Control of Underactuated Mechanical Systems with Drift Using Higher-Order Averaging Theory

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Abstract This paper uses a recently developed generalized averaging theory [18] to develop stabilizing control laws for a large class of nonlinear systems with drift. These control laws exponentially stabilize in the average.

1 Introduction

This paper introduces a new technique to stabilize a large class of nonlinear systems with drift. Our result is based on a recently developed generalized averaging theory [18] that uses nonlinear Floquet theory together with series expansions to arbitrarily approximate the flow of a time-periodic vector field. We apply this averaging method to *1-homogeneous* systems with drift. These nonlinear systems, which are described in a companion paper [19], are a generalization of the simple mechanical systems of Lewis and Murray [7], and include many physical nonlinear systems with drift. Our method exponentially stabilizes these sytems in the average. The work reported here can be considered an extension of our previous work for driftless systems [17] to systems with drift.

There has been recent success on the use of *motion control algorithms* and series expansion methods to obtain (exponentially) stabilizing control laws for *simple mechanical systems*, which are characterized by Lagrangians with kinetic and potential energy terms only [3, 5, 8]. Given that 1-homogeneous control systems are a generalization of simple mechanical control systems, the methods reported in this paper can be seen as an extension of this recent work. We also introduce a novel form of averaged feedback which did not appear in these prior works.

Other prior work on feedback stabilization of nonlinear systems with drift has tended to focus on specific canonical control forms, [11, 2], which are actually special instances of simple mechanical or 1-homogeneous control systems [19]. There has also been work on utilizing homogeneous transformations for exponential stabilization with respect to a homogeneous norm [10]. There exists many other methods for stabilization of systems with drift, however our interest is in obtaining exponentially

stabilizing controllers, for which there are few methods. We note that Floquet theory and averaging have recently been applied to the problem of stabilizing nonlinear systems. However, many of these methods have been restricted to special application domains. For example, [15] uses feedback techniques and Floquet analysis to stabilize a free joint manipulator. In [14], Floquet theory is used to stabilize driftless systems evolving on simple Lie groups. In contrast to these specialized applications, our methods are quite general.

Section 2 summarizes our recent work on a generalized averaging theory. The particular structure of 1-homogeneous control systems is discussed in Section 3. Techniques to stabilize 1-homogeneous systems are presented in Section 4. Section 5 illustrates the method with a simple exampe.

2 A Generalized Averaging Theory

The flow of the differential equation,

$$\dot{x} = X(x, t; \epsilon) = \epsilon \widehat{X}(x, t), \quad x(0) = x_0, \quad (1)$$

with X smooth in x, and T-periodic, i.e., $X(x, t; \epsilon) = X(x, t + T; \epsilon)$ can be analyzed by a non-linear version of Floquet theory. This approach represents the flow as the composition of a periodic flow and the evolution of an averaged vector field. These components can be approximated to arbitrary order by appropriate series expansions. We will apply these theorems in Section 4 to develop generalized expressions for the averages of 1-homogeneous systems under periodic control.

Theorem 1 (Nonlinear Floquet Theorem) *[18]*
Let $\Phi^X_{0,t}$ be the flow of the time-periodic differential equation (1). If the monodromy map has a logarithm, then the flow $\Phi^X_{0,t}$ can be represented as a composition of flows $\Phi^X_{0,t} = P(t) \circ \exp(Zt)$, where P is T-periodic, and $\exp(Zt)$ denotes the flow of the nonlinear autonomous vector field Z.

The monodromy map is the flow of X at time T, e.g., $\Phi^X_{0,T}$. It coincides with the flow of the *averaged* autonomous vector field, Z, at time T, e.g., $\exp(ZT)$.

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Theorem 2 *[18] If the monodromy map has a fixed point, the actual flow has a periodic orbit whose stability properties are determined by the stability properties of the monodromy map.*

Corollary 1 *[18] If the flow of system (1) has a fixed point x^*, as does the monodromy map, then stability of the fixed point may be determined by the monodromy map.*

Proposition 1 *[18] The logarithm of the monodromy map may be used instead of the monodromy map to determine the stability properties of the actual flow. The logarithm is given by the autonomous vector field Z.*

An (exponentially) asymptotically stable fixed point for the monodromy map implies an (exponentially) asymptotically stable orbit or fixed point for the system (1). Averaging theory seeks to find suitable approximations to the infinite series expansions given by $P(t)$ and Z. The approximations are given by truncations of the series expansions for $P(t)$ and Z: the mth-order truncation will be denoted by $\mathrm{Trunc}_m(\cdot)$; see [18] for the structure of these truncations.

