Invalidity of the Standard Locality Condition for Hidden-Variable Models of Bohm-EPR Experiments

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Abstract John Bell and others used a locality condition to establish inequalities that they believe must be satisfied by any local hidden-variable model for the spin probability distribution for two entangled particles in a Bohm-EPR experiment. We show that this condition is invalid because it contradicts the product rule of probability theory for any model that exhibits the expected property of perfect correlation. This breaks the connection between Bell inequalities and the existence of any local hidden-variable model of interest. As already known, these inequalities give necessary conditions for the existence of third/fourth-order joint probability distributions for the spin outcomes from three/four separate Bohm-EPR experimental set-ups that are consistent with the second-order joint spin distributions for each experiment after marginalization. If a Bell inequality is violated, as quantum mechanics theory predicts and experiments show can happen, then at least one third-order joint probability is negative. However, this does not imply anything about the existence of local hidden-variable models for the second-order joint probability distributions for the spin outcomes of a *single* experiment. The locality condition does seem reasonable under the widely-applied frequentist interpretation of the spin probability distributions that views them as real properties of a random process that are manifested through their relative frequency of occurrence, which gives conditioning in the probabilities for the spin outcomes a causal role. In contrast, under the Bayesian interpretation of probability, probabilistic conditioning on one particle's spin outcome in the product rule is viewed as information to make probabilistic predictions of the other particle's spin outcome. There is nothing causal and so no reason to develop a locality condition.

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1 INTRODUCTION

The standard locality condition is believed to be a requirement for any hidden-variable model that explains the probability distribution of spin outcomes from a pair of entangled particles in the singlet state that enter separate SG (Stern-Gerlach) devices in a Bohm-EPR experiment. This condition, often called factorizability or factorability, was first stated by Clauser and Horne [1974] and then supported by Bell [1976, 1981] as a probabilistic form of his deterministic locality condition (Bell [1964]). It states that the spin outcomes for the two particles are probabilistically independent conditional on the hidden variables and it plays a major role in studies of quantum nonlocality and quantum entanglement (Brunner et al [2014] provide a review). We show that this locality condition is invalid under the requirement that the hidden-variable model must exhibit the expected perfect correlation for the spin outcomes of two aligned SG devices, which is a consequence of the total spin being zero in the singlet state, because the condition is inconsistent with the product rule of probability theory. It fails to preserve cogent information coming from the correct probabilistic conditioning. The derivation of the Bell inequalities based on the standard locality condition for a hidden variable model behind the spin outcomes is therefore invalid.

As found by others (Suppes and Zanotti [1981], Fine [1982a,b]) and shown in Appendix A using a simple proof that is independent of Bell's locality condition, Bell's original inequality (Bell [1964]) involving three covariances of pairs of spin variables does have an important role unrelated to local hidden-variable models: it provides a necessary condition for the existence of a third-order joint probability distribution for three spin variables that is compatible through marginalization with the second-order joint distributions for the three possible spin pairs. An explicit expression is presented for this third-order distribution. Bell's original inequality is also shown to be logically equivalent to the 1969 CHSH Bell inequality (Clauser et al [1969]) that involves four covariances of pairs of four spin variables. This inequality must be satisfied in order for a fourth-order joint probability distribution for the four spin variables to exist when four of their second-order joint distributions are specified. When any Bell inequality is violated, a third-order joint probability distribution for three spin outcomes fails to be valid because some of its probabilities are negative. This does not imply, however, that there cannot be a local hidden-variable model for the probability distribution from QM theory for the two entangled spin outcomes in a *single* Bohm-EPR experiment.

In the original justifications for the locality condition (Clauser and Horne [1974], Bell [1976, 1981]), a commonly-applied interpretation of probability in QM plays a crucial role where probability distributions are viewed as real properties of inherently random events that are exhibited through their relative frequency of occurrence. In statistics, this is known as the frequentist interpretation. With this perspective, the standard locality condition reflects the plausible requirement that the probability of one particle's spin measurement does not depend on the other particle's spin measurement when conditioned on the hidden variables for the two particles. However, under the alternative Bayesian interpretation of probability as a multi-valued logic for quantitative plausible reasoning (Cox [1946], Jaynes [1983, 2003], Beck [2010]), this argument loses its force. Probabilistic conditioning on a spin outcome of one particle in the product rule is viewed simply as information to make probabilistic predictions of the spin outcome of the other particle.

The Bayesian point of view for probability in QM has been advocated in recent years because it leads to less puzzling interpretations of QM theory than the frequentist perspective for quantum entanglement and quantum nonlocality. For understanding QM, Jaynes [1990a], Grandy [2009], Goyal and Knuth [2011] and Beck [2018], for example, prefer the Cox-Jaynes approach to Bayesian probability as a logic, while Caves et al [2002], Fuchs [2003], Pitowsky [2003] and Fuchs et al [2014], for example, prefer the alternative Ramsey-DeFinetti-Savage approach to subjective Bayesian probability (e.g. Fishburn [1986]), which is based on betting odds and is often referred to in the QM literature as Quantum Bayesianism or QBism.

In Section 2, the Bohm-EPR experimental setup is briefly presented. Then in Section 3, the standard locality condition for a hidden-variable model to explain the entangled spin probability distribution in Bohm-EPR experiments is investigated and its special factorizing of the joint probability of the spin outcomes is shown to violate the product rule of probability theory. Although this invalidates the derivation of the Bell inequalities as a necessary condition for a local hidden-variable model, it is noted in Section 3 that they do have a role in necessary and sufficient conditions for the existence of joint spin distributions across three or four Bohm-EPR experiments. Section 4 gives concluding remarks.

