Competitive Online Optimization under Inventory Constraints

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ABSTRACT

This paper studies online optimization under inventory (budget) constraints. While online optimization is a well-studied topic, versions with inventory constraints have proven difficult. We consider a formulation of inventory-constrained optimization that is a generalization of the classic one-way trading problem and has a wide range of applications. We present a new algorithmic framework, CR-Pursuit, and prove that it achieves the optimal competitive ratio among all deterministic algorithms (up to a problem-dependent constant factor) for inventory-constrained online optimization. Our algorithm and its analysis not only simplify and unify the state-of-the-art results for the standard one-way trading problem, but they also establish novel bounds for generalizations including concave revenue functions. For example, for one-way trading with price elasticity, CR-Pursuit achieves a competitive ratio within a small additive constant (i.e., 1/3) to the lower bound of $\ln\theta + 1$, where θ is the ratio between the maximum and minimum base prices.

CCS CONCEPTS

• **Theory of computation → Online algorithms**; *Design and analysis of algorithms*; • **Applied computing** → *Decision analysis*.

KEYWORDS

Inventory Constraints; Revenue Maximization; Online Algorithms; One-way Trading; Price Elasticity

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1 INTRODUCTION

In this paper, we focus on an important class of online optimization problems that has proven challenging: *online optimization under inventory (budget) constraints (OOIC)*. In these problems, a decision maker has a fixed amount of inventory, e.g., airlines selling flight tickets, and must make sequential decisions without knowledge of future revenue functions or the stopping time T. Further, the strict inventory constraint means that current actions have consequences for future rounds. As a result of this entanglement, positive results have only been possible for inventory constrained online optimization in special cases to this point, e.g., linear revenue function considered in the one-way trading problem [2].

In the one-way trading problem [2], a trader owns some assets (e.g., dollars) and aims to exchange them into other assets (e.g., yen) as much as possible, depending on the price (e.g., exchange rate). There is a long history of work on one-way trading, e.g., [2, 4], and OOIC includes both the classic one-way trading problem and variations with concave revenue functions and price elasticity.

Applications. Beyond that, OOIC also captures a variety of other applications. Three examples that have motivated our interest in OOIC are (i) power contingency reserve markets [1], (ii) network spectrum trading [3], and (iii) online advertisement, further illustrated in the main work.

We formulate OOIC as follows:

$$\text{OOIC}: \quad \max \quad \sum_{t=1}^{T} g_t(v_t) \tag{1}$$

$$\text{s.t.} \quad \sum_{t=1}^{T} v_t \le \Delta, \tag{2}$$

$$\text{var.} \quad v_t \ge 0, \forall t \in [T]. \tag{3}$$

We are interested in the online setting where at each time $t \in [T]$, upon observing revenue function $g_t(\cdot)$, an irrevocable quantity decision v_t needs to be made, yielding revenue $g_t(v_t)$. The overall objective is to maximize the aggregate revenue, while respecting the inventory constraint $\sum_{t\in[T]} v_t \le \Delta$. We assume that $g_t(\cdot), \forall t \in [T]$, satisfy the following conditions:

- $g_t(v)$ is concave, increasing, and differentiable over $[0, \Delta]$;
- $g_t(0) = 0$;
- $p(t) \triangleq g_t'(0) > 0$ and $p(t) \in [m, M]$.

Algorithm 1 CR-Pursuit(π) Online algorithm

Input: $\pi > 1$, Δ
Output: $\bar{v}_t, t \in [T]$
while t is not the last slot **do**
  Obtain $\eta_{OPT}\left(\sigma^{[1:t]}\right)$ by solving the convex problem OOIC given the input until t, i.e., $\sigma^{[1:t]}$
  Obtain a $\bar{v}_t \in [0, \Delta]$ that satisfies (4)
end while

The first condition is a smoothness condition on the revenue function and a natural diminishing return assumption. It also limits our discussion in the more interesting setting where at each time, selling more could never decrease revenue. The second condition implies that selling nothing yields no revenue. The third condition limits the marginal revenue at the origin (named base price hereafter) and ensures that it is beneficial to sell, since the base price is positive. Denote the family of all possible revenue functions at time t as $\mathcal{G}$. We assume m and M are known beforehand to the decision maker and denote $\theta = M/m$.

A deterministic online algorithm $\mathcal{A}$ is *π-competitive* if

$$\pi = \max_{\sigma \in \Sigma} \frac{\eta_{OPT}(\sigma)}{\eta_{\mathcal{A}}(\sigma)},$$

where Σ is the set of all possible inputs ($g_t(\cdot)$, $t \in [T]$) and $\eta_{OPT}(\sigma)$ and $\eta_{\mathcal{A}}(\sigma)$ are the revenues generated by the optimal offline algorithm *OPT* and the online algorithm $\mathcal{A}$, respectively. This value π is the *competitive ratio* (CR) of the algorithm $\mathcal{A}$.