Theorem 3 *[18] The mth-order truncation of the logarithm of the monodromy map gives an $(m+1)^{th}$-order flow approximation for finite time, i.e., for time $O(1)$*

$$\exp(Zt) = \exp(\mathrm{Trunc}_m(Z)t) + O(\epsilon^{m+1}).$$

Theorem 4 *[18] An mth-order truncation of the time-periodic Floquet mapping is of order $(m+1)$-close to the time-periodic Floquet mapping on the time scale $o(1)$.*

$$P(t) = \mathrm{Trunc}_m(P(t)) + O(\epsilon^{m+1})$$

2.1 Averaging of Systems with Drift

We consider the case of *vibrational control* of a system with drift. Vibrational control inputs are high amplitude, high frequency, i.e.,

$$\dot{x} = X(x) + (1/\epsilon)F(x, t/\epsilon), \tag{2}$$

with ϵ small and where $F(\cdot, t)$ is T-periodic. The system is not in the canonical form required by averaging theory, however, under certain assumptions, the *variation of constants* (VOC) transformation gives equations in the standard form required by averaging theory. First, transform time $t/\epsilon \mapsto \tau$, to obtain

$$dx/d\tau = \epsilon X(x) + F(x, \tau). \tag{3}$$

In the VOC approach, $X(x)$ is seen to be a perturbation to the primary vector field $F(x, \tau)$. Define the vector field,

$$Y(y, \tau) = \epsilon\left(\left(\Phi_{0,\tau}^F\right)^* X\right)(y) \tag{4}$$

where $\Phi_{0,\tau}^F$ is the flow of the vector field F. According to the VOC, the solution $x(t)$ is given exactly by

$$x(\tau) = \Phi_{0,\tau}^F(y(\tau)), \tag{5}$$

where $\{y(t), t \in [0, T]\}$ is the solution to the system

$$dy/d\tau = Y(y, \tau), \quad y(0) = x_0, \tag{6}$$

with $Y(y, \tau)$ given in Eq. (4). In order to make sense of the vector field Y, we will introduce a periodicity assumption, to be proven later (c.f. Proposition 4).

Assumption 1 *The evolution of the vector field $F(x, t)$ is periodic of period T, i.e., $\Phi_{0,t}^F = \Phi_{0,t+T}^F$.*

For many problems, periodicity implies that one can compute an autonomous average of Y,

$$dz/d\tau = Z^{\mathrm{avg}}(z) \tag{7}$$

where $Z^{\mathrm{avg}} \equiv \frac{1}{T}\log(\Phi_{0,T}^Y)$ is the logarithm of the monodromy map, as per Theorem 1. We say that the sytem (7) is m^{th}-order averaged if Z^{avg} is related by its mth-order truncation, c.f. Theorem 3.

Theorem 5 *The average of the system (2), under Assumption 1, is given by the averaged system (7). This holds for any order of truncation.*

Proof: Floquet theory decomposes the flow of Y into, $\Phi_{0,t}^Y = P(t) \circ \exp(Zt)$, where $P(t)$ is a T-periodic mapping and $\exp(Zt)$ is an autonomous flow. The flow of the sytem (2) is,

$$\Phi_{0,\tau}^{\epsilon X+F} = \Phi_{0,\tau}^F \circ \Phi_{0,\tau}^Y,$$

per the variation of constants. Applying Theorem 1,

$$\Phi_{0,\tau}^{\epsilon X+F} = \Phi_{0,\tau}^F \circ P(\tau) \circ \exp(Z\tau)$$

As $\Phi_{0,\tau}^F$ and $P(t)$ are T-periodic, they can both be absorbed into one periodic mapping, $\hat{P}(t) \equiv \Phi_{0,t}^F \circ P(t)$,

$$\Phi_{0,\tau}^{\epsilon X+F} = \hat{P}(\tau) \circ \exp(Z\tau),$$

Either $Z^{\mathrm{avg}} = Z$, or it is a truncated version, $Z^{\mathrm{avg}} = \mathrm{Trunc}_m(Z^{\mathrm{avg}})$. ∎

A transformation from τ back to time t yields

$$\Phi_{0,t}^{X+\frac{1}{\epsilon}F} = \hat{P}(t/\epsilon) \circ \exp(Zt/\epsilon). \tag{8}$$

Therefore, all of the previous theorem still hold when applied to systems with drift of the form assumed here. In particular, the theorems detailing the proximity of the flow of truncated averages to the actual flow (Thms. 3 and 4), the determination of stable orbits (Thm. 2), and the stability of fixed points (Cor. 1).