2 PROBABILITY DISTRIBUTION FOR SINGLET STATE OF TWO ENTANGLED PARTICLE SPINS

Consider the experiment that was proposed by Bohm [1951] (pp. 614-619) as an alternative way to frame the argument put forth by Einstein et al [1935] that QM as it stands is an incomplete theory: there is a source that generates a particle that has a net spin of zero but it immediately splits into two spin-$\frac{1}{2}$ charged particles, labeled A and B, that freely move apart in opposite directions with spins in opposite directions due to their "entanglement" by the creation process. They each eventually enter a SG device, with corresponding labels A and B, whose longitudinal axes lie along the line of motion of the particles. These devices can be rotated about their axes but their orientation is assumed known for each test. The "up" direction of devices A and B are denoted by unit directional vectors $\mathbf{a}$ and $\mathbf{b}$, respectively, and they lie in a plane orthogonal to the longitudinal axes of the devices. The spins are rotated by the magnetic field in each SG device. Each spin outcome is unpredictable but binary: the particle is either deflected up or down. The corresponding outcomes are denoted by the binary variables $A(\mathbf{a}) = 1$ (*spin up*) or -1 (*spin down*), and $B(\mathbf{b}) = 1$ or -1, that is, "$A(\mathbf{a}) = 1$", for example, denotes the proposition, or event, that for particle A entering SG device A oriented in direction $\mathbf{a}$, the spin outcome is up.

QM theory, based on applying Born's rule to a spin wave function Ψ_{ss} for the *singlet state*, provides a joint probability distribution over the four possible pairs of spin values given by $(A(\mathbf{a}), B(\mathbf{b})) = (1,1),(1,-1),(-1,1)$ and $(-1,-1)$ (Sakurai [2011]):

$$\begin{aligned} P[A(\mathbf{a}), B(\mathbf{b})] &= \frac{1}{4}\left[1 - A(\mathbf{a})B(\mathbf{b})\mathbf{a}\cdot\mathbf{b}\right] \\ &= \frac{1}{2}\sin^2(\theta_{ab}/2)\,,\ \text{if } A(\mathbf{a}) = B(\mathbf{b}) \\ &= \frac{1}{2}\cos^2(\theta_{ab}/2)\,,\ \text{if } A(\mathbf{a}) = -B(\mathbf{b}) \qquad (1) \end{aligned}$$

where θ_{ab} is the angle between the unit vectors $\mathbf{a}$ and $\mathbf{b}$. Although $\mathbb{P}[\ldots]$ will be used to denote the probability of an event or proposition, it is convenient to use a shorter notation for probability distributions where $P[A(\mathbf{a}), B(\mathbf{b})]$

in (1) denotes a probability function that gives the probability that $(A(\mathbf{a}), B(\mathbf{b}))$ equals a binary pair from the set $\{-1, 1\} \times \{-1, 1\}$; for example,
$P[\alpha, \beta] = \mathbb{P}[A(\mathbf{a}) = \alpha \text{ and } B(\mathbf{b}) = \beta \,|\mathbf{a}, \mathbf{b}, \Psi_{ss}]$
where the conditioning indicates that $\mathbf{a}$ and $\mathbf{b}$ are assumed given and the two particles are in the singlet state defined by the wave function Ψ_{ss}. This conditioning is left as understood in the shorter notation. Similarly, $P[A(\mathbf{a})\,|B(\mathbf{b})]$ denotes a conditional probability where
$P[\alpha\,|\beta] = \mathbb{P}[A(\mathbf{a}) = \alpha\,|B(\mathbf{b}) = \beta, \mathbf{a}, \mathbf{b}, \Psi_{ss}]$
when the appropriate values (α, β) of the two spin variables are substituted for $(A(\mathbf{a}), B(\mathbf{b}))$.

The marginal and conditional distributions corresponding to (1) may then be deduced for $A(\mathbf{a})$ and $B(\mathbf{b})$ (and, conversely, taking their product implies the joint distribution in (1)):

$$P[A(\mathbf{a})] = \sum_{B(\mathbf{b})=-1}^{+1} P[A(\mathbf{a}), B(\mathbf{b})] = 1/2,$$

$$P[B(\mathbf{b})] = \sum_{A(\mathbf{a})=-1}^{+1} P[A(\mathbf{a}), B(\mathbf{b})] = 1/2 \tag{2}$$

$$\begin{aligned} P[A(\mathbf{a})\,|B(\mathbf{b})] &= P[B(\mathbf{b})\,|A(\mathbf{a})] \\ &= P[A(\mathbf{a}), B(\mathbf{b})]/P[A(\mathbf{a})] \\ &= \frac{1}{2}\,(1 - A(\mathbf{a})B(\mathbf{b})\mathbf{a}\cdot\mathbf{b}) \\ &= \sin^2(\theta_{ab}/2)\,,\ \text{if } A(\mathbf{a}) = B(\mathbf{b}) \\ &= \cos^2(\theta_{ab}/2)\,,\ \text{if } A(\mathbf{a}) = -B(\mathbf{b}) \end{aligned} \tag{3}$$

We note that when $\mathbf{b} = \mathbf{a}$ so that $\theta_{ab} = 0$, then $A(\mathbf{a}) = -B(\mathbf{a})$ for sure. Similarly, when $\mathbf{b} = -\mathbf{a}$ so that $\theta_{ab} = \pi$, then $A(\mathbf{a}) = B(-\mathbf{a})$ for sure. The spin outcomes of positively ($\mathbf{b} = \mathbf{a}$) and negatively ($\mathbf{b} = -\mathbf{a}$) aligned SG devices are therefore predicted with certainty to be the opposite ($A(\mathbf{a}) = -B(\mathbf{a})$) and the same ($A(\mathbf{a}) = B(-\mathbf{a})$), respectively. We refer to these two properties as *perfect correlation*. They are expected, irrespective of QM theory, because two spin-$\frac{1}{2}$ particles in the singlet state have a total spin of zero before entering the SG devices, which either have their orientations the same so their spin outcomes should be the opposite, or have their orientations the opposite so their spin outcomes should be the same.

