

TESTING FOR SEPARABILITY IS HARD

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ABSTRACT. This paper shows that it is computationally hard to decide (or test) if a consumption data set is consistent with separable preferences.

1. INTRODUCTION

The assumption of separable preferences is ubiquitous in economics. Economists assume separability of preferences, virtually without ever testing this assumption empirically. Here I argue that there is a reason for such lack of empirical scrutiny: The problem of deciding if a data set is consistent with separable preferences is computationally hard. There cannot exist a test that is practical on large data sets, and that serves to test for separability.

Every empirical study on consumption assumes, explicitly or implicitly, that preferences are separable. For example, data on supermarket purchases are used in isolation from other consumption decisions. Or data on consumption in one year is used without regard for intertemporal consumption decisions. The choice among different goods is analyzed while ignoring any consumption/leisure tradeoffs, and independently of the allocation of financial assets. All such analyses, which depend on certain compartmentalizations in the economy, rely on the assumption on separability. It is hard to imagine a paper in applied economics that does not make some use of separability.

A few authors have proposed tests for separability. Varian (1983) has a test that involves solving a system of polynomial inequalities. Cherchye, Demuyneck, and De Rock (2011) provide a computational approach to handling Varian's system of inequalities. Quah (2012) has a test which is finite, meaning that one would need to check if the data fit a finite number of configurations. These tests are all hard to take

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to data because they are computationally hard, and may be infeasible in large datasets.

My contribution here is to show that separability is inherently hard. Specifically, that is NP complete. This implies that it is as hard as any problem in the class NP, a class of problems that contains all the natural decision problems studied in computer science. NP complete problems are widely regarded as intractable.

A result similar to mine has already appeared in Cherchye, Demuyne, and De Rock (2011). The main difference is that their result is asymptotic in the number of goods as well as the number of observations. Cherchye et al. proved that separability is hard if one has a large data set *and many goods*. In my view, it is important to establish the result with a fixed number of goods because most studies in economics use only a handful of goods. The construction used in the proof of the theorem below uses 9 goods, the same number as in the classical study on consumption by Deaton (1974).¹ The number 9 is not a limitation of old data sets and classical studies; recent studies on consumption also use a small number of goods (for example Cherchye, Demuyne, and De Rock (2011) use 15 goods). More broadly speaking, asymptotics on the size of the dataset simply seem more fundamental than on the number of goods.

Finally, another difference with Cherchye, Demuyne, and De Rock (2011) is that they focus on conditions for concave separability. My result is on separability alone.

2. TESTING SEPARABILITY

We take consumption space to be \mathbf{R}_+^{n+m} . The set of available goods is partitioned in two, and we write a consumption bundle as $x = (z, o) \in \mathbf{R}_+^{n+m}$. There are n goods of “type z ” and m of “type o .”

A *data set* is a collection (x_k, p_k) , $k = 1, \dots, K$ in which for every k $x_k \in \mathbf{R}_+^{n+m}$ is a consumption bundle purchased at prices $p_k \in \mathbf{R}_{++}^{n+m}$. Let $x_k = (z_k, o_k) \in \mathbf{R}_+^n \times \mathbf{R}_+^m$.

A data set (x_k, p_k) , $k = 1, \dots, K$ is *rationalizable by separable preferences* if there are monotone increasing functions $u : \mathbf{R}_+^n \rightarrow \mathbf{R}$ and $v : \mathbf{R}^{1+m} \rightarrow \mathbf{R}$ such that

$$v(u(z), o) < v(u(z_k), o_k)$$

for all $(z, o) \in \mathbf{R}_+^{n+m}$ with $p_k^z \cdot z + p_k^o \cdot o \leq p_k^z \cdot z_k + p_k^o \cdot o_k$ and $(z, o) \neq (z_k, o_k)$.

¹This means that the problem is hard already with 9 goods. But the construction can probably be improved to use an even smaller number of goods.