High Frequency and Small Motions. It is possible to consider instead the case of high frequency oscillatory control of a system with drift,

$$\dot{x} = X(x) + F(x, t/\epsilon). \tag{9}$$

with ϵ small and where $F(\cdot, t)$ is T-periodic. A transformation of time converts the system to the standard form required by averaging theory,

$$dx/d\tau = \epsilon X(x) + \epsilon F(x, t). \qquad (10)$$

The previous theory reviewed above can still be applied, however the dependence of the formulas on ϵ will differ. Equation (10) corresponds to the case of small motions [5]. The control analysis that will be done for the vibrational control case also applies to these two forms.

3 Structure of 1-Homogeneous Systems

This section summarizes the class of systems with drift to which our theory applies. For more details see [19].

Vector Bundles. Let Q be a (differentiable) fibre bundle, $function \pi QM$ with fibre F. A *section* $\sigma : M \to Q$ is a smooth map satisfying $\pi \circ \sigma = \mathrm{id}_M$. The space defined by the collection of all sections on the fiber bundle Q is denoted by $\Gamma(Q)$. It is itself a fiber bundle.

A *vector bundle* (E, π, M, V) is fiber bundle whose typical fiber is a vector space. Given a vector bundle $\pi : E \to M$, the *zero section*, denoted by σ_0, is a smooth mapping which maps points in the base space to the zero vector based at that point,

$$\sigma_0(x) = 0_x \in E_x, \quad \forall\, x \in M.$$

The manifold structure of E is obtained from local charts (U, ψ), which are also called *local trivializations*. In a local trivialization, the vector bundle is a direct product space, $\psi(U) \subset M \times V$. Often, when giving coordinate representations, we will simply refer to a local trivialization as $E \cong M \times V$.

We review some elementary concepts related to the tangent bundle, TE, of E. The *vertical bundle over E*, denoted by VE, is the subbundle of TE given by the union of $T\pi^{-1}(0_q)$ for all $q \in Q$. A vector in TE is *vertical* if it lies in the kernel of $T\pi$.

Definition 1 *There is a canonical isomorphism between $E \times_M E$ and VE, called the* vertical lift. *It is given by,*

$$v^{\mathrm{lift}} = \left.\frac{d}{dt}\right|_{t=0} (u_r + tv_r), \quad r \in M, \ u, v \in E. \qquad (11)$$

Geometric Homogeneity and Vector Fields. Homogeneity is determined using the dilation operator, δ_t, which dilates the vector fiber,

$$\delta_t : E \to E, \quad (r, u) \mapsto (r, e^t u). \qquad (12)$$

The dilation operator satisfies $(\delta_t)^p = \delta_{pt}$. Corresponding to the dilation is its infinitesimal generator, Δ, a vector field on E. In a local trivialization, the generator is,

$$\Delta = u^i \frac{\partial}{\partial u^i} \qquad (13)$$

Our working definition of homogeneity follows.

Definition 2 *A mapping between vector bundles Ψ : $E_1 \to E_2$ is* homogeneous of order p if $\Psi \circ \delta_t = \delta_{pt} \circ \Psi$.

The two vector bundles need not have the same base nor fiber. We assume that Ψ is a smooth function defined over E, and that all vector bundles are finite dimensional. The notion of homogeneity extends to the space of vector fields on E, $\mathcal{X}(E)$, via the generator Δ.

Definition 3 *A vector field $X \in \mathcal{X}(E)$ is said to be homogeneous of order p if, $[\Delta, X] = pX$, for $p > -2$.*

The only smooth vector field of homogeneous order less than -1 is the zero vector field.

Proposition 2 *Given $X, Y \in \mathcal{X}(E)$ homogeneous of order p and q, respectively, $[X, Y]$ is homogeneous of order $p + q$.*

Corollary 2 *Given a section of the vector bundle E, its vertical lift is homogeneous of order -1.*

The converse to the corollary also holds.

Proposition 3 *[6] All vector fields of homogeneous order -1 are the vertical lift of a section of E.*

Corollary 3 *If $X, Y \in \mathcal{X}(E)$ are vertical lifts, then $[X, Y] = 0$.*

When $X, Y \in \mathcal{X}(E)$ are vertical lifts, the Jacobi identity implies the symmetry of the Jacobi-Lie bracket $[X, [\Gamma, Y]] = [Y, [\Gamma, X]]$, for any $\Gamma \in \mathcal{X}(E)$. Consequently, a symmetric product may be defined.

