TOWARDS REDUCING THE GAP BETWEEN PMEPR OF MULTICARRIER AND SINGLE CARRIER SIGNALS

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ABSTRACT

It has recently been shown that by altering the sign of each subcarrier in a multicarrier system significant reduction in the peak to mean envelope power (PMEPR) can be obtained. In fact, the PMEPR can even be made a constant independent of the number of subcarriers n. However, finding the best sign requires a search over 2^n possible signs which is computationally prohibitive. In this paper, we first propose a greedy algorithm to choose the signs based on p-norm minimization and we prove that it can achieve a PMEPR of order $\log n$. We further decrease the PMEPR by enlarging the search space considered by the greedy algorithm. By ignoring peaks with probability less than 10^{-3}, simulation results show that the PMEPR of a multicarrier system with 128 subcarriers each one modulated by 64QAM constellations is reduced to 3.4. This implies that at the cost of one bit of information per subcarrier (i.e., not sending information over the sign of each subcarrier) and modest computational complexity in the transmitter, the PMEPR can be reduced from 12.5 to 3.4 which is within 1.6 dB of the PMEPR of a single carrier system with 64QAM modulation.

1. INTRODUCTION

Multicarrier modulation has been proposed for high speed wireless and wireline communications in different standards such as IEEE 802.11a,g, xDSL, and Digital Video/Audio Broadcasting. The main advantage of this modulation over single carrier modulations is the simplicity of channel equalization for frequency selective channels. However, the main drawback of multicarrier systems is their high peak to mean envelope power ratio (PMEPR) as n subcarriers may add up constructively and produce large peaks of order n. In practice n is large (e.g. of the order of hundred) and therefore, the power amplifier should be highly linear which significantly hampers the power efficiency of the power amplifies which in return significantly reduces the battery life time.

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The complex envelope of a multicarrier signal with n subcarriers may be represented as,

$$s_C(\theta) = \sum_{i=1}^{n} c_i e^{j\theta i}, \qquad 0 \le \theta < 2\pi, \tag{1}$$

where $C = (c_1, \ldots, c_n)$ is the complex modulating vector (or a codeword) with entries from a given complex constellation. The admissible modulating vectors are called codewords and the ensemble of all possible codewords constitute the code $\mathcal{C}$. Then, the PMEPR of each codeword C in the code family $\mathcal{C}$ may be defined as,

$$\mathrm{PMEPR}_{\mathcal{C}}(C) = \max_{0 \le \theta < 2\pi} \frac{|s_C(\theta)|^2}{E\{\|C\|_2^2\}}. \tag{2}$$

Similarly, $\mathrm{PMEPR}_{\mathcal{C}}$ is defined as the maximum of Eq. (2) over all codewords in $\mathcal{C}$. If c_i's are chosen independently and identically from some constellation with average power E_{av} then $E\{\|C\|_2^2\} = nE_{av}$.

Even though the worst case PMEPR is of the order n when c_i's are chosen from a constellation such as QAM, it is shown that with high probability the PMEPR of a random codeword is $\log n$ almost surely [1, 2, 3]. This implies that the PMEPR is not as bad as what is predicted by the worst case and its distribution should be taken into consideration. Fig. 1 compares the complementary cumulative distribution function (CCDF) of PMEPR for a multicarrier system with $n = 128$ and using 64QAM constellation with that of a single carrier system. By ignoring peaks with probability below 10^{-3},[1] the PMEPR of the multicarrier system is 12.5 as opposed to 2.3 for the single carrier system. This shows a 7.35 dB gap between the PMEPR of these two systems.

Because of this large gap, many schemes have been proposed for PMEPR reduction such as coding, clipping, selective mapping, and partial transmit sequence (PTS) [4, 5, 6, 7, 8, 9, 10]. In [11], it has been shown that adjusting the sign

[1] Throughout the paper, in order to compare the simulation results, we approximate the PMEPR of a scheme by the value η such that $\Pr(\mathrm{PMEPR} > \eta) = 10^{-3}$. We basically ignore peaks with probability below 10^{-3} in our simulations.