The conditional probabilities in (3) directly imply a correlation between the outcomes at the two SG devices, which is also exhibited by the result that the covariance is given by:

$$\begin{aligned} \langle A(\mathbf{a})B(\mathbf{b})\rangle &\triangleq \mathbb{E}[A(\mathbf{a})B(\mathbf{b})] \\ &\triangleq \sum_{A(\mathbf{a})=-1}^{+1}\ \sum_{B(\mathbf{b})=-1}^{+1} A(\mathbf{a})B(\mathbf{b})P[A(\mathbf{a}), B(\mathbf{b})] \\ &= \sin^2(\theta_{ab}/2) - \cos^2(\theta_{ab}) \\ &= -\cos\theta_{ab} = -\mathbf{a}\cdot\mathbf{b} \end{aligned} \tag{4}$$

and so does not factorize into $\langle A(\mathbf{a})\rangle\langle B(\mathbf{b})\rangle (= 0)$. The underlying reasons for this correlation between the spin outcomes have been widely discussed, often under the topics of quantum non-locality, quantum entanglement, Bell inequalities and hidden variables. It is noted that John Bell made numerous contributions to these topics that are collected together in Bell [1987].

As discussed in Beck [2018], the indeterminacy in the spin outcomes and the meaning of their correlation depends critically on whether probability is interpreted using a frequentist or a Bayesian perspective. Under the frequentist interpretation, the indeterminacy is assumed to be due to *inherent randomness* and the correlation implied by (3) is a real effect because probabilistic conditioning on one particle's spin measurement is viewed as corresponding to a causal influence on the other particle's spin. Under the Bayesian interpretation, the indeterminacy is assumed to be due to *missing information* and the correlation is because the conditioning provides information for predicting the other particle's spin outcome. There is no implication that a measurement on one particle has any effect on the spin of the other particle.

3 HIDDEN VARIABLES, LOCALITY AND BELL INEQUALITIES

The question of whether a hidden-variable model can explain the joint probability distribution in (1) for the pair of spin outcomes has a long history in which a prominent role is played by Bell inequalities, so named because the first such inequality was published by Bell [1964]. These inequalities are usually expressed in terms of covariances on three or four pairs of spin outcomes for Bohm-EPR experiments. They were originally derived under the assumption of hidden variables behind the indeterminacy of the spin outcomes in these experiments where the uncertainty in the values of the hidden variables is represented by a probability distribution. A specific *locality condition* is critical to these derivations. The inequalities are not satisfied by the covariances from QM that are given in (4) under some choices of the orientations of the two SG devices. The experimental evidence based on using sample covariances from multiple experiments to approximate the theoretical covariances in the Bell inequalities is usually taken as implying that the inequalities are violated. This has produced a widely-held belief that the QM distribution in (1) cannot be explained by a local hidden-variable model, so we briefly review this argument.

3.1 Locality Condition and Covariance Inequalities

Bell [1976, 1981] assumed that if local hidden variables (that is, properties associated with the particles) can explain the correlations exhibited in (1), then the following probabilistic *locality condition* (also called the *factorizability* or *factorability condition*) must apply:

For any $\boldsymbol{\lambda}$ in a set $\boldsymbol{\Lambda}$ of possible hidden variables and for all cases with $A(\mathbf{a}), B(\mathbf{b}) \in \{+1, -1\}$,

$$P[A(\mathbf{a}), B(\mathbf{b}) \,|\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] = P[A(\mathbf{a}) \,|\boldsymbol{\lambda}, \mathbf{a}]P[B(\mathbf{b}) \,|\boldsymbol{\lambda}, \mathbf{b}] \quad (5)$$

where $\boldsymbol{\lambda}$ denotes the hypothesized local hidden variables for the two entangled particles ("local" because they are properties associated only with each particle and "hidden" because their values not known explicitly). This factorization implies that $A(\mathbf{a})$ and $B(\mathbf{b})$ are independent conditional on $\boldsymbol{\lambda}$ and so the conditional mean of the product is the product of the conditional means:

$$\langle A(\mathbf{a})B(\mathbf{b}) \,|\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}\rangle = \langle A(\mathbf{a}) \,|\boldsymbol{\lambda}, \mathbf{a}\rangle\langle B(\mathbf{b}) \,|\boldsymbol{\lambda}, \mathbf{b}\rangle \quad (6)$$

The independences of $P[A(\mathbf{a}) \,|\boldsymbol{\lambda}, \mathbf{a}]$ from $\mathbf{b}$ and from $B(\mathbf{b})$ that are implicit in (5) are often called *parameter* and *outcome independence*, respectively (Shimony [1990]).

Bell's justification of the locality condition in (5) is that the spin outcome $B(\mathbf{b})$ and orientation $\mathbf{b}$ of SG device B should have no causal influence over the spin outcome $A(\mathbf{a})$ at SG device A because of their physical separation. Actually, (5) was first introduced by Clauser and Horne [1974] where it was described as expressing no action at a distance between the two measuring devices, a justification consistent with Bell's. Notice that there is an implicit assumption made by these authors that the probabilities in (5) are real properties of the experimental setup and that probabilistic conditioning represents a causal influence on the spin outcomes. The probability for $A(\mathbf{a})$ should therefore not be conditional on $B(\mathbf{b})$, as it would be if the full product rule of probability theory was applied. This crucial assumption means that Bell and Clauser and Horne, and many others writing on the topics of locality and entanglement in QM, are implicitly using the frequentist interpretation of probability.