Definition 4 *The symmetric product of vertical lifts using the vector field $\Gamma \in \mathcal{X}(E)$ is defined to be,*

$$\langle X : Y \rangle^\Gamma \equiv [X, [\Gamma, Y]],$$

where $X, Y \in \mathcal{X}(E)$ are lifted vector fields.

We will simply write $\langle X : Y \rangle$ without reference to the vector field Γ when the context is clear. This definition is a generalization of the symmetric product definition found in Lewis and Murray [7].

Gradations of Homogeneous Spaces. Define the vector subbundle of homogeneous order k to be

$$\mathcal{P}_k \equiv \{ X \in \mathcal{X}(E) \mid X \text{ is of homogeneous degree } k. \}.$$

The following properties hold: (1) $[\mathcal{P}_i, \mathcal{P}_j] \subset \mathcal{P}_{i+j}$, and (2) $\mathcal{P}_k = \{0\}$, $\forall k < -1$. Accordingly, we may define the following union of homogeneous spaces,

$$\mathcal{M}_k = \oplus_{i=-1}^{i \le k} \mathcal{P}_i \qquad (14)$$

which inherit the properties of its constitutive sets, (1) $[\mathcal{M}_i, \mathcal{M}_j] \subset \mathcal{M}_{i+j}$, and (2) $\mathcal{M}_i = \{0\}$ $\forall i < -1$. It can be seen that $\mathcal{M}_{-1} = \mathcal{P}_{-1}$, meaning that it is the most "basic" nontrivial space. Consequently, the spaces $\mathcal{M}_k$

form a gradation. For the systems that we will study, the vector field Γ is restricted to not exceed homogeneous order 1, e.g. $\Gamma \in \mathcal{M}_1$. Therefore, $\left\langle X^{\text{lift}} : Y^{\text{lift}} \right\rangle^{\Gamma} \in \mathcal{M}_{-1}$ is again a lifted vector field. Most importantly, this implies that the symmetric product commutes with other lifted vector fields.

We may now define a *1-homogeneous control system* on E. The affine control system, with control inputs u^a,

$$\dot{x} = X(x) + Y_a^{\text{lift}}(x)u^a, \qquad a = 1\ldots m, \qquad (15)$$

is a 1-homogeneous control system if $X \in \mathcal{M}_1$. The input vector fields lie in $\mathcal{M}_{-1}$ by virtue of being lifts. This is a generalization of simple mechanical systems that also incorporates most forms of mechanical systems.

3.1 Averaging and Homogeneity

We consider control inputs that combine state feedback and time-periodic vibrational terms; $u^a(x,t) = f^a(x) + (1/\epsilon)v^a(t/\epsilon)$, with $v^a(\cdot)$ T-periodic. Substituting these controls into (15) gives,

$$\dot{x} = X(x) + Y_a^{\text{lift}}(x)f^a(x) + \frac{1}{\epsilon}Y_a^{\text{lift}}(x)v^a(t/\epsilon)$$

To meet our requirements, the state feedback, $f^a : E \to \mathbb{R}$, can consist of terms that are at most homogeneous order 2; state feedback which is an arbitrary function of the base space M and linear in the vector fiber V easily satisfies this requirement. The state terms are absorbed into one term,

$$\dot{x} = X_S(x) + \frac{1}{\epsilon}Y_a^{\text{lift}}(x)v^a(t/\epsilon) \qquad (16)$$

where $X_S \in \mathcal{M}_1$. The system may be transformed into the form required by averaging as per the discussion in Section 2.1. In order to average the system according to Theorem 5, Assumption 1 must hold. The following Proposition ensures this condition.

Proposition 4 *[4] The flow* $\Phi_{0,\tau}^{Y_a^{\text{lift}}(x)v^a(\tau)}$ *is T-periodic.*

Applying the VOC formula (Eq. (4)), we obtain,

$$Y(y,\tau) = \left(\Phi_{0,\tau}^{Y_a^{\text{lift}}(x)v^a(\tau)}\right)^* (\epsilon X_S). \qquad (17)$$