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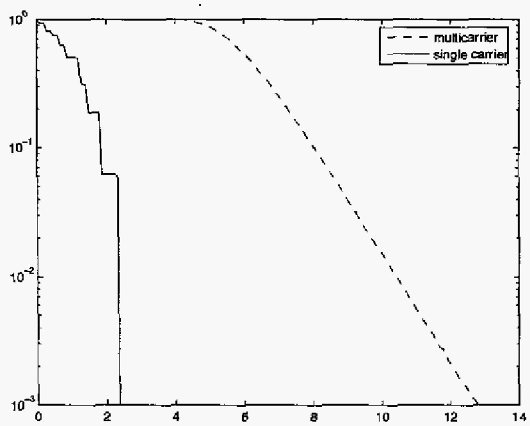

Fig. 1. Comparison of Pr(PMEPR $> \lambda$) for a multicarrier system with $n = 128$ and a single carrier system using 64QAM constellation for 5000 random codewords.

of each subcarrier is a promising technique for PMEPR reduction of multicarrier signals and leads to the proof for the existence of nonvanishing to zero rate codes with PMEPR bounded by a constant. The main idea is to choose a sign ϵ_i for each subcarrier to minimize the maximum of the signal. Hence, given the codeword $C = (c_1, \ldots, c_n)$, the following problem should be solved,

$$\min_{\epsilon} \max_{0 \leq \theta \leq 2\pi} |\sum_{i=1}^{n} \epsilon_i c_i e^{j\theta i}| \tag{3}$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\epsilon_i \in \{+1, -1\}$.

Of course finding the solution for the combinatorial optimization problem in (3) has exponential complexity. In [11], an algorithm is proposed to find the signs with linear complexity which guarantees the PMEPR of $c \log n$ where c is a constant independent of n. Recently, in [12], it is shown that by searching over a small subset of 2^n signs, PMEPR of order $\log n$ can be achieved with $n^{\log n}$ complexity. The main goal of this paper is to investigate polynomial time algorithms to choose the signs and further reduce the gap between the PMEPR of multicarrier and single carrier signals.

The first contribution of the paper is to propose a greedy algorithm to choose the signs that guarantees the PMEPR of $c \log n$ where c is a constant independent of n. This is done by using a p-norm minimization as opposed to minimizing the conditional probability for the derandomization method. This algorithm has the advantage of having less computation at each stage of recursion as we just compute the p-norm of a vector and its performance is comparable to the performance of the algorithm proposed in [11].

We then propose two refinements of our algorithm to further improve the reduction of PMEPR by increasing the computational complexity of the algorithm. These methods are based on enlarging the search and pruning the tree of 2^n signs using the p-norm metric.

The paper is organized as follows: Section 2 deals with the greedy algorithm with order n complexity and PMEPR guarantee of order $\log n$. Section 3 introduces the improvements on the aforementioned algorithm with the additional computational complexity.

2. A GREEDY ALGORITHM TO CHOOSE THE SIGNS

Following [11], we first change the problem in (3) and instead of looking at the maximum of $s_C(\theta)$ over $0 \leq \theta \leq 2\pi$, we look at its maximum over uniform samples of θ at $\theta_p = \frac{2\pi p}{kn}$ for $p = 1, \ldots, kn$ where $k > 1$ is the oversampling factor. Therefore, the problem can be stated as:

$$\min_{\epsilon} \max_{1 \leq p \leq 2kn} |\sum_{i=1}^{n} \epsilon_i a_{pi}| \tag{4}$$

where a_{pi} is defined as,

$$a_{pi} = \begin{cases} \mathrm{Re}\{c_i e^{j\theta_p i}\} & 1 \leq p \leq kn, \\ \mathrm{Im}\{c_i e^{j\theta_p i}\} & kn+1 \leq p \leq 2kn \end{cases} \tag{5}$$

In other words, we would like to solve

$$\min_{\epsilon_i^2=1, i=1,\ldots,n} \|A\epsilon\|_\infty = \min_{\epsilon_i^2=1, i=1,\ldots,n} \|\sum_{i=1}^{n} a_i \epsilon_i\|_\infty \tag{6}$$

where $A^t = [a_i]$ is a $2kn \times n$ real matrix and $a_i = [a_{p,i}]$. Without loss of generality we assume that $|a_{i,p}| < 1$ (which can be done by scaling the constellation).

It is known that for large n and for any codeword C, there exists a choice of sign vector ϵ such that the PMEPR is bounded by a constant independent of n. Moreover, randomly choosing signs will lead to a PMEPR of $\log n$ with high probability. The challenge however is to find a sign vector efficiently that "guarantees" the PMEPR to be either bounded by $\log n$ or constant.