Bell [1964] originally stated a locality condition that is a deterministic version of (5) that he called the "vital assumption" that the outcome $B(\mathbf{b})$ for the particle at SG device B should not depend on the setting $\mathbf{a}$ for SG device A and he used the notation $B(\mathbf{b}, \boldsymbol{\lambda})$. Similarly, he used $A(\mathbf{a}, \boldsymbol{\lambda})$. He also imposed a deterministic counterpart of one of the perfect correlation conditions: $A(\mathbf{a}, \boldsymbol{\lambda}) = -B(\mathbf{a}, \boldsymbol{\lambda})$. Later in Bell [1976, 1981], he again states that the outcomes at the two separated SG devices should not have any causal influence over each other but now expresses this fact in the probabilistic form in (5), presumably so that the outcomes at SG devices A and B need not necessarily be known when the orientations $\mathbf{a}$ and $\mathbf{b}$ and the hidden variable $\boldsymbol{\lambda}$ are known, as he appears to assume in his 1964 paper. Bell's original deterministic model involving $A(\mathbf{a}, \boldsymbol{\lambda})$ and $B(\mathbf{b}, \boldsymbol{\lambda})$ may be viewed as a special probabilistic case where in (5):

$$\mathbb{P}[A(\mathbf{a}) = \alpha \,|\boldsymbol{\lambda}, \mathbf{a}] = \frac{1}{2}\,[1 + \alpha A\,(\mathbf{a}, \boldsymbol{\lambda})]$$

$$\mathbb{P}[B(\mathbf{b}) = \beta \,|\boldsymbol{\lambda}, \mathbf{b}] = \frac{1}{2}\,[1 + \beta B\,(\mathbf{b}, \boldsymbol{\lambda})]$$

Bell [1964, 1976, 1981] and Clauser and Horne [1974] show that their assumed locality condition leads to the following inequality in terms of three covariances:

$$|\langle A(\mathbf{a})B(\mathbf{b})\rangle - \langle A(\mathbf{a})B(\mathbf{c})\rangle| - \langle A(\mathbf{b})B(\mathbf{c})\rangle \leq 1 \quad (7)$$

where there is a total of *three* different Bohm-EPR experiments involving orientations for A and B of $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{c})$ and $(\mathbf{b}, \mathbf{c})$.

Clauser et al [1969] also derived another inequality using a similar approach to that in Bell [1964] based on Bell's deterministic locality condition. The CHSH inequality involves covariances for a total of *four* Bohm-EPR experiments with orientations for SG devices A and B of $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{c})$, $(\mathbf{b}, \mathbf{d})$ and $(\mathbf{c}, \mathbf{d})$:

$$\begin{aligned} &|\langle A(\mathbf{a})(B(\mathbf{b}) - B(\mathbf{c}))\rangle + \langle A(\mathbf{d})(B(\mathbf{b}) + B(\mathbf{c}))\rangle| \\ &\leq |\langle A(\mathbf{a})(B(\mathbf{b}) - B(\mathbf{c}))\rangle| + |\langle A(\mathbf{d})(B(\mathbf{b}) + B(\mathbf{c}))\rangle| \\ &\leq 2 \end{aligned} \quad (8)$$

The first inequality in (8) is obvious. A simple proof of the second one based on (5) is given in Goldstein et al [2011], which does not use the aforementioned property of perfect correlation.

The Bell and CHSH inequalities in (7) and (8) are actually logically equivalent if, following Bell [1964], we assume that perfect correlation holds. Set $\mathbf{d} = \mathbf{b}$ in (8) so that $\langle A(\mathbf{d})B(\mathbf{b})\rangle = -1$ because then $A(\mathbf{d}) = A(\mathbf{b}) = -B(\mathbf{b})$, implying that the final term in (8) is:

$$|\langle A(\mathbf{d})B(\mathbf{b})\rangle + \langle A(\mathbf{d})B(\mathbf{c})\rangle| = 1 - \langle A(\mathbf{b})B(\mathbf{c})\rangle,$$

giving (7). Conversely, if we sum two Bell inequalities, one given by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ as in (7), and the other by replacing them in order with $\mathbf{d}$, $\mathbf{b}$ and $-\mathbf{c}$ in (7), then using $B(-\mathbf{c}) = A(\mathbf{c}) = -B(\mathbf{c})$ from perfect correlation, we get the CHSH inequality in (8).

It is well known since their introduction that the Bell inequalities in (7) and (8) are violated for some choices of the directions by the QM distribution in (1). To first show this for (7), substitute the QM covariance expression in (4) into each term, then (7) can be expressed as:

$$|\cos\theta_{ab} - \cos\theta_{ac}| + \cos\theta_{bc} \leq 1$$

By taking $\theta_{ab} = \theta_{bc} = \pi/4$, so that $\theta_{ac} = \pi/2$, the left-hand side is $\sqrt{2}$ and the inequality is violated. Similarly, substituting the QM covariance expression in (4) into each term in (8), the CHSH inequality can be expressed as:

$$|\cos\theta_{ab} - \cos\theta_{ac} + \cos\theta_{db} + \cos\theta_{dc}| \leq 2$$

then for $\theta_{ab} = \theta_{db} = \theta_{dc} = \pi/4$, so that $\theta_{ac} = 3\pi/4$, the left-hand side is $2\sqrt{2}$ and the inequality is violated. Actually, it is readily argued using continuity that there is a continuum of allowable values for the angles for which both inequalities are not satisfied.

To examine whether the experimental data on spin outcomes implies that the Bell inequalities are violated for some device orientations, each theoretical covariance, which corresponds to a single experimental set-up, is estimated by using sample covariances over many experiments with the same set-up. These empirical estimates of each theoretical covariance are then substituted into the CHSH inequality in (8). Most of these experiments have been performed using polarization of photons rather than spin-$\frac{1}{2}$ particles (e.g. Aspect et al [1981, 1982], Shalm et al [2015]), so the experiments involve the spin-1 version of (1) where $\frac{1}{2}\theta_{ab}$ is replaced by θ_{ab}, but recently electron spins have been used (Hensen et al [2015]). The combined experimental evidence strongly suggests that the CHSH inequality for photons and electrons is violated for certain choices of the orientations of the measuring devices, consistent with QM. It has then been concluded that local hidden variables that would explain the probability distribution (1) do not exist. This conclusion depends critically on the assumption that if local hidden variables exist, then the locality condition in (5) must hold, which leads to the Bell inequalities, and so their violation in experiments implies that (5) is violated for some choices of orientations.

