SUPPLEMENTAL MATERIAL:
Resonantly induced friction and frequency combs in driven nanomechanical systems

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I. THERMALLY INDUCED NONLINEAR FRICTION

Here we discuss the temperature change and the resulting change of the vibration eigenfrequency of a resonantly driven nanomechanical resonator. In the units used in the main text, where we set the effective mass equal to unity, the displacement at the mode antinode has dimension $[q] = \mathrm{g}^{1/2}\mathrm{cm}$, whereas the resonant force has dimension $[F] = \mathrm{g}^{1/2}\mathrm{cm/s}^2$. If we consider a flexural mode in a quasi one-dimensional beam or a string, the displacement as a function of the coordinate x along the beam is $u(x,t) = \rho_{1D}^{-1/2}\phi(x)q(t)$, where ρ_{1D} is the density per unit length and $\phi(x)$ gives the shape of the mode, $\int \phi^2 dx = 1$. The energy in the driving field is $-\int dx f(x,t)u(x,t)$, where $f(x,t)$ is the "true" force per unit length. If we think of the force-induced term in the equation of motion as $[\rho_{1D}\ddot{u}(x)]_F = f(x,t)$, then we have for the force in the equation for q [Eq. (3) of the main text] the expression $F(t) = \rho_{1D}^{-1/2}\int dx\, f(x,t)\phi(x)$. Experiments on nanomechanical systems can be usually well described if one assumes that the force $f(x,t)$ can be factored into a space- and time-dependent parts, $f(x,t) = \tilde{f}_{\rm sp}(x)f_{\rm t}(t)$. Then

$$F(t) = \rho_{1D}^{-1/2} f_{\rm t}(t) \int dx\, f_{\rm sp}(x)\phi(x). \qquad (1)$$

The power dissipated by the force per unit length is $f(x,t)\partial_t u(x,t)$. For a uniform isotropic resonator, the full equation for the increment of the temperature field is

$$C_r\partial_t\delta T = k_r\partial_x^2\delta T + f(x,t)\partial_t u(x,t)/S, \qquad (2)$$

where C_r is the specific heat of the resonator per unit volume, k_r is the thermal conductivity, and S is the cross-section area. We assume here that the temperature is constant across the resonator; an extension to a more general case, including the Zener thermoelastic relaxation [1] (see also [2]) is beyond the scope of this paper.

Equation (2) has to be complemented by the boundary conditions. Often it is assumed that the temperature at the boundary of a nanoresonator is fixed by the support [3], a condition that applies if the support has a large mass and a high thermal conductivity, for example. Equation (2) can be then solved by expanding $\delta T(x,t)$ in the orthogonal eigenmodes $T_n(x)$ of the temperature field in the absence of the drive,

$$(k_r/C_r)\partial_x^2 T_n = -\lambda_n T_n, \quad \int dx T_n(x)T_m(x) = \delta_{nm}$$

(the analysis can be easily extended to a more complicated geometry of the resonator and to more complicated boundary conditions than $T_n = 0$).

The major contribution to the temperature change comes from the mode $T_n(x)$ that has the form close to that of $f(x)\phi(x)$. It depends on the boundary conditions for the temperature field, the spatial structure of the displacement field of the mode $\phi(x)$, and also the coordinate dependence of the driving field.

For dielectric nanoresonators the thermal conductivity is comparatively low. At room temperature $k_r \sim 10^6$ erg/(cm · s · K), and the specific heat is $C_r \sim 10^7$ erg/(cm^3 · K). Then for the resonator length $l_r \sim 10$ μm, the relaxation time of low-lying thermal modes is $\tau_T \sim C_r l_r^2/k_r \sim 10^{-5}$ s. This time significantly exceeds the period (reciprocal frequency) of the vibrational modes, which is typically below $10^{-6} - 10^{-7}$ s. Then the temperature field averages out the oscillating terms in $f(x,t)\partial_t u(x,t)$ in Eq. (2). At the same time, τ_T is typically much shorter than the relaxation times of low-lying vibrational modes, which often exceeds the vibration period by a factor $> 10^4$. In this important case the temperature adiabatically follows the vibration amplitude.

The driving-induced temperature change is then of the form $\delta T(x,t) = \sum c_n(t)T_n(x)$ with

$$c_n = \rho_{1D}^{-1/2}(SC_r)^{-1}\lambda_n^{-1}\int dx\, T_n(x)\phi(x)[f(x,t)\,\dot{q}]_{\rm av}, \quad (3)$$

where $[\ldots]_{\rm av}$ indicates averaging over the vibration period. For low-lying vibrational modes and for a weakly nonuniform driving force $f(x,t)$ the major contribution to $\delta T(x,t)$ comes from low-lying temperature modes, with $\lambda_n \sim 1/\tau_T$. Then the magnitude of the temperature change averaged over the resonator is

$$\delta T \sim l_r^2 k_r^{-1} S^{-1}\,[F(t)\dot{q}(t)]_{\rm av}.$$

We note that the assumption of the temperature being constant in the resonator cross-section requires that $C_r l_\perp^2/k_r$ ($l_\perp$ is the typical transverse dimension) be much shorter than the vibration period, the condition well satisfied for the typical $l_\perp \lesssim 0.1$ μm.

