0)$ and $\mathcal{B}$'s radius of gyration $\rho_c = 1.4735\,b$. Now consider two edge triplets, (e_1, e_4, e_7) and (e_3, e_5, e_8). These triplets correspond to equilateral triangles of the same size, and hence have the same stiffnesses parameters: $\sigma_1 = 3/2$ and $\mu = f_T b/\sqrt{3}$. For each triplet, we wish to place the fixels such that the concurrency point of the contact normals is as close to the centroid as possible. For (e_1, e_4, e_7), the optimal concurrency point coincides with the centroid, and the corresponding quality measure value is given by $Q_{\dot{q}} = 0.2659 f_T/b$. For (e_3, e_5, e_8), it is impossible to place the concurrency point at the centroid. The closest location (see Fig. 2) is at a distance $0.4226\,b$ from the centroid and determines the optimal fixture for this triplet, with a quality measure value $Q_{\dot{q}} = 0.2457 f_T/b$.

5.3 Optimal Four-Fixel Fixtures

To find the optimal fixture of a polygonal object $\mathcal{B}$ by 4 fixels, we can consider all combinations of three and four edges of $\mathcal{B}$. For a given edge combination, fixel arrangements can be parametrized by $s = (s_1, s_2, s_3, s_4)$, where s_i is the distance from a fixel to a reference point along the relevant edge. Then Ref. [12] shows that the set of all values of s that determine stable equilibrium fixtures for the edge combination can be written as the union $S_1 \cup S_2$, where S_j are *bounded convex polytopes.* We hence can seek the optimal fixture separately in each of the simple sets S_j.

According to Ref. [12], the stiffness matrix of a stable fixture $s \in S_j$ on an edge combination is given, with respect to a body frame $\mathcal{F}_B$, by $K_{11} = \sum_{i=1}^4 k_i n_i n_i^T$, $K_{12} = \sum_{i=1}^4 k_i z_i(s) n_i$ and $K_{22} = \sum_{i=1}^4 k_i z_i(s)^2$, where k_i are contact stiffness constants, n_i are inward unit contact normals, and $z_i(s)$ are the moments of n_i with respect to the body frame's origin. Similar to 3-fixel fixtures, the fact that n_i have constant directions implies that the scaled stiffness matrix of any stable fixture on the edge combination can be cast in the form (13) in the *same* frame $\mathcal{F}_B$, provided $\mathcal{F}_B$ is based at $\mathcal{B}$'s centroid and oriented properly. The same fact also indicates that σ_i ($\sigma_1 \leqslant \sigma_2$), the eigenvalues of K_{11}, are *constant.* The parameter μ and $p_c = (\xi, \eta)$, the fixture's center of compliance, depend on s. From Section 5.1, it can be verified that $\mu(s) = \sum_{i=1}^4 k_i t_i(s)^2$, where $t_i(s) = z_i(s) + n_i^T J p_c(s)$ is the moment of n_i with respect to the center of compliance.

In Ref. [10] we show that measure, given by $Q_{\dot{q}}(s) = \lambda_{\min}(\widetilde{K}(s))$, satisfies

$$Q_{\dot{q}}(s) \leqslant \min\{\sigma_1, \frac{\mu(s)}{\rho_c^2 + \rho(s)^2}\}, \tag{14}$$

where ρ_c is $\mathcal{B}$'s radius of gyration and $\rho = (\xi^2 + \eta^2)^{1/2}$ the distance between the fixture's center of compliance and $\mathcal{B}$'s centroid. This upper bound allows the following qualitative observations on the optimal fixture associated with a given edge combination. First, for the edge combination, $Q_{\dot{q}}$ can *never* exceed the constant σ_1. Since σ_1 is the smaller eigenvalue of $K_{11} = \sum_{i=1}^4 k_i n_i n_i^T$, the contact normals are preferred to be *evenly oriented* to increase σ_1. In particular, if the stiffness constants are uniform, i.e., $k_i = k$, then $\sigma_1 \leqslant 2k$, and $\sigma_1 = 2k$ precisely when the contact normals are 90° apart. Second, the parameter μ, and hence the bound (14) increase monotonically with $|t_i|$, the moments of the contact normals about the center of compliance. This indicates that the fixels should *spread apart* with respect to the center of compliance, so as to make $|t_i|$ large. Finally, as ρ, the distance between the fixture's center of compliance and $\mathcal{B}$'s centroid, increases, the upper bound decreases monotonically. Thus, ρ should necessarily be as *small* as possible, and most desirably, should be zero. The optimal fixture is therefore determined by the trade-off among these three factors.

The problem of finding the optimal 4-fixel fixture of a polygon involves maximizing the smallest eigenvalue of the scaled stiffness matrix $\widetilde{K}$ over the convex polytopes S_i associated with each edge combination. A procedure presented in Ref. [10] solves this problem by seeking the zero of a scalar function, whose evaluation is an *indefinite quadratic program.* A notable merit of this procedure is that it guarantees to find the *globally* optimal fixture. While indefinite quadratic programming is difficult for problems of large size, it presents no major computational difficulty for our optimal fixturing applications since there are only 4 independent variables.

We now present two examples of optimal 4-fixel fixtures. For simplicity, we assume that the fixels are identical, with unit stiffness constants.

Example 5.2. Let a rectangle be fixtured by 4 fixels. It can be shown that regardless of the rectangle's shape and dimension, the globally optimal fixel arrangements always place the fixels at the edges' endpoints which belong to a pair of the rectangle's diagonally opposite corners [10]. We observe that in these fixtures, while the contact normals are evenly oriented because of the object's special shape, the fixels indeed spread apart to the greatest extent.

Example 5.3. Consider the quadrilateral in Fig. 3. The coordinates of the object's vertices are $(0,0)$, $(b,0)$, $(0.7b, 0.6b)$, and $(0.15b, 0.45b)$. We choose a weighting function $\nu(r) = (1/4)\delta(r - r_j)$ where r_j are the object's vertices. Using the procedure that is based on indefinite quadratic programming and described in Ref. [10], we numerically found the optimal fixel arrangement as shown. In this fixture, the two fixels on the edge AB are located at the edge's endpoints. The fixel on edge AC is located at a distance $0.824|AC|$ from vertex A, while the fixel on edge BD is at a distance $0.6943|BD|$ from vertex B. The optimal quality measure value is $Q_{\dot{q}} = 1.6838$. Since $\sigma_1 = 1.6838$ and $\sigma_2 = 2.3162$, it follows that $Q_{\dot{q}}$ achieves the upper bound in (14) for the optimal fixture.

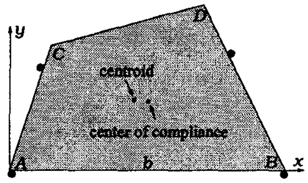

Fig. 3. 4-fixel optimal fixturing of a quadrilateral

6 Conclusion

We addressed the practically important problem of planning minimum-deflection grasps and fixtures. The approach was based on a quality measure that characterizes the grasped or fixtured object's worst-case deflection due to disturbing wrenches lying in the unit wrench ball. In developing the quality measure, frame-invariant norms of rigid body velocities and wrenches were for the first time employed. By considering minimum-deflection fixturing of polygonal objects by three and four fixels, we demonstrated that our approach can be effectively applied to planning grasps and fixtures where deflection significantly influences performance.

Acknowledgement: This work was supported by the National Science Foundation under grant number IRI-9503835.

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