3.2 Invalidity of the Locality Condition

Actually, the factorization (conditional independence) in (5) is fundamentally invalid because it is never possible for local hidden variables to provide all of the information that correct probabilistic conditioning provides, which is given by the product (Bayes) rule from probability theory:

$$\begin{aligned} &P[A(\mathbf{a}), B(\mathbf{b})\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] \\ =&P[A(\mathbf{a})\,|\,B(\mathbf{b}), \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}]P[B(\mathbf{b})\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] \\ =&P[A(\mathbf{a})\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}]P[B(\mathbf{b})\,|\,A(\mathbf{a}), \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] \end{aligned} \tag{9}$$

The consistency of the locality condition in (5) with (9) requires that $P[A(\mathbf{a})\,|\,B(\mathbf{b}), \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] = P[A(\mathbf{a})\,|\,\boldsymbol{\lambda}, \mathbf{a}]$ provided $P[B(\mathbf{b})\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}]$ is not zero. This can be shown by summing over $A(\mathbf{a}) = +1$ and -1 in (5), which shows that $P[B(\mathbf{b})\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] = P[B(\mathbf{b})\,|\,\boldsymbol{\lambda}, \mathbf{b}]$. Then equating (5) and (9) and cancelling these two probabilities as a common factor assuming it is not zero, gives the stated result. Actually, Bell [1976] explicitly states this equivalence as his locality condition and then shows it implies (5). However, we now show that this equivalence is not compatible with the required perfect correlation for any local hidden variable model for the spin behavior, which comes from the constraint that the two particles are created with opposite spin directions.

Suppose first that $\mathbb{P}[B(\mathbf{b}) = 1\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] > 0$. Fix direction $\mathbf{a}$ and consider the probability for $A(\mathbf{a}) = 1$ given that $B(\mathbf{b}) = 1$. First consider the case $\mathbf{b} = \mathbf{a}$, then from perfect correlation, $A(\mathbf{a}) = -B(\mathbf{a}) = -B(\mathbf{b}) = -1$, so $\mathbb{P}[A(\mathbf{a}) = 1\,|\,B(\mathbf{b}) = 1, \boldsymbol{\lambda}, \mathbf{a} = \mathbf{b}] = 0$, regardless of the value of $\boldsymbol{\lambda}$, which is a property of the particles set only by their source and has nothing to do with the settings $\mathbf{a}$ and $\mathbf{b}$ of the SG devices. If the case $\mathbf{b} = -\mathbf{a}$ is chosen instead, then from perfect correlation, $A(\mathbf{a}) = -B(\mathbf{a}) = B(-\mathbf{a}) = B(\mathbf{b}) = 1$, so $\mathbb{P}[A(\mathbf{a}) = 1\,|\,B(\mathbf{b}) = 1, \boldsymbol{\lambda}, \mathbf{a} = -\mathbf{b}] = 1$. Thus, representing the probability $\mathbb{P}[A(\mathbf{a}) = 1\,|\,B(\mathbf{b}) = 1, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}]$ as $\mathbb{P}[A(\mathbf{a}) = 1\,|\,\boldsymbol{\lambda}, \mathbf{a}]$ is not valid because its value depends on the direction $\mathbf{b}$, which provides relevant information for the probability that $A(\mathbf{a}) = 1$ and so it cannot be dropped from the conditioning.

If, on the other hand, $\mathbb{P}[B(\mathbf{b}) = 1\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] = 0$, then $\mathbb{P}[B(\mathbf{b}) = -1\,|\,\boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] = 1$ so consistency of (5) with (9) implies $\mathbb{P}[A(\mathbf{a}) = 1\,|\,B(\mathbf{b}) = -1, \boldsymbol{\lambda}, \mathbf{a}, \mathbf{b}] = \mathbb{P}[A(\mathbf{a}) = 1\,|\,\boldsymbol{\lambda}, \mathbf{a}]$. Fix direction $\mathbf{a}$ and consider the two separate cases: $\mathbf{b} = \mathbf{a}$ and $\mathbf{b} = -\mathbf{a}$, then the left-hand side of the last probability equality is 1 and 0, respectively, whereas the right-hand side of the equality is independent of $\mathbf{b}$. Thus, we see again that the locality condition (5) is not consistent with the product rule of probability theory. When discussing the Bohm-EPR experiment, Jaynes [1989] noted that (9) is the correct factorization of the joint distribution according to the product rule but its implied incompatibility with the locality condition (5) under the requirement of perfect correlation does not seem to have been previously noticed.

We see now that for the deterministic case in Bell [1964], his “vital assumption” is also not compatible with perfect correlation. A similar proof to the probabilistic case shows that A cannot depend only on $\mathbf{a}$ and $\boldsymbol{\lambda}$ for if $B(\mathbf{b}) = 1$, then $\mathbf{b} = \mathbf{a}$ implies $A(\mathbf{a}, \boldsymbol{\lambda}) = -1$ but $\mathbf{b} = -\mathbf{a}$ implies $A(\mathbf{a}, \boldsymbol{\lambda}) = 1$, and this dependence on $\mathbf{b}$ holds regardless of the value of $\boldsymbol{\lambda}$, which is assumed to be a property of the particles unrelated to the choice of the orientations of the SG devices. Thus, both the deterministic and probabilistic locality conditions are inconsistent with perfect correlation. The same conclusion also follows directly from the aforementioned fact that the deterministic locality condition can be viewed as a special case of the probabilistic one.