The temperature change causes a change of the vibration frequency. There are several mechanisms of this

effect [4]. One of them is the coupling of the mode to the phonons in the nanoresonator that is nonlinear in the mode strain. This coupling is fairly general. It emerges already from the combination of the standard cubic coupling of the considered low-frequency mode (in particular, a flexural mode) to acoustic phonons and the geometric nonlinearity, but it also comes from other terms in the nonlinear Hamiltonian of the vibrations in the resonator.

Phenomenologically, the mechanism can be described by taking into account the term in the free energy density of the nanoresonator $\delta\mathcal{F}$, which is quadratic in the linear strain tensor $\hat{\epsilon}(\mathbf{r})$ and linear in the temperature change $\delta T(\mathbf{r})$. A simplified form of this term in the one-dimensional model for a flexural mode is

$$\delta\mathcal{F} = -\gamma_{\mathcal{F}} \int dx\, \delta T(x)(\partial_x^2 u)^2, \tag{4}$$

where $\gamma_{\mathcal{F}}$ is the coupling constant; it is determined by the thermal expansion coefficient, the specific heat, and the resonator geometry [4]. The elastic part of the free energy in the harmonic approximation can be written as $\mathcal{F}_E = \frac{1}{2}\gamma_\omega \int dx\, [\partial_x^2 u]^2$ with γ_ω determined in the standard way by the elasticity and the geometry [5]; this term gives the vibration frequency ω_0 for constant temperature. It corresponds to the potential energy of the mode written as $\omega_0^2 q^2/2$.

Then the change of the vibration frequency due to the temperature change is

$$\delta\omega_0 = -(\omega_0 \rho_{1D})^{-1} \gamma_{\mathcal{F}} \int dx\, \delta T(x)(\partial_x^2 \phi)^2. \tag{5}$$

From Eqs. (1), (3), and (5) we find that, for a slow thermal relaxation, the resonant driving induced force in the equation for $q(t)$ is

$$f_T = G_T [F(t)\dot{q}(t)]_{\rm av} q(t), \qquad G_T = 2\gamma_{\mathcal{F}} (\rho_{1D} S C_r)^{-1} \sum_n \lambda_n^{-1} \int dx\, T_n(x)\phi(x) f_{\rm sp}(x)$$
$$\times \int dy\, T_n(y)(\partial_y^2 \phi)^2 \left[\int dx f_{\rm sp}(x)\phi(x)\right]^{-1}. \tag{6}$$

The coefficient G_T gives the coefficient $2m\omega_0\lambda_\omega\lambda_T$ in Eq. (2) of the main text, with the account taken of the spatial dependence of the temperature change.

It should be noted that the coupling (4) also leads to the standard nonlinear friction, with the friction force that corresponds to $q^2\dot{q}$ or $\dot{q}^3$ in the phenomenological picture [4]. However, in the considered case of slow thermal relaxation this force has an extra factor $\propto (\tau_T\omega_0)^{-2}$. Therefore it can be small compared to the force f_T.

In prestressed nanoresonators, an important mechanism of the coupling of the frequency and temperature changes is related to the change of the tension due to thermal expansion, cf. [3] and references therein. It can be analyzed in a way similar to that described above and leads to a qualitatively similar result. If the thermal expansion coefficient is positive, this mechanism leads to the decrease of the vibration frequency with an increasing drive strength, as does the geometric nonlinearity.

II. NONLINEAR OSCILLATIONS IN THE ROTATING FRAME AND THE FREQUENCY COMB IN THE POWER SPECTRUM

In this section we discuss the power spectrum of the oscillator when the resonantly induced friction force is negative and compensates the damping, so that, in the rotating frame, the oscillator vibrates with a given value of its Hamiltonian $H_{\rm RWA}$, i.e., with a given quasienergy. This analysis is not limited to nanomechanical resonators. It applies to any resonantly driven weakly nonlinear oscillator with Duffing (or Kerr, as it is called in quantum optics) nonlinearity.