Theorem. *The problem of deciding if a dataset has a separable rationalization is NP-complete.*

3. PROOF

3.1. Notation and definitions. By e_i we denote the i th unit vector in \mathbf{R}^n , that is the vector that has zero in every entry except the i th, in which it has a one. Write e_{12} for $e_1 + e_2$. The *embedding* of \mathbf{R}^n into \mathbf{R}^m , with $n < m$ is the function that maps a vector $x \in \mathbf{R}^n$ into the vector $(x, 0, \dots, 0)$ in \mathbf{R}^m , which coincides with x_i in the first n entries and then has a zero in the remaining entries.

A *graph* is a set X together with a binary relation $R \subseteq X \times X$. We write $x R y$ for $(x, y) \in R$. A sequence x_1, \dots, x_K in X is a *path* from x_1 to x_K if

$$x_K R x_{K-1} R x_{K-2} \cdots x_2 R x_1.$$

A graph, or the binary relation R , is *acyclic* if for every pair x and x' , with $x \neq x'$, if there is a path from x to x' then there is no path from x' to x .

3.2. Construction. We shall reduce from three-satisfiability. Consider a formula with L clauses, C_1, \dots, C_L , involving the variables x_1, \dots, x_I .

The strategy for the reduction is as follows. We introduce a pair of bundles z_i^1 and z_i^2 for each variable i . These bundles are not comparable by revealed preferences, but they are embedded into a configuration of prices and bundles such that any rationalizing preference must reflect an assignment of truth/falsehood to each variable that makes all the clauses C_l true. This is accomplished using separability: in fact separability is crucial to make the construction work with a fixed number of goods. The bundles z_i^1 and z_i^2 live in \mathbf{R}^2 and they are shifted by adding different amounts of the other goods so that they can play the same role in different clauses. By separability, the comparison between z_i^1 and z_i^2 must be the same in all shifted instances.

We first (Step 1) develop the construction for a single clause. Then (Step 2) we tie the different clauses together. The formal construction follows.

For each variable x_i , define the following vectors in \mathbf{R}_+^2 : $z_i^1 = (i, 1/i)$ and $z_i^2 = (i + 1/2, 1/(i + 1/2))$. Let $\tau : \{x_1, \dots, x_I\} \rightarrow \{0, 1\}$ be a truth-table for the variables x_1, \dots, x_I . Define a binary relation B on $\{z_i^q : q = 1, 2; i = 1, \dots, I\}$ from τ by $z_i^1 B z_i^2$ if $\tau(x_i) = 1$ and $z_i^2 B z_i^1$ if $\tau(x_i) = 0$.

Note that each of the vectors in $\{z_i^q : q = 1, 2; i = 1, \dots, K\}$ lie on the boundary of the convex set $A = \{(\theta_1, \theta_2) : \theta_2 \geq 1/\theta_1 \text{ and } \theta_1 > 0\}$.

Define p_i^q to be such that the hyperplane

$$\{(\theta_1, \theta_2) : p_i^q \cdot (\theta_1, \theta_2) = 1\}$$

supports A at z_i^q . (A simple calculation reveals that $p_i^1 = (1/(2i), i/2)$ and $p_i^2 = (1/(2i+1), i/2 + 1/4)$, but this does not play a role in the sequel.)

As a consequence of these definitions, we obtain the following

Lemma 1. *For all $i, i' = 1, \dots, I$, and all $q, q' = 1, 2$, we have*

$$p_i^q \cdot z_i^q = 1 < p_i^q \cdot z_{i'}^{q'}$$

when $i' \neq i$ and/or $q' \neq q$.

To make the sequel easier to follow, we write $\rho(z_i^q)$ for p_i^q . Define two positive numbers, ε and M as follows. Let ε be such that

$$(1) \quad 1 < \rho(z_i^q) \cdot z_i^q + \varepsilon \rho(z_i^q) \cdot (1, 1) < \rho(z_i^q) \cdot z_{i'}^{q'}$$

when $i' \neq i$ and/or $q' \neq q$. In second place, let M be such that

$$(2) \quad \rho(z_i^q) \cdot z_{i'}^{q'} + \varepsilon \rho(z_i^q) \cdot (1, 1) < M$$

when $i' \neq i$ and/or $q' \neq q$.