Perfect correlation is not needed to prove the CHSH inequality (e.g. Goldstein et al [2011]). If it is dropped, however, then the class of hidden variable models is inconsistent with the QM distribution in (1) at the outset. Such a class of models should therefore be of little interest. Thus, the standard locality condition does not apply to any class of hidden variable models that can explain the spin probability distribution in the Bohm-EPR experiment. This fact is independent of how the probabilities are interpreted since all probabilities must satisfy the product rule, by definition.

We end with a comment on the GHZ version of Bell's Theorem (Greenberger et al [1989]). This is a "no-go" theorem for a hidden-variable model behind the joint distribution for four spin-$\frac{1}{2}$ particles. The proof only uses a deterministic case of the joint distribution, which is factorized in a similar way to (5). Notice that we have shown that the factorization of the joint distribution given by the locality condition in (5) is invalid by using perfect correlation, which is the deterministic case of the joint spin distribution for two spin-$\frac{1}{2}$ particles. It seems that the GHZ factorization is invalid in the same way but a detailed proof is left for future work.

3.3 Bayesian Perspective on Locality Condition

As noted previously, the justification for taking (5) as expressing a locality condition ultimately rests on the frequentist interpretation of probability. From the perspective of the Bayesian interpretation of probability, the conditioning in (9) is viewed as relevant information for probabilistic predictions of the spin outcomes, rather than a causal influence on these outcomes (Beck [2018], Jaynes [1989]). Thus, there is no motivation for the locality condition in (5), nor is there any concern about much discussed aspects of quantum entanglement such as superluminal propagation of effects or the need to show that there can be no instantaneous signaling by well-separated operators each using an SG device in a Bohm-Bell experimental setup.

The underlying reason for the the global correlation between the spins implied by (1) is not some physical interaction between the two particles but rather it is the information that the particles are assumed to be created with their spins in opposite directions, that is, there is assumed prior knowledge that the two particles are in the singlet state, giving a constraint of zero total spin before the two particles enter their respective SG devices. It is possible then to develop a local hidden-variable model for (1) that utilizes this information constraint by paying special attention to the meaning of conditioning in QM probabilities, and that does not violate EPR locality because no physical influence at a distance is required. Such a model will be presented in a future publication.

3.4 Significance of the Bell Inequalities

Given that the standard locality condition in (5) is invalid, the Bell inequalities are irrelevant to the question of whether a hidden-variable model is possible for (1). It is a remarkable fact, however, that the Bell inequalities in (7) and (8) have a fundamental role in probability theory that is unrelated to the locality condition or hidden variables. As noted by Hess and Philipp [2005], it was shown in the mathematical literature before Bell's work (e.g. Bass [1955], Vorob'ev [1962]) that the inequality in (7) is part of a necessary condition for the existence of a valid third-order probability distribution $P[A(\mathbf{a}), A(\mathbf{b}), B(\mathbf{c})]$ for the three binary stochastic variables $A(\mathbf{a})$, $A(\mathbf{b}) = -B(\mathbf{b})$ and $B(\mathbf{c})$ that gives by marginalization three specified second-order probability distributions for the three possible pairs of these variables.

In a similar spirit, Fine [1982a] showed that the existence of a valid fourth-order probability distribution $P[A(\mathbf{a}), A(\mathbf{d}), B(\mathbf{b}), B(\mathbf{c})]$ that is consistent with the specified probability distributions $P[A(\mathbf{a}), B(\mathbf{b})]$, $P[A(\mathbf{a}), B(\mathbf{c})]$, $P[A(\mathbf{d}), B(\mathbf{b})]$ and $P[A(\mathbf{d}), B(\mathbf{c})]$ directly implies the eight inequalities on the second-order joint probabilities that were first derived by Clauser and Horne [1974] based on the standard locality condition. Furthermore, these eight probability inequalities can be expressed in terms of covariances, leading to two CHSH inequalities implied by (8) by removing the absolute values, as well as six others that are just permutations of the directions $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ in these two inequalities. The CHSH inequality in (8) is therefore a necessary condition for the existence of a valid fourth-order joint distribution for the four spin variables. As shown in Appendix A, if (8) is violated, a third-order joint distribution fails to exist because it has negative probabilities and so the fourth-order joint does not exist as well (a valid fourth-order joint implies all four third-order joints exist from marginalizations over each of the four variables).

Fine [1982b] proved that satisfaction of all of the CHSH inequalities is also a sufficient condition for $P[A(\mathbf{a}), A(\mathbf{d}), B(\mathbf{b}), B(\mathbf{c})]$ to exist. In the same paper, he proved that a set of four inequalities that include Bell's original equality in (7) gives a necessary and sufficient condition that a third-order probability distribution $P[A(\mathbf{a}), A(\mathbf{b}), B(\mathbf{c})]$ exists that has three specified second-order joint distributions $P[A(\mathbf{a}), A(\mathbf{b})]$, $P[A(\mathbf{a}), B(\mathbf{c})]$, and $P[A(\mathbf{b}), B(\mathbf{c})]$ as marginals. Further discussion is given in Appendix A to provide more insight into these results.

We make the following observation to conclude this section. If any of the Bell inequalities like (7) or (8) are violated, then third or fourth-order distributions do not exist that are compatible with three or four specified second-order distributions given by (1), so obviously there can be no hidden variables that explain these higher order distributions. Notice, however, that these distributions involve three or four separate Bohm-EPR experimental setups. There seems to be no compelling argument put forth that a joint distribution should exist for multiple spins coming from different experimental setups. *More importantly, violation of the Bell inequalities does not imply that there cannot be a hidden-variable model for the joint spin distribution in* (1) *for a single such experiment.* Indeed, such models have been given before, for example, the model of Kochen and Specker [1967] for two-dimensional Hilbert spaces (see also Harrigan and Spekkens [2010]), although they are not demonstrably local.