The spectral density of fluctuations of the displacement $q(t)$ of the resonantly driven oscillator near the driving frequency ω_F has the form

$$S(\omega) = \frac{1}{2t_l}\left|\int_{-t_l}^{t_l} dt\, q(t) e^{i\omega t}\right|^2$$
$$\approx \frac{1}{8t_l\omega_F}\left|\int_{-t_l}^{t_l} dt\, [q_0(t) + ip_0(t)] e^{i(\omega-\omega_F)t}\right|^2 \tag{7}$$

where it is implied that $t_l \to \infty$. We have assumed that $|\omega - \omega_F| \ll \omega_F$ and expressed $q(t)$ in terms of the slowly varying in time quadratures $q_0(t), p_0(t)$ of the oscillator, or equivalently, the coordinate and momentum in the rotating frame, see the main text. In other words, we are using the rotating wave approximation, in which the variables $q_0(t), p_0(t)$ do not contain fast-oscillating terms $\propto \exp(\pm i\omega_F t)$.

The Hamiltonian equations for $q_0(t), p_0(t)$, see Eq. (4) and Fig. 1(a) of the main text, show that q_0, p_0 are periodic functions of time, for a given value of $H_{\rm RWA}$. There-

fore we can write

$$[q_0(t) + ip_0(t)]_H = \sum_m z_m e^{im\Omega t}. \quad (8)$$

Here, $[\cdot]_H$ indicates that the value is evaluated for a given $H_{\rm RWA}$; $\Omega \equiv \Omega(H_{\rm RWA})$ is the oscillation frequency in the rotating frame, i.e., the frequency of the oscillations of the amplitude and phase of the forced vibrations, $\Omega \ll \omega_F$. The Fourier components z_m are also determined by $H_{\rm RWA}$. For a resonantly driven Duffing oscillator they were discussed in the context of quantum theory of interstate switching in the range of bistability of the oscillator [6]. However, the expressions for z_m were not presented and have not been later discussed for the large-amplitude state in the range of bistability, nor have they been discussed where the oscillator has only one stable state.

From Eqs. (7) and (8), the power spectrum of the driven oscillator for a given $H_{\rm RWA}$ is $S(\omega) = S_H(\omega)$,

$$S_H(\omega) = \frac{\pi}{2\omega_F} \sum_m |z_m|^2 \delta(\omega - \omega_F + m\Omega). \quad (9)$$

The spectrum (9) is a frequency comb. It consists of a set of equidistant peaks separated by Ω. The intensity (area) of the peaks is given by the Fourier components z_m; note that, generally, $z_{-m} \neq z_m^*$.

The calculation of the spectrum can be conveniently done by switching from the variables q_0, p_0 to variables $Q_0 = q_0/\zeta, P_0 = p_0/\zeta$ with $\zeta = (4F/3\gamma)^{1/3}\omega_F^{1/2}$, see Eq. (10) of the main text. The Hamiltonian equations of motion for the variables Q_0, P_0 in dimensionless time $\tau = \beta^{1/3}(\delta\omega)t$ read

$$[dQ_0/d\tau]_H = \partial_{P_0} h_{\rm RWA} \equiv P_0(Q_0^2 + P_o^2 - \beta^{-1/3}), \quad (10)$$
$$[dP_0/d\tau]_H = -\partial_{Q_0} h_{\rm RWA} \equiv -Q_0(Q_0^2 + P_0^2 - \beta^{-1/3}) + 1,$$

where $\beta = 3\gamma F^2/32\omega_F^3(\delta\omega)^3$ is the scaled intensity of the driving force. Here we have used the explicit form of the Hamiltonian $h_{\rm RWA}$ given by Eq. (10) of the main text.

As a next step, we introduce an auxiliary variable $X(\tau)$ defined by the expression

$$X(\tau) = Q_0^2(\tau) + P_0^2(\tau) - \beta^{-1/3}. \quad (11)$$

From the expression for $h_{\rm RWA}$ we find

$$X^2(\tau) = 4Q_0(\tau) + 4h_{\rm RWA} + \beta^{-2/3}. \quad (12)$$

From Eqs. (10) - (12) we obtain an equation for $X(\tau)$ in the form

$$\frac{dX}{d\tau} = 2P_0$$
$$= \pm\frac{1}{2}\left\{(a_1 - X)(X - a_2)[(X - a_3)^2 + a_4^2]\right\}^{1/2}, \quad (13)$$

with $a_1 > a_2$ being the real roots and $a_3 \pm ia_4$ being the complex roots of the equation

$$(x^2 - \beta^{-2/3} - 4h_{\rm RWA})^2 - 16x - 16\beta^{-1/3} = 0. \quad (14)$$

As seen from this equation, $a_3 = -(a_1 + a_2)/2$. Also, $a_1 > |a_2|$.