In [11], a deterministic algorithm is proposed to design the signs using derandomization. The algorithm chooses the signs recursively based on the knowledge of all a_i's. In fact, at j'th step, we choose the sign that minimizes the conditional probability that $\|A\epsilon\|_\infty$ is greater than some threshold λ and given $\epsilon_1, \ldots, \epsilon_{j-1}$. Since finding the conditional probability is quite messy, we can use the Chernoff bound instead. This leads to the following algorithm (see [11] for details),

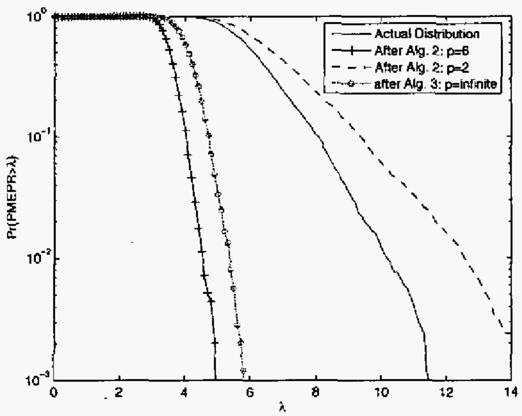

Fig. 2. Comparison of Pr(PMEPR $>\lambda$) for $n = 128$ and using Algorithm 2 for different value of p and for 5000 codewords.

Algorithm 1. *For any codeword $C = (c_1, \ldots, c_n)$, let a_{pi} be as in (5). Then $\epsilon_1 = 1$, and ϵ_j's are recursively determined as the minus sign of*

$$\sum_{p=1}^{2kn} \sinh\left\{\alpha^* \sum_{r=1}^{j-1} \epsilon_r a_{pr}\right\} \sinh(\alpha^* a_{pj}) \prod_{r=j+1}^{n} \cosh\{\alpha^* a_{pr}\}.$$

for $j = 2, \ldots, n$, where $\alpha^* = \sqrt{\frac{2\log 4kn}{n}}$.

It is shown that the resulting PMEPR will be less than $c \log n$ for any n where c is a constant independent of n which depends on the constellations [11].

The only drawback of Algorithm 1 is that the computation at each step involves taking cosine hyperbolic kn times which may increase the computation. In order to simplify the computation of Algorithm 1 at each step, one may try choosing the signs in a greedy manner in which at each step the sign that minimizes $\| \sum_{i=1}^{j} a_j \epsilon_j \|_\infty$ is chosen given $\epsilon_1, \ldots, \epsilon_{j-1}$. Interestingly, we can improve the performance by changing the infinity norm to norm p. Fig. 2 shows the performance of this method using different norms. It is clear that for $n = 128$, using $p = 6$ or 7 leads to quite a large improvement.

We can in fact justify this behavior analytically. The main result of this section is to obtain a bound on the PMEPR obtained from greedily minimizing the metric $\| \sum_{j=1}^{i} \epsilon_j a_j \|_p^p$. In particular, we show that the optimal p is $\log 2kn$, which yields a PMEPR of $c \log n$ for any n. Here is the algorithm:

Algorithm 2: Let $\epsilon_1 = 1$, and having chosen $\epsilon_2, \ldots, \epsilon_{k-1}$, then

$$\epsilon_k = \arg \min_{\epsilon_k \in \{+1,-1\}} \| \sum_{j=1}^{k} a_i \epsilon_i \|_p^p \qquad (7)$$

The next Theorem provides a worst case guarantee on the PMEPR when p is even. We conjecture that the result holds for p odd as well.

Theorem 1: For any p greater than 2, and assuming all the entries of $A = [a_{i,j}]$ are $|a_{i,j}| \leq 1$, Algorithm 2 ensures that

$$\|A\epsilon\|_\infty \leq (2kn)^{1/p} \sqrt{pn} \qquad (8)$$

for any n. If $p = \log 2kn$, then the upper bound is $e\sqrt{n \log kn}$.

Proof: We present the proof when p is even for simplicity. If p is odd, we can follows a similar approach. Assume $\epsilon_1, \ldots, \epsilon_{k-1}$ have already been determined. We define the sequence $B_{r-1}^p = \frac{1}{2kn} \| \sum_{j=1}^{r-1} a_j \epsilon_j \|_p^p$. Using algorithm 2, we now find a bound on B_k based on B_{k-1}. We first denote $\sum_{j=1}^{k-1} a_j \epsilon_j = (y_1, \ldots, y_{2kn})^t$ and $a_k = (x_1, \ldots, x_{2kn})^t$. Hence we may write,

$$\begin{aligned} 2knB_r^p &= \min\left\{ \sum_{j=1}^{2kn} (y_j - x_j)^p, \sum_{j=1}^{kn} (y_j + x_j)^p \right\} \\ &\leq \frac{1}{2} \left(\sum_{j=1}^{2kn} (y_j + x_j)^p + (y_j - x_j)^p \right) \\ &\leq \frac{1}{2} \left(\sum_{j=1}^{2kn} (y_j + 1)^p + (y_j - 1)^p \right) \qquad (9) \end{aligned}$$

The last equality follows from the fact that $|x_i| \leq 1$ and also using the inequality

$$(y_j + x_j)^p + (y_j - x_j)^p \leq (y_j + 1)^p + (y_j - 1)^p \quad (10)$$

for $p \geq 1$ and $|x_j| \leq 1$. The bound can be proved using the convexity of the left hand side of (10) and therefore its maximum is attained on the boundary.