Also found in [4] is a series expansion for the pull-back term in Eq. (17) using a theorem of Agrachev and Gamkrelidze [1],

$$(\Phi_{0,t}^g)^*f = f + \sum_{k=1}^{\infty} \int_0^T \cdots \int_0^{s_{k-1}} \left(\text{ad}_{g(s_k)} \cdots \right.$$
$$\left. \text{ad}_{g(s_1)}f\right) ds_k \ldots ds_1$$

where the $\{s_j\}$ represent time and $\text{ad}_g f = [g, f]$. Due to the homogeneous structure of our class of systems, only the first two terms of the summation are nonvanishing. Hence, (17) takes the form:

$$Y = \epsilon X_S + \epsilon V_{(1)}^{(a)}(t) \left[Y_a^{\text{lift}}, X_S\right] - \frac{1}{2}\epsilon V_{(1,1)}^{(a,b)}(t) \left\langle Y_a^{\text{lift}} : Y_b^{\text{lift}} \right\rangle,$$
$$(18)$$

where the $V_{(\cdot)}^{(\cdot)}(t)$ terms are called *averaging coefficients*, and Definition 4 uses $\Gamma = X_S$ for the symmetric product. The simplest averaging coefficients are,

$$V_{(n)}^{(a)}(t) \equiv \int_0^t \int_0^{s_{n-1}} \ldots \int_0^{s_2} v^a(s_1)ds_1 \cdots ds_n. \qquad (19)$$

When time-averaged they become *averaged coefficients*. Cases of multiple upper and lower indices denote products of this type of integral. E.g., $V_{(1,1)}^{(a,b)}(t)$ has the form

$$V_{(1,1)}^{(a,b)}(t) = V_{(1)}^{(a)} V_{(1)}^{(b)} = \left(\int_0^t v^a(s_1)ds_1\right)\left(\int_0^t v^b(s_1)ds_1\right).$$

Additionally define the following, $\widetilde{V}_{(n)}^{(a)} \equiv V_{(n)}^{(a)} - \overline{V_{(n)}^{(a)}}$ and for the multi-index version $\widetilde{V}_{(N)}^{(A)} \equiv V_{(N)}^{(A)} - \overline{V_{(N)}^{(A)}}$ where $(A) = (a_1, a_2, \ldots, a_{|A|})$ and $(N) = (n_1, n_2, \ldots, n_{|N|})$. The overbar $\overline{}$ denotes time averaging, $\overline{f(t)} = \frac{1}{T}\int_0^T f(\tau)\,d\tau$. The $\frown$ symbol will denote integrals within the product structure. For example,

$$V_{(\widehat{0,0},1)}^{(\widehat{a,b},c)}(t) = \left(\int_0^t V_{(0,0)}^{(a,b)}(\tau)\,d\tau\right)\left(V_{(1)}^{(c)}(t)\right)$$

4 Averaging and Control

Although the standard form for a 1-homogeneous control system is not in the form required by averaging, Section 2 demonstrated that the variations of constants resulted in an average-able system, eq. (18). The vector field, Y, determines the differential equation,

$$\frac{dy}{d\tau} = \epsilon X_S + \epsilon V_{(1)}^{(a)}(t)\left[Y_a^{\text{lift}}, X_S\right] - \frac{1}{2}\epsilon V_{(1,1)}^{(a,b)}(t)\left\langle Y_a^{\text{lift}} : Y_b^{\text{lift}} \right\rangle,$$

which is T-periodic according to Proposition 4. We may now apply averaging theory. Notably, we may use averaged expansions obtained from a truncated approximation according to Theorem 3.

4.1 Averaged Expansions

First order averaging gives the autonomous differential equation,

$$\frac{dz}{d\tau} = \epsilon X_S + \epsilon \overline{V_{(1)}^{(a)}}\left[Y_a^{\text{lift}}, X_S\right] - \frac{1}{2}\epsilon \overline{V_{(1,1)}^{(a,b)}}\left\langle Y_a^{\text{lift}} : Y_b^{\text{lift}} \right\rangle$$

The first order truncated periodic Floquet mapping is $P(\tau) \approx \text{Id} + O(\epsilon)$, however, an improved version capturing an additional order of ϵ (ref. [18]) gives,

$$P(\tau) = \text{Id} + \epsilon \int_0^t \widetilde{V}_{(1)}^{(a)}(\tau)\,d\tau \left[Y_a^{\text{lift}}, X_S\right]$$
$$- \frac{1}{2}\epsilon \int_0^t \widetilde{V}_{(1,1)}^{(a,b)}(\tau)\,d\tau \left\langle Y_a^{\text{lift}} : Y_b^{\text{lift}} \right\rangle + O(\epsilon^2)$$
$$(20)$$

The second order averaged vector field is found in Table 1, and the second order periodic Floquet mapping is actually given by (20). A future publication will detail higher order expansions. The factorial growth in the expansions cannot be avoided.