3.2.1. Step 1: The construction for a single clause. Consider a single clause C . Say that $C = y_i \vee y_j \vee y_h$, with $y \in \{x, \bar{x}\}$. Let $\hat{z}_i^q = z_i^q$ and $\hat{p}_i^q = p_i^q$ if $y_i = x_i$, and $\hat{z}_i^q = z_i^{3-q}$ and $\hat{p}_i^q = p_i^{3-q}$ if $y_i = \bar{x}_i$. Define \hat{z}_j^q , \hat{z}_h^q , \hat{p}_j^q , and \hat{p}_h^q analogously.

We are going to map the clause C into a dataset in \mathbf{R}^6 .

Consider the observations $\{(w_i, r_i) : i = 1, 2, 4, 5, 7, 8\}$ and the set of bundles $\{w_i : i = 1, \dots, 9\} \subset \mathbf{R}_+^8$ defined as follows:

$$\begin{aligned} w_1 &= \hat{z}_i^2 + e_3 & r_1 &= \rho(\hat{z}_i^2) + Me_3 + 2M(e_5 + e_6) \\ w_2 &= \hat{z}_j^1 + \varepsilon(e_1 + e_2) + e_4 & r_2 &= \rho(\hat{z}_j^1) \\ w_3 &= \hat{z}_j^1 + e_3 + e_7 \\ w_4 &= \hat{z}_j^2 + e_3 + e_7 & r_4 &= \rho(\hat{z}_j^2) + Me_3 + 2M(e_4 + e_6) \\ w_5 &= \hat{z}_h^1 + \varepsilon(e_1 + e_2) + e_5 & r_5 &= \rho(\hat{z}_h^1) \\ w_6 &= \hat{z}_h^1 + e_3 + e_8 \\ w_7 &= \hat{z}_h^2 + e_3 + e_8 & r_7 &= \rho(\hat{z}_h^2) + Me_3 + 2M(e_4 + e_5) \\ w_8 &= \hat{z}_i^1 + \varepsilon(e_1 + e_2) + e_6 & r_8 &= \rho(\hat{z}_i^1) \\ w_9 &= \hat{z}_i^1 + e_3 \end{aligned}$$

This means that bundles w_i are purchased at prices r_i , for $i = 1, 2, 4, 5, 7, 8$. The bundles w_3, w_6, w_9 are added for convenience.

Let X be the set of products taken from the first two, and the last four, entries of the vectors w_k . That is, X is the set of pairs (z, o) , with $z \in \mathbf{R}^2$ and $o \in \mathbf{R}^6$, such that there is z', o', w_k and w_l with $w_k = (z, o')$

and $w_l = (z', o)$. Write X_z for the projection of X onto \mathbf{R}^2 , and X_o for the projection of X onto \mathbf{R}^6 ; so $X = X_z \times X_o$.

The following tables contain the results of calculating $r_k \cdot w_t$ (so that $r_k \cdot w_t$ is the content of the cell with row r_k and column w_t).