We are interested in the oscillating trajectory with $a_2 \leq X(\tau) \leq a_1$; the sign of $dX/d\tau$ in Eq. (13) is changed at the turning points a_1, a_2. It follows from Eq. (13) (see [7], 3.145.2) that $X(\tau)$ is expressed in terms of the Jacobi elliptic functions as

$$X(\tau) = \frac{a_1b_2 + a_2b_1 - (a_1b_2 - a_2b_1)cn(\tau'|m_h)}{b_1 + b_2 + (b_1 - b_2)cn(\tau'|m_h)}$$
$$b_1 = |a_3 + ia_4 - a_1|, \qquad b_2 = |a_3 + ia_4 - a_2|$$
$$m_h = [(a_1 - a_2)^2 - (b_1 - b_2)^2]/4b_1b_2 \quad (15)$$

where $\tau' = (b_1b_2)^{1/2}\tau/2$.

The Jacobi elliptic function $\mathrm{cn}(\tau'|m_h)$ has a real period $4K(m_h)$, where $K(m_h)$ is the complete elliptic integral of the first kind. Therefore the period of vibrations with a given quasienergy $H_{\rm RWA}$ is

$$\mathbb{T}(H_{\rm RWA}) \equiv \frac{2\pi}{\Omega} = 8K(m_h)/\beta^{1/3}(b_1b_2)^{1/2}(\delta\omega). \quad (16)$$

This expression describes the dependence of the spacing between the frequency comb lines Ω on the quasienergy, for given intensity and frequency of the drive.

From Eq. (8), the Fourier components z_m that determine the intensity of the comb lines are given by the expression

$$z_m = \frac{\zeta}{4K}\int_0^{4K} d\tau'(Q_0 + iP_0)\exp(-im\pi\tau'/2K) \quad (17)$$

where $K \equiv K(m_h)$; in what follows we use the conventional notation K for the elliptic integral, it should not be confused with the parameter K used in the main text for the ratio of the friction and gain coefficients.

Equations (11) - (15) show that Q_0, P_0 are elliptic functions. As functions of τ', along with the real period $4K(m_h)$, they have the imaginary period $4iK'(m_h) \equiv 4iK(1 - m_h)$. Therefore the Fourier components (17) can be calculated by integrating over a rectangular contour in the complex τ'-plane that goes from $-2K$ to $2K$, then to $2K + 4iK'$, then to $-2K + 4iK'$, and then back to $-2K$. The integrals over the vertical sections of the contour cancel, whereas on the upper horizontal section Q_0, P_0 are the same as on the real axis.

Inside the contour, $X(\tau)$ has two poles. They are located at a purely imaginary $\tau' = \tau_p'^{(1,2)}$ given by the equation

$$\mathrm{cn}(\tau_p'^{(1,2)}) = -\frac{b_1 + b_2}{b_1 - b_2}, \quad |\tau_p'^{(1)}| < |\tau_p'^{(2)}|. \quad (18)$$

Near the pole $X(\tau)$ has the form

$$X(\tau) \approx (-1)^{j+1}\, i\frac{(b_1b_2)^{1/2}}{\tau' - \tau_p'^{(j)}} \qquad (j = 1, 2), \quad (19)$$

From Eqs. (12), (13), and (19), $Q_0 + iP_0$ has a pole of order 2 at $\tau' = \tau_p'^{(2)}$, with

$$Q_0 + iP_0 \approx 2Q_0 \approx -\frac{1}{2}\frac{b_1 b_2}{(\tau' - \tau_p'^{(2)})^2}. \qquad (20)$$

One can see from the expression for the Hamiltonian $h_{\rm RWA}$ that near the pole there are no corrections to Eq. (20) that would be $\propto (\tau' - \tau_p'^{(2)})^{-1}$. Therefore

$$z_m = -\frac{m\pi^2\zeta b_1 b_2}{8K^2}\,\frac{\exp(-im\pi\tau_p'^{(2)}/2K)}{1-\exp(2m\pi K'/K)}. \qquad (21)$$

Expression (21) reduces the problem of calculating the frequency comb to a solution of the 4th order polynomial equation (14) and the transcendental equation (18). By construction, $0 < {\rm Im}\ \tau_p'^{(2)} < 4K'$. Therefore for large $m > 0$ the intensities of the comb lines, which are $\propto |z_m|^2$, fall off as $\exp[-m\pi(4K' - {\rm Im}\tau_p'^{(2)})/K]$. On the other hand, for $m < 0$ and $|m| \gg 1$ they fall off as $\exp[-|m|\pi{\rm Im}\tau_p'^{(2)}/K]$. For not too small $H_{\rm RWA} - (H_{\rm RWA})_{\rm min}$ the spectral comb displays several pronounced equidistant spectral lines. We note that the $|z_m|^2 \neq |z_{-m}|^2$ and the comb is therefore asymmetric.

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