2 SELECTED RESULTS

The class of online algorithms that make up the CR-Pursuit framework, denoted as CR-Pursuit(π) is presented succinctly in Alg. 1. Essentially, CR-Pursuit(π) aims at keeping the *offline-to-online revenue ratio* to be $\pi > 1$ at all time, i.e., its output at time t (denoted as $\bar{v}_t$) satisfies

$$\sum_{\tau=1}^{t} g_\tau\left(\bar{v}_\tau\right) = \frac{1}{\pi}\eta_{OPT}\left(\sigma^{[1:t]}\right), \quad \forall t \in [T]. \tag{4}$$

We remark that such $\bar{v}_t$ always exists, because (i) $g_t(\cdot)$ is a continuous and increasing function and (ii) the quantity required to maintain π is in $[g_t(0), g_t\left(\hat{v}_t\right)]$, where $\hat{v}_t$ is the maximizer of $g_t(v)$.

While CR-Pursuit(π) can be defined for any π, the solution obtained by CR-Pursuit(π) may violate the inventory constraint in OOIC and be infeasible. This motivates the following definition.

Definition 2.1. CR-Pursuit(π) is feasible if $\Phi_\Delta\left(\pi\right) \leq \Delta$, where

$$\Phi_\Delta\left(\pi\right) \triangleq \max_{\sigma \in \Sigma} \sum_{t=1}^{T} \bar{v}_t(\sigma), \tag{5}$$

and $\bar{v}_t(\sigma)$ is the output of CR-Pursuit(π) at time t under input σ.

Our first result is that it suffices to focus on CR-Pursuit for achieving optimal competitive ratio.

Theorem 2.2. *Let π^* be the unique solution to the characteristic equation $\Phi_\Delta\left(\pi\right) = \Delta$. Then CR-Pursuit($\pi^*$) is feasible and π^* is the optimal competitive ratio of deterministic online algorithms.*

2.1 General Concave Revenue Functions

Finding the right π under a given setting is not trivial. Our next result provides an approximation to the optimal competitive ratio for families of concave functions $\mathcal{G}$.

Theorem 2.3. *Recall that $\mathcal{G}$ is the set of all possible $g(\cdot)$ and let $\hat{v} \in [0, \Delta]$ be the maximizer of $g(\cdot)$. Let $c(\mathcal{G}) = \sup_{g \in \mathcal{G}} \frac{g'(0)}{g(\hat{v})/\hat{v}}$, then the optimal competitive ratio π^* for the family of concave functions $\mathcal{G}$ satisfies*

$$\ln\theta + 1 \leq \pi^* \leq c(\mathcal{G})\left(\ln\theta + 1\right).$$

Note that when we restrict our attention to the family of linear revenue functions, as in one-way trading problem, we have $c = 1$, matching the known optimal competitive ratio.

2.2 One-way Trading with Price Elasticity

We next focus on practical settings where we can further improve this approximation ratio. In particular, we consider the one-way trading problem, except with price elasticity, i.e., the set of all possible revenue functions can be expressed as

$$\mathcal{G} = \{g_t(v) | g_t(v) = (p(t) - f_t(v))v, p(t) \in [m, M], \\ f_t(v) \in [0, +\infty), \forall v \in [0, \Delta], f_t(0) = 0\},$$

where f_t is a non-negative convex function representing price elasticity. Under this setting where we further impose parametric constraints on the revenue function g_t, we obtain improved results on finding a competitive ratio π.

Theorem 2.4. *Let $\bar{\pi} = \left(\ln\theta + 1\right)^2 / \left(\ln\theta + 3/4\right) < \ln\theta + 4/3$. The online algorithm CR-Pursuit($\bar{\pi}$) is feasible and is thus $\bar{\pi}$-competitive.*

Note that $\bar{\pi} < \ln\theta + 4/3$, which is very close to the lower bound of $\ln\theta + 1$. It also improves beyond the result of $2(\ln\theta + 1)$ if we follow the characterization in the previous result. This improvement is due to a sharper bound on the quantity required to upkeep a competitive ratio for this particular setting.

3 BEYOND THE WORST CASE MENTALITY

Our CR-Pursuit framework focuses only on achieving competitiveness under worst case inputs, limiting its application. Intuitively, a "better" online algorithm would be more opportunistic and sell more of its inventory when the incoming revenue function is "not adversarial". By design, CR-Pursuit is pessimistic: it only maintains a fixed competitive ratio π^* for the whole trading period, even if some inputs are not adversarial. One way to improve the performance of CR-Pursuit is to instead *adaptively* choose π_t to maintain at time t, the smallest attainable competitive ratio at time t.

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