	w_1	w_2	w_3
r_1	$1 + M$	$\rho(\hat{z}_i^2) \cdot \hat{z}_j^1 + \varepsilon \rho(\hat{z}_i^2) \cdot e_{12}$	$\rho(\hat{z}_i^2) \cdot \hat{z}_j^1 + M$
r_2	$\rho(\hat{z}_j^1) \cdot \hat{z}_i^2$	$1 + \varepsilon \rho(\hat{z}_j^1) \cdot e_{12}$	1
r_4	$\rho(\hat{z}_j^2) \cdot \hat{z}_i^2 + M$	$\rho(\hat{z}_j^2) \cdot \hat{z}_j^1 + \varepsilon \rho(\hat{z}_j^2) \cdot e_{12} + 2M$	$\rho(\hat{z}_j^2) \cdot \hat{z}_j^1 + M$
r_5	$\rho(\hat{z}_h^1) \cdot \hat{z}_i^2$	$\rho(\hat{z}_h^1) \cdot \hat{z}_j^1 + \varepsilon \rho(\hat{z}_h^1) \cdot e_{12}$	$\rho(\hat{z}_h^1) \cdot \hat{z}_j^1$
r_7	$\rho(\hat{z}_h^2) \cdot \hat{z}_i^2 + M$	$\rho(\hat{z}_h^2) \cdot \hat{z}_j^1 + \varepsilon \rho(\hat{z}_h^2) \cdot e_{12} + 2M$	$\rho(\hat{z}_h^2) \cdot \hat{z}_j^1 + M$
r_8	$\rho(\hat{z}_i^1) \cdot \hat{z}_i^2$	$\rho(\hat{z}_i^1) \cdot \hat{z}_j^1 + \varepsilon \rho(\hat{z}_i^1) \cdot e_{12}$	$\rho(\hat{z}_i^1) \cdot \hat{z}_j^1$

	w_4	w_5
r_1	$\rho(\hat{z}_i^2) \cdot \hat{z}_j^2 + M$	$\rho(\hat{z}_i^2) \cdot \hat{z}_h^1 + \varepsilon \rho(\hat{z}_i^2) \cdot e_{12} + 2M$
r_2	$\rho(\hat{z}_j^1) \cdot \hat{z}_j^2$	$\rho(\hat{z}_j^1) \cdot \hat{z}_h^1 + \varepsilon \rho(\hat{z}_j^1) \cdot e_{12}$
r_4	$1 + M$	$\rho(\hat{z}_j^2) \cdot \hat{z}_h^1 + \varepsilon \rho(\hat{z}_j^2) \cdot e_{12}$
r_5	$\rho(\hat{z}_h^1) \cdot \hat{z}_j^2$	$1 + \varepsilon \rho(\hat{z}_h^1) \cdot e_{12}$
r_7	$\rho(\hat{z}_h^2) \cdot \hat{z}_j^2 + M$	$\rho(\hat{z}_h^2) \cdot \hat{z}_h^1 + \varepsilon \rho(\hat{z}_h^2) \cdot e_{12} + 2M$
r_8	$\rho(\hat{z}_i^1) \cdot \hat{z}_j^2$	$\rho(\hat{z}_i^1) \cdot \hat{z}_h^1 + \varepsilon \rho(\hat{z}_i^1) \cdot e_{12}$

	w_6	w_7	w_8	w_9
r_1	$\rho(\hat{z}_i^2) \cdot \hat{z}_h^1 + M$	$\rho(\hat{z}_i^2) \cdot \hat{z}_h^2 + M$	$\rho(\hat{z}_i^2) \cdot \hat{z}_i^1 + \varepsilon \rho(\hat{z}_i^2) \cdot e_{12} + 2M$	$\rho(\hat{z}_i^2) \cdot \hat{z}_i^1 + M$
r_2	$\rho(\hat{z}_j^1) \cdot \hat{z}_h^1$	$\rho(\hat{z}_j^1) \cdot \hat{z}_h^2$	$\rho(\hat{z}_j^1) \cdot \hat{z}_i^1 + \varepsilon \rho(\hat{z}_j^1) \cdot e_{12}$	$\rho(\hat{z}_j^1) \cdot \hat{z}_i^1$
r_4	$\rho(\hat{z}_j^2) \cdot \hat{z}_h^1 + M$	$\rho(\hat{z}_j^2) \cdot \hat{z}_h^2 + M$	$\rho(\hat{z}_j^2) \cdot \hat{z}_i^1 + \varepsilon \rho(\hat{z}_j^2) \cdot e_{12} + 2M$	$\rho(\hat{z}_j^2) \cdot \hat{z}_i^1 + M$
r_5	1	$\rho(\hat{z}_h^1) \cdot \hat{z}_h^2$	$\rho(\hat{z}_h^1) \cdot \hat{z}_i^1 + \varepsilon \rho(\hat{z}_h^1) \cdot e_{12}$	$\rho(\hat{z}_h^1) \cdot \hat{z}_i^1$
r_7	$\rho(\hat{z}_h^2) \cdot \hat{z}_h^1 + M$	$1 + M$	$\rho(\hat{z}_h^2) \cdot \hat{z}_i^1 + \varepsilon \rho(\hat{z}_h^2) \cdot e_{12}$	$\rho(\hat{z}_h^2) \cdot \hat{z}_i^1 + M$
r_8	$\rho(\hat{z}_i^1) \cdot \hat{z}_h^1$	$\rho(\hat{z}_i^1) \cdot \hat{z}_h^2$	$1 + \varepsilon \rho(\hat{z}_i^1) \cdot e_{12}$	1