4.2 Sinusoidal Inputs for Indirect Actuation

From a controls perspective, the averaged coefficients play an important role. By modulating their values it is possible to effect controlled flow in the direction of the Lie brackets or symmetric products they multiply. Accordingly, there has been much study into the combinations of oscillatory inputs that will uniquely activate select brackets (i.e. *approximate inversion*).

Once the class of periodic input functions has been selected, the idea is to parametrize them so as to have the averaged coefficients be linear in the parameters in a favorable manner. Below are two results that can be found in [20], with $\omega \in \mathbb{Z}^+$.

Theorem 6 *[20] The inputs*

$$v^a(t) = \alpha_{ab}^a \omega \cos(\omega t) \quad \text{and} \quad v^b(t) = \alpha_{ab}^b \omega \cos(\omega t),$$

excite the symmetric product $\left\langle Y_a^{\text{lift}} : Y_b^{\text{lift}} \right\rangle$. *The response will scale according to the product* $\alpha_{ab}^a \alpha_{ab}^b$.

Theorem 7 *[20] In order to excite the symmetric product* $\left\langle Y_a^{\text{lift}} : \left\langle Y_b^{\text{lift}} : Y_a^{\text{lift}} \right\rangle \right\rangle$, *use the inputs*

$$v^a(t) = \alpha_{aba}^a \omega^2 \cos(\omega t) \quad \text{and} \quad v^b(t) = \alpha_{aba}^b \omega \cos(\omega t).$$

The response will scale according to $(\alpha_{aba}^a)^2 \alpha_{aba}^b$.

4.3 Feedback Stabilization

To summarize, we have obtained formulas for the response of 1-homogeneous control systems to an oscillatory control at some arbitrary order. We may analyze the effects of the control inputs on the expansions, leading to an α-parametrized form. Now, we must determine a stabilization feedback strategy. For convenience, introduce a multi-index $\{a_i\} = \{a_1, a_2, \dots a_{k-1}, a_k\}$. with length $|a_i| = k$ that follows the lexigraphical ordering

$$\{a_i\} < \{b_i\} \text{ if } \begin{cases} |a_i| < |b_i| \text{ or} \\ |a_i| = |b_i| \text{ and } \exists k : a_i \leq b_i, \\ \forall i \in \{1, \dots, k-1\} \text{ and } a_k < b_k \end{cases}$$

List the Jacobi-Lie brackets as they appear in the averaged vector field by this ordering (symmetric products are also brackets). Once ordered, let $\widehat{Y}_j$ denote the

Jacobi-Lie brackets, and let $\mathrm{T}^j(\alpha)$ denote their corresponding averaged coefficients. With this ordering, the averaged equations can be put into the form:

$$\begin{aligned} \dot{z} &= X_S(z) + \mathrm{T}^j(\alpha)\widehat{Y}_j(z) \\ &= X_S(z) + B(z)H(\alpha) \end{aligned} \tag{21}$$

where the matrices B and H are,

$$B(z) = [\widehat{Y}_1 \dots \widehat{Y}_N] \text{ and } H(\alpha) = [(\mathrm{T}^1)^T \dots (\mathrm{T}^N)^T]^T.$$

If a system is found to be small-time locally controllable using a set of α-parametrizable Jacobi-Lie brackets, then the averaged system will be fully controllable. Stabilization of the average will imply stability of the actual system. We may stabilize the mondramy map using discrete feedback (c.f. Thm 2 and Cor 1). Alternatively, we may stabilize the logarithm of the monodromy map with continuous feedback (c.f. Prop. 1).

Discretized Feedback We use state error as feedback to modulate the parameters α, converting the problem to periodic discrete feedback. It is very similar to the motion control algorithms [3, 5, 8].

Theorem 8 *Consider a system (16) which is small-time locally controllable at* $x^* \in E$. *Let* $v^k(t)$ *be the set of* α *parametrized, T-periodic input functions where* $k = \{1, \dots, m\}$ *and* $\alpha \in \mathbb{R}^{n-m}$. *Let* $z(t)$, *be the averaged system response to the inputs. Given the averaged system (21), assuming that the m directly controlled states have been linearly stabilized and that the linearization of H, with respect to* α *at* $\alpha = 0$ *and* $z = x^*$, *is invertible on the $(n - m)$ dimensional subspace to control, there exists a* $K \in \mathbb{R}^{(n-m) \times n}$ *such that for*

$$\alpha = -\Lambda K z(T\lfloor t/T \rfloor)$$

where $\Lambda^{(n-m) \times (n-m)}$ *is invertible and* $\lfloor \cdot \rfloor$ *denotes the floor function, the average system response is stabilized.*