4 CONCLUSIONS

The Bohm-EPR experiment serves as an important test bed to examine some of the mysteries exhibited by QM theory. The principal interest in this paper is the old question of whether a local hidden-variable model could explain the probability distribution in (1), which is derived from Born's rule for the spin outcomes at separated SG (Stern-Gerlach) devices of two particles with entangled spins in the singlet state. The overall conclusion here is that this is still an open question because violations of Bell inequalities are irrelevant to it. Bell's locality (factorability/factorizability) condition that he and others have used to establish these inequalities contradicts the product rule of probability theory under the requirement that the local hidden-variable model exhibit the perfect correlation implied by (1) for aligned SG devices. This contradiction holds regardless of how probability is interpreted. The justification for this locality condition rests crucially on the frequentist interpretation of probability where conditioning in the probabilities for the spin outcomes is viewed as playing a causal role. Therefore, observing the spin outcome at one device is inferred to immediately influence the spin outcome at the other device, although no plausible mechanism has been presented for such an instantaneous effect.

On the other hand, using a Bayesian interpretation of probability, the probabilistic spin correlations mean that the spin outcome (actual or hypothesized) at one SG device provides information relevant to the probability for predicting the spin outcome at the other device, and there is no motivation to postulate any non-local effect. Indeed, it will be shown in a future publication that the Bayesian perspective allows a hidden-variable model to be constructed for the QM joint spin distribution that clearly respects EPR locality because no physical influence at a distance is required.

As shown by others, Bell inequalities do have an important role unrelated to local hidden-variable models: they give necessary and sufficient conditions for the existence of a third-order (or fourth-order) joint probability distribution for the spin outcomes that has as marginal distributions three (or four) QM second-order joint probabilities. For choices of the orientations of the SG devices where a Bell inequality is not satisfied, at least one of the third-order joint probabilities becomes negative.

Repeated experiments with a pair of orientations $(\mathbf{a}, \mathbf{b})$ can be used to check the joint probability distribution for $A(\mathbf{a})$ and $B(\mathbf{b})$ from QM by using the sample moments to estimate the corresponding theoretical moments in equation (A8). However, performing a set of separate experiments for each of three or four pairs of orientations, $(\mathbf{a}, \mathbf{b})$, $(\mathbf{a}, \mathbf{c})$ and so on, and then substituting the sample moments into the Bell inequalities (7) or (8), cannot be used to check for the existence of hidden variables underlying the second-order joint spin distribution for a single Bohm-EPR experimental setup. Furthermore, there seems to be no reason to think that a joint distribution should even exist that includes spin outcomes from different experimental setups.

In summary, the standard locality condition in QM rests on the frequentist interpretation of probability, which treats probability distributions as real properties of random phenomena that control the long-term behavior of apparently random events through some invisible hand. This locality condition violates the product rule of probability theory for any hidden-variable model that satisfies the perfect correlation property for a pair of entangled spins. This suggests that the frequentist interpretation is not applicable to probabilities in QM. From the perspective of Edwin Jaynes, the frequentist interpretation is an example of what he calls the *Mind-Projection Fallacy* where models of reality are confused with reality (Jaynes [1990a,b, 2003]).

If the Bayesian interpretation of probability is chosen, there is no motivation for the locality condition. Probability distributions are viewed pragmatically as probability models for predicting outcomes that are uncertain because there is insufficient information available for precise predictions, and conditioning in probability distributions is viewed as information to be used in inferences and not as representing a causal influence. In general, the adoption of the Bayesian point of view leads to less puzzling interpretations of QM theory than the frequentist perspective in understanding quantum entanglement and quantum non-locality in the Bohm-EPR experiment (Beck [2018]).

A Appendix on Bell inequalities and existence of third and fourth order distributions

Let A, B and C be any three binary stochastic variables whose possible values are $\{-1, 1\}$. If a third-order joint distribution exists for A, B and C, then it can be expressed in the form:

$$\mathbb{P}[A=\alpha,\ B=\beta,\ C=\delta] = \frac{1}{8}\,[1+\alpha\langle A\rangle+\beta\langle B\rangle+\delta\langle C\rangle+ \alpha\beta\langle AB\rangle+\beta\delta\langle BC\rangle+\alpha\delta\langle CA\rangle+\alpha\beta\delta\langle ABC\rangle]$$

where α, β, δ = +1 or -1, or, equivalently, using the shorthand notation for probability functions:

$$P[A,B,C] = \frac{1}{8}\,[1+A\langle A\rangle+B\langle B\rangle+C\langle C\rangle+AB\langle AB\rangle+ BC\langle BC\rangle+CA\langle CA\rangle+ABC\langle ABC\rangle] \tag{A.1}$$

This result can be shown by noting that it is the unique solution for the eight probabilities defining the distribution that satisfies normalization and the seven moment equations.

Suppose now that the values of the seven moments are specified in the interval $[-1, 1]$ and we ask if (A.1) gives a valid probability distribution. This depends on whether all

Table 1: Probability distributions $P[A, B]$, $P[B, C]$ and $P[C, A]$.