Define the graph (X, R) by letting $w R w'$ if and only if $w = w_k$ for some k and $r_k \cdot w_k > r_k \cdot w'$. Careful (if tedious) inspection of the calculations above, (and using inequalities (1) and (2)) reveal that that

$$(3) \quad R = \{(w_1, w_2), (w_2, w_3), (w_4, w_5), (w_5, w_6), (w_7, w_8), (w_8, w_9)\}.$$

Let τ be a truth table for which our clause $C = y_i \vee y_j \vee y_h$ is true. That is: $\tau(y) = 1$ for at least one $y \in \{y_i, y_j, y_h\}$. Let B be the binary relation induced by τ (see the definition above).

Note that B is an acyclic binary relation on $\{\hat{z}_k^q : q = 1, 2, k = i, j, h\}$. None of the vectors in $\{\hat{z}_k^q : q = 1, 2, k = i, j, h\}$ is larger than the other (in the usual order on \mathbf{R}^2). There is therefore a function $u : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ for which $u(z) > u(z')$ whenever $z B z'$ or $z > z'$.

Let R' be defined by: $w R' w'$ if (1) $w R w'$ or (2) if there are $z, z' \in X_Z$ and o such that $w = (z, o)$, $w' = (z', o)$ and $u(z) > u(z')$. Say that pair

(w, w') is a 1-edge if $w R' w'$ for reason (1) and a 2-edge if $w R' w'$ for reason (2).

Lemma 2. (X, R') is acyclic.

Proof. Suppose, towards a contradiction, that there is a cycle. The cycle cannot consist purely of 2-edges because each 2-edge implies an increase in $u(z)$. Therefore some of the edges must consist be 1-edges. Inspection of the graph (X, R) reveals that all of the edges in R must then be part of this cycle. The reason is that edges can connect (z, o) and (z', o') with $o \neq o'$ only if they belong to R . Then a cycle can only be closed if it involves *all* of the edges in R . Such a cycle would define a path from w_9 to w_1 , from w_3 to w_4 , and from w_6 to w_7 . Each of these paths would involve only 2-edges. By definition of 2-edges, then $u(w_1) > u(w_9)$, $u(w_4) > u(w_3)$, and $u(w_7) > u(w_6)$. But $u(w_1) > u(w_9)$ can only be true if $\hat{z}_i^2 B \hat{z}_i^1$. Similarly, we obtain that $\hat{z}_j^2 B \hat{z}_j^1$ and $\hat{z}_h^2 B \hat{z}_h^1$. This contradicts that C is true under the truthtable τ . \square

Lemma 3. There is a function v such that $v(u(z), o) > v(u(z'), o')$ whenever $(z, o) R' (z', o')$ or $u(z) \geq u(z')$ and $o > o'$.