Proof: The proof was essentially given in [21], but will be quickly sketched. Given the assumptions on the system, the averaged system (21) is controllable. Linearization with respect to z and α yields

$$\dot{z} = Az + B \left. \frac{\partial H}{\partial \alpha} \right|_{\alpha=0} \alpha = Az + B\Upsilon\alpha.$$

Choosing α constant over a period, the above system can be directly integrated to obtain a discrete, linear system

$$z(k+1) = \hat{A}z(k) + \hat{B}\alpha.$$

The assumptions imply that $\hat{B}$ has a pseudo-inverse, Λ, for the $(n-m)$-dimensional subspace to stabilize. Choose K so that the eigenvalues of $\hat{A} - \hat{B}\Lambda K$ lie in the unit circle. This stabilizes the discrete system (i.e. the monodromy map), and the continuous system with piecewise constant feedback. ∎

Continuous Feedback Alternatively, Proposition 1, implies that if the averaged vector field is stabilized, then

we may infer stability of the original system. Integration over an input period, as required by the discretized feedback strategy, may be avoided.

Theorem 9 *Consider a system of the form (16) which is small-time locally controllable at $x^* \in E$. Let $v^a(t)$ be the corresponding set of α parametrized, T-periodic inputs functions where $a = 1 \ldots m$ and $\alpha \in \mathbb{R}^{n-m}$. Lastly denote by $z(t)$, the averaged system response to the inputs. Given the averaged system (21), assuming that the m directly controlled states have been linearly stabilized and that the linearization of H, with respect to α at $\alpha = 0$ and $z = x^*$, is invertible on the $(n-m)$ dimensional subspace to control, then there exists a $K \in \mathbb{R}^{(n-m)\times n}$ such that for*

$$\alpha = -\Lambda K z(t)$$

where $\Lambda^{(n-m)\times(n-m)}$ is invertible, we have stabilized the average system response.

Proof: Same as Thm. 8, without discretization. ∎

Comments. These theorems stabilize an equilibrium point of the averaged system. To track a trajectory, replace $x(t)$ with $x(t) - x_d(t)$; the system must be locally controllable along the trajectory. If the α-parametrized control input functions do not vanish at the equilibrium, then by Theorem 2, the flow of the actual system stabilizes to an orbit around the fixed point. If, on the other hand, the input functions do vanish at the equilibrium, then Corollary 1 implies that the flow of the actual system stabilizes to the fixed point (i.e. the orbit collapses to the fixed point). For the discretized feedback, the Nyquist criteria is a limiting factor in tracking a trajectory for the indirectly controlled states.

The main difficulty in the feedback procedure is the fact that the feedback cannot be the instantaneous values, but must be the averaged values of the system. Trajectories of the actual flow are related to the averaged flow by the Floquet mapping,

$$x(t) = P(t)(z(t)) \tag{22}$$

We may solve for the average $z(t)$, using the current state $x(t)$. Since $P(t)$ is given by a series expansion, we can easily compute its inverse.

For the discretized feedback strategy this is not a critical factor to consider due to the fact that $P(t)$ is periodic, i.e., $P(kT) = P(0) = \text{Id}$, $k \in \mathbb{Z}^+$. The directly stabilized states do need the average to be used as feedback. In the case that the actual state values $x(t)$ are used as feedback for the directly controlled subsystem, this averaging method will place an upper bound on the feedback gains. The oscillatory inputs should be faster than the natural dynamics of the directly stabilized subsytem, otherwise there will be attenuation of the oscillatory signal.

In an experimental setup, one may utilize averages computed in realtime as continuous feedback. The bene-fit of this latter approach is that the averaging process may serve to filter out any noise in the sensor signals. It may also attenuate the feedback of external disturbances. As the continually computed average, $\bar{x}(t) = \frac{1}{T}\int_{t-T}^{t} x(\tau)\,d\tau$, mbey not be equal to the average $z(t) = P^{-1}(t)(x(t))$, there may be some differences. When performing averaging of sensed measurements, examine $P(t)$ to determine which states require averaging.