$P[A,B]$	$A = +1$	$A = -1$	$P[B,C]$	$B = +1$	$B = -1$	$P[C,A]$	$C = +1$	$C = -1$
$B = +1$	$\frac{1}{2}$	0	$C = +1$	0	$\frac{1}{2}$	$A = +1$	$\frac{1}{2}$	0
$B = -1$	0	$\frac{1}{2}$	$C = -1$	$\frac{1}{2}$	0	$A = -1$	0	$\frac{1}{2}$

eight probabilities lie in the interval $[0, 1]$. Clearly, these probabilities are bounded above by 1 because the magnitude of each of the eight terms inside the brackets in (A.1) is bounded above by 1. However, depending on the specified values of the moments, $P[A, B, C]$ may be negative. Necessary and sufficient conditions for $P[A, B, C]$ to be a valid probability distribution for specified values of the seven moments are the eight conditions on the moments implied by the right-hand side of (A.1) being non-negative as A, B and C range over -1 and 1. These eight conditions come in four pairs where each pair corresponds to switching the sign of each component of the triplet (A,B,C), namely pairs $(1, -1, -1)$ and $(-1, 1, 1)$, $(1, 1, 1)$ and $(-1, -1, -1)$, $(1, -1, 1)$ and $(-1, 1, -1)$, and $(1, 1, -1)$ and $(-1, -1, 1)$. If we sum each such pair, we get four Bell inequalities involving only the second-order moments:

$$\langle AB\rangle + \langle BC\rangle + \langle CA\rangle \geq -1 \tag{A.2}$$

$$\langle AB\rangle - \langle BC\rangle - \langle CA\rangle \geq -1 \tag{A.3}$$

$$-\langle AB\rangle + \langle BC\rangle - \langle CA\rangle \geq -1 \tag{A.4}$$

$$-\langle AB\rangle - \langle BC\rangle + \langle CA\rangle \geq -1 \tag{A.5}$$

These four inequalities can be reduced to two equivalent inequalities that each have a similar form to the original 1964 Bell inequality in (7) by combining (A.2) and (A.4), and (A.3) and (A.5):

$$|\langle AB\rangle + \langle AC\rangle| - \langle BC\rangle \leq 1 \tag{A.6}$$

$$|\langle AB\rangle - \langle AC\rangle| + \langle BC\rangle \leq 1 \tag{A.7}$$

Satisfaction of these two Bell inequalities is therefore necessary for a valid third-order distribution for (A,B,C). If either of them is violated, then a valid third-order distribution for (A,B,C) does not exist with the three specified second moments. These conditions are also sufficient for a compatible $P[A, B, C]$ to exist if the means $\langle A\rangle$, $\langle B\rangle$ and $\langle C\rangle$ are all zero, which can be seen by choosing $\langle ABC\rangle = 0$ in (A.1). This result was first shown by Suppes and Zanotti [1981].

If the three second-order distributions $P[A, B]$, $P[B, C]$ and $P[C, A]$ are given, which imply the marginal distributions $P[A]$, $P[B]$ and $P[C]$, then it follows that the first six moments are specified because

$$P[A, B] = \frac{1}{4}\left[1 + A\langle A\rangle + B\langle B\rangle + AB\langle AB\rangle\right],$$

$$P[A] = \frac{1}{2}\left[1 + A\langle A\rangle\right], \text{ etc.} \tag{A.8}$$

Then with an arbitrary choice of the third moment $\langle ABC\rangle$, the third-order distribution $P[A, B, C]$ given by (A.1) will be a valid one provided all eight probabilities are non-negative. The necessary conditions for this to hold are then satisfaction of (A.6) and (A.7).

The conditions in (A.2) to (A.5) can also be expressed in terms of probabilities by using (A.8) to replace the moments. This gives the equivalent four conditions in Theorem 2 of Fine [1982b], who showed that they are necessary and sufficient for a compatible $P[A, B, C]$ to exist, giving the following theorem:

Existence Theorem: *A necessary and sufficient condition for a valid third-order probability distribution $P[A, B, C]$ to exist that gives by marginalization three specified second-order distributions $P[A, B]$, $P[A, C]$ and $P[B, C]$ is that the four inequalities* (A.2) – (A.5)*) are all satisfied, or, equivalently, the two inequalities in* (A.6) *and* (A.7) *are satisfied.*

If we have four binary stochastic variables A, B, C and D with a fourth-order distribution $P[A, B, C, D]$, then by marginalization, the two third-order distributions for (A,B,C) and (B,C,D) exist. The Existence Theorem then implies that (A.7) holds and so does (A.6) with A replaced by D. Summing these two inequalities then implies an inequality of the form of the 1969 CHSH inequality in (8):

$$|\langle AB\rangle - \langle AC\rangle| + |\langle DB\rangle + \langle DC\rangle| \leq 2 \tag{A.9}$$

This inequality is therefore a necessary condition for the existence of a fourth-order distribution for (A,B,C,D) that is compatible with the second-order distributions for the pairs (A,B), (A,C), (D,B) and (D,C). If inequality (A.9) is not satisfied, then at least one third-order joint probability is negative and so a valid fourth-order joint distribution $P[A, B, C, D]$ does not exist. If it did, then all four third-order marginal distributions would exist, contradicting the assumed violation of inequality (A.9). Three other similar but distinct necessary conditions can be obtained by examining permutations of A, B, C and D in (A.9). Using (A.8), these four inequalities can also be expressed in terms of second-order joint and first-order marginal probabilities to give the counterparts of the four inequalities in Clauser and Horne [1974] and Fine [1982b]. Fine shows that satisfaction of these four inequalities is also sufficient for the existence of a compatible fourth-order joint distribution $P[A, B, C, D]$.

Here is a simple example motivated by one in Vorob'ev [1962] that illustrates the Existence Theorem. Consider binary stochastic variables A, B and C that have second-order joint distributions as in Table 1. The first and second moments of A, B and C are $\langle A\rangle = \langle B\rangle = \langle C\rangle = 0$ and $\langle AB\rangle = \langle CA\rangle = 1$ and $\langle BC\rangle = -1$. Substitution of these moments shows that inequality (A.7) is satisfied but that inequality (A.6) is violated. Therefore, a valid third-order distribution $P[A, B, C]$ does not exist and there must be at least one negative probability. The mathematical expression for it in (A.1) gives:

$$P[A, B, C] = \frac{1}{8}\left[1 + AB - BC + CA + ABC\mu_3\right] \tag{A.10}$$

where $\mu_3 = \langle ABC\rangle$. This third moment is bounded in magnitude by 1 so the probability for $A = -1$ and $B = C = 1$ is negative.

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