Proof. Let R'' be a binary relation on $u(\mathbf{R}_+^2) \times \mathbf{R}_+^4$ defined by $(s, o) R'' (s', o')$ if $(s, o) > (s', o')$ or if there is z and z' such that $s = u(z)$ and $s' = u(z')$ and $(z, o) R (z, o)$. By Lemma 2 and the observation that non of the vectors in X_o is comparable in the usual Euclidean order, the relation R'' is acyclic. The set $u(X_z) \times X_o \cup \mathbb{Q} \cap u(\mathbf{R}_+^2) \times \mathbb{Q}_+^4$ is countable and order dense, so there is a function v as required by the statement of the lemma. \square

3.2.2. Step 2: The construction for L clauses. For each clause C_l , define the bundles w_t and prices r_k as above. Let w_t^l be defined as the vector in \mathbf{R}^9 obtained by embedding w_t and adding $M2^l e_9$. Let r_k^l be the sum of e_9 and the embedding of r_k in \mathbf{R}^9 . As a result of these definitions, $r_k^l \cdot w_t^k = r_k \cdot w_t + M2^l$. Importantly, for a fixed l , the comparison of $r_k^l \cdot w_k^k$ and $r_k^l \cdot w_t^k$ is the same as the comparison of $r_k \cdot w_k$ and $r_k \cdot w_t$ performed in Step 1.

Define $X^l = X_z^l \times X_o^l$ from w_1^l, \dots, w_9^l in the same way as $X = X_z \times X_o$ was defined. Define R^l from w_t^l and r_k^l in the same way as R was defined. Note that $r_k^l \cdot w_t^k$ only differ from $r_k \cdot w_t$ in the constant $M2^l$, so the graphs (X, R) and (X^l, R^l) are the same once we identify w_k with w_k^l .

Let $\bar{X} = \cup_{l=1}^L X^l$. Define a binary relation \bar{R} on \bar{X} by $w \bar{R} w'$ iff there is some l and some t such that $w = w_t^l$ and $r_t^l \cdot w_t^l > r_t^l \cdot w'$. Note that \bar{R} has the following properties

- (1) \bar{R} coincides with R^l on X^l ($R^l = \bar{R} \cap X^l \times X^l$);

- (2) if $w \in X^l$ and $w' \in X^{l'}$, with $l' < l$, then $w \bar{R} w'$ and it is false that $w' \bar{R} w$.

Let τ be a truthtable for which all clauses C^l are true. Such a truth table defines a binary relation B on $\bar{X}_z = \cup_l X_z^l$. The binary relation is acyclic, as there are no pairs of subsequent edges in B . As in Step 1, there is a function $u : \mathbf{R}_+^2 \rightarrow \mathbf{R}$ for which $u(z) > u(z')$ whenever $z B z'$ or $z > z'$.

Let \bar{R}' be defined by: $w R' w'$ if (1) $w R w'$ or (2) if there are $z, z' \in \bar{X}_z$ and o such that $w = (z, o)$, $w' = (z', o)$ and $u(z) > u(z')$.

Lemma 4. (\bar{X}, \bar{R}') is acyclic.

Proof. By the second property of \bar{R} , there cannot exist a cycle that contains an edge going from $w \in X^l$ to $w' \in X^{l'}$ with $l \neq l'$. Therefore any cycle must contain only vertexes in some X^l . By Lemma 1, there is no such cycle. \square

The following result, which finishes the proof, follows from Lemma 4 in a similar way to how Lemma 3 follows from Lemma 1.

Lemma 5. *There is a function v such that $v(u(z), o) > v(u(z'), o')$ whenever $(z, o) \bar{R}' (z', o')$ or $u(z) \geq u(z')$ and $o > o'$.*

3.3. Some remarks on the proof. 1) The construction for one clause can be summarized in the diagram depicted in Figure 1. The horizontal axis represents \mathbf{R}^2 and the vertical axis \mathbf{R}^6 . The directions of the arrows reflect the binary relation R : $w_1 R w_2$ is denoted by the arrow pointing from w_1 to w_2 , and so on. Note that the pairs of bundles w_9 and w_1 , w_3 and w_4 , and w_6 and w_7 share their o component. These pairs are the only ones that share an o component.