5 Example

For brief demonstration of the feedback possibilities, we present a second order nonholonomic integrator. The unlifted control vector fields are,

$$Y_1(x) = \left\{ \begin{array}{c} 1 \\ 0 \\ q_2 \end{array} \right\}, \quad Y_2(x) = \left\{ \begin{array}{c} 0 \\ 1 \\ 0 \end{array} \right\}, \tag{23}$$

where $x = (q, \dot{q}) \in E = TQ$, with $q = (q_1, q_2, q_3)$. The drift term correponds to integration of the second order system, $\ddot{q} = Y_1(x)u^1 + Y_2(x)u^2$. The control inputs $u^a(t)$ decomposed into state feedback and time-periodic terms,

$$u^1(x,t) = -(k_p q_1 + k_v \dot{q}_1) + \tfrac{1}{\epsilon}v^1(t/\epsilon)$$
$$u^2(x,t) = -(k_p q_2 + k_v \dot{q}_2) + \tfrac{1}{\epsilon}v^2(t/\epsilon).$$

The system is STLC at the origin; the symmetric product between Y_1 and Y_2 is a vector field with constant contribution to the third state. Consequently, the inputs

$$v^1(t) = \alpha^1 \cos(t), \quad v^2(t) = \alpha^2 \cos(t),$$

will be used, and only first order averaging is required. Different choices of α^i will correspond to the different feedback strategies that are possible. Define the feedback of the averaged third state to be $e(t) = -(k_1 z_3(t) + k_2 \dot{z}_3(t))$. Now,

$$\alpha^1 = e(t_k) \quad \text{and} \quad \alpha^2 = -1,$$

where $t_k = T\lfloor t/T \rfloor$, will correspond to orbit stabilization with discretized feedback, since zero error will still result in oscillatory actuation. The parametrization,

$$\alpha^1 = \text{sign}(e(t_k))\sqrt{|e(t_k)|} \quad \text{and} \quad \alpha^2 = -\sqrt{|e(t_k)|},$$

will correspond to point stabilization with discretized feedback, since zero error will give no control actuation. Finally,

$$\alpha^1 = \text{sign}(e(t))\sqrt{|e(t)|} \quad \text{and} \quad \alpha^2 = -\sqrt{|e(t)|},$$

will correspond to point stabilization with continuous feedback. The different responses of system (23) to the feedback strategies are plotted in Figure 1, and correspond to (a) discrete feedback/orbit stabilization, (b) discrete feedback/point stabilization, (c) continuous feedback/point stabilization, and (d) improved continuous feedback/point stabilization (Eq. (20)). The parameters are $k_p = 3, k_v = 4, k_1 = 0.5, k_2 = 1.9, \epsilon = 1/7, \omega = 1$. The period is $T = 2\pi/7$, giving a frequency just over one hertz.
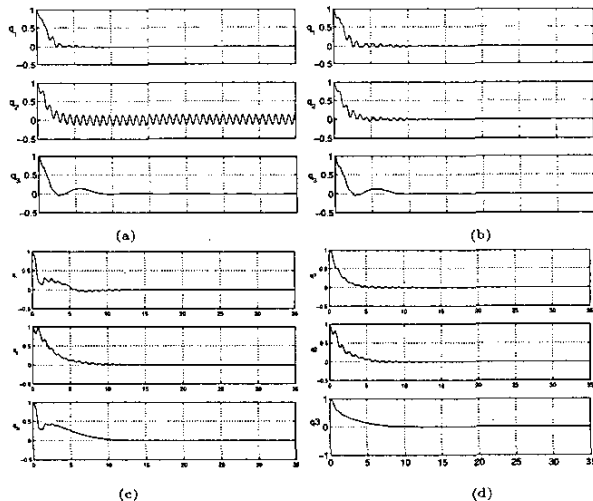
Figure 1: Stabilization

6 Conclusion

We applied [18] to 1-homogeneous control systems, and showed how averaging theory may be used to stabilize a large class of underactuated mechanical systems with drift. By proving feedback stabilization for systems evolving on a vector bundle, E, the theory collapses to known instances from the literature for various choices of E [19], e.g., for example we recover [2, 3, 5, 11, 8].

These ideas have been successfully used to stabilize systems with drift. In [20], can be found a discretely orbit stabilized second order five-state nonholonomic integrator. In [16], we discretely stabilize trajectories of the snakeboard, a constrained mechanical system with symmetries and drift. In [12], we experimentally verified these ideas with a robotic fish using continuous feedback of the sensed average. In [9], McIsaac and Ostrowski employed discretized feedback based on the sensed average for trajectory tracking of an experimental robotic eel. Although they were unable to theoretically prove stability, the theory in this paper can be used to do so.

Since the method results in a controllable linear approximation to the nonlinear system, ideas from robust control theory may be utilized to explicitly determine uncertainty bounds. In work related to ours (it also uses averaging), but for driftless systems, Morin and Samson [13] demonstrate a discretized feedback strategy that is robust to unmodelled dynamics. This is a topic of further research.

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