2) The presence of the bundles w_2 , w_5 and w_8 may need an explanation. We need to use them for the following reason. Consider the case of w_2 . We want to have $w_1 R w_3$, but not that $w_1 R w_4$. This is difficult because w_3 and w_4 differ only in the z component. By introducing w_2 , which dominates w_3 but not w_4 , we can achieve the desired relations.

3) I have taken a shortcut in the proof by introducing exponential quantities w_i^l . They are there to make sure that certain quantities are large enough, and are easily avoided.

4) The definition of the bundles z_k^q and supporting prices $\rho(z_i^q)$ may involve using irrational numbers, which is questionable from an algorithmic viewpoint. Since the inequalities in Lemma 1 are strict, these numbers can be replaced with rational numbers to have the construction only operate with “discrete” objects. (In a similar fashion, the

primitive dataset should only involve consumption bundles and prices with rational entries.)

5) The main contribution here is to do the construction for a fixed number of goods. If one is free to use any number of different goods to capture the different edges, then it is easy to recreate any given graph as a revealed preference binary relation. When the number of goods is fixed, not all graphs can be revealed preference relations (the best known example is the case when there are two goods, in which the weak axiom of revealed preference suffices for rationalizability). So it is important to be able to work with a rather specific graph, the one depicted in Figure 1. The ability to do the reduction for a fixed number of goods relies, among other things, on using 3SAT.

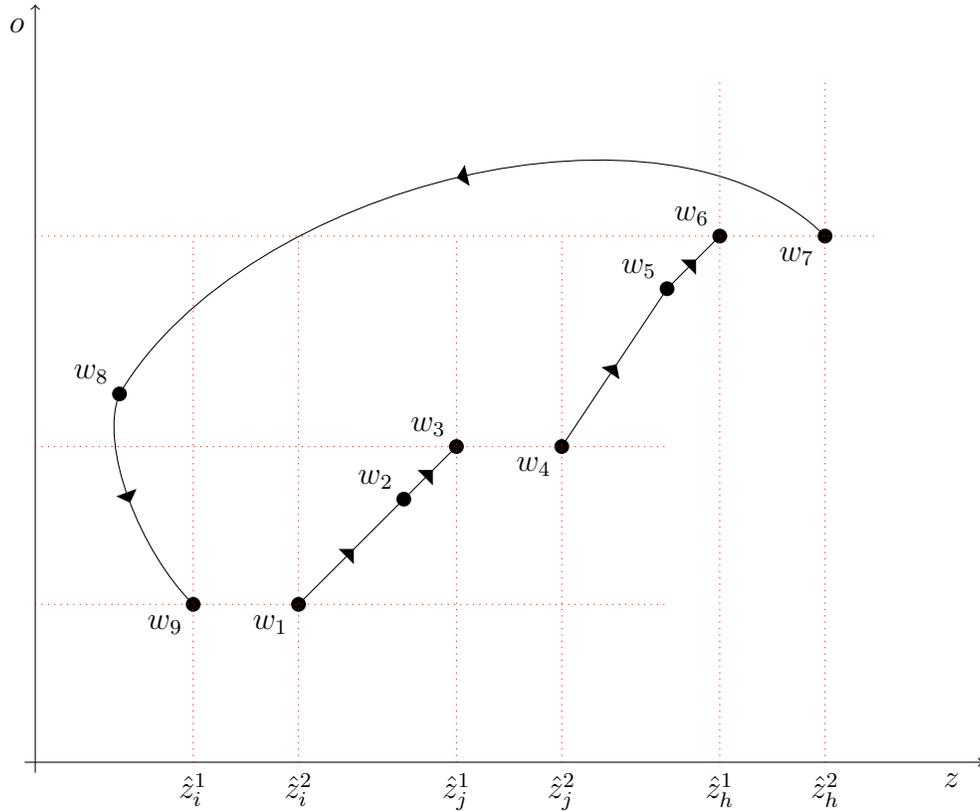


FIGURE 1. The construction for a single clause.

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