We can further bound (9) by using the inequality,

$$\begin{aligned} & \frac{1}{2kn} \sum_{j=1}^{2kn} (x_j + 1)^p + (x_j - 1)^p \\ = \; & \frac{1}{2kn} \sum_{j=1}^{2kn} \sum_{r=0}^{p} \binom{s}{r} x_j^r (1 + (-1)^{p-r}) \\ \leq \; & \sum_{r=0}^{p} \binom{s}{r} \left(\frac{1}{2kn} \sum_{j=1}^{2kn} x_j^p \right)^{r/p} (1 + (-1)^{p-r}) \\ \leq \; & \left(\left(\frac{1}{2kn} \sum_{j=1}^{2kn} x_j^p \right)^{1/p} + 1 \right)^p \\ & + \left(\left(\frac{1}{2kn} \sum_{j=1}^{2kn} x_j^p \right)^{1/p} - 1 \right)^p \end{aligned}$$

Therefore,

$$knB_r^p \leq kn\left\{(B_{r-1}+1)^p + (B_{r-1}-1)^p\right\} \quad (11)$$
$$\leq 2kn\left(B_{r-1}^2 + p\right)^{p/2} \quad (12)$$

where the last inequality follows by expanding the right hand side of (11) and using the fact that

$$\binom{2n}{2j} \leq \binom{n}{n-j} \times (2n)^{k-j} = \binom{n}{j} \times (2n)^{k-j}.$$

We can therefore obtain a recursive bound for $B_r^2 \leq B_{r-1}^2 + p$. Noting that $B_1 \leq 1$, we conclude that $B_n \leq \sqrt{np}$, and therefore,

$$\|\sum_{i=1}^{n} a_i\epsilon_i\|_\infty \leq \|\sum_{i=1}^{n} a_i\epsilon_i\|_p$$
$$\leq \left(knp^{p/2}n^{p/2}\right)^{1/p}$$
$$= (2kn)^{1/p}\sqrt{pn}. \quad (13)$$

Finally, letting $p = \log 2kn$, the theorem follows. □

Theorem 1 implies that if the norm p is properly chosen, the PMEPR of the resulting codeword is guaranteed to be less than $c \log n$ where c is a constant independent of n.

In fact, if we just allow the designer to find ϵ_i causally, i.e., based on $a_1, \ldots, a_i$ and not using $a_{i+1}, \ldots, a_n$, the problem of choosing the signs can be formulated as a mathematical game [1]. Following Spencer's terminilogy, at the k'th stage the "pusher" chooses a_k such that $\|a_k\|_\infty \leq 1$ and then the "chooser" decides on the sign ϵ_k. The value of the game at the k'th stage is $\|\sum_{j=1}^{k} a_j\epsilon_j\|_\infty$. Based on a result of [13], we can state the following corollary.

Corollary 2: Considering any real $kn \times n$ matrix A with entries bounded by one, any algorithm that chooses ϵ_i's causally, cannot achieve a PMEPR of less than $\log n$ for large n.

In fact any suboptimal algorithm for the pusher to find a_k's leads to a lower bound for the problem of causally choosing ϵ_i's. In [13], an algorithm is also proposed to design the signs causally. Here is the algorithm:

Algorithm 3: Let $\epsilon_1 = 1$, and having chosen $\epsilon_2, \ldots, \epsilon_{k-1}$, then

$$\epsilon_k = \underset{\epsilon_k}{\operatorname{argmin}} \cosh\left(\sum_{i=1}^{k} a_i\epsilon_i\right) \quad (14)$$

where $\cosh(X)$ for the vector $X^t = (x_1, \ldots, x_m)$ is defined as $\sum_{i=1}^{m} \cosh x_i$.

In [13], it is further proved that for a square $n \times n$ matrix, the algorithm can guarantee that $\|A\epsilon\|_\infty \leq \sqrt{2n\log n}$. The proof can be easily extended to the case of a $kn \times n$ matrix.

Fig. 3 compares the performance of Algorithms 1, 2 and 3 for a system with 128 subcarriers and 64 QAM. It is observed that Algorithm 3 has the worst performance and if p is properly chosen, algorithm 1 and 2 have quite close resulting PMEPR distribution. Furthermore, the PMEPR has been reduced to 4.8 using Algorithm 1 or 2. In the next section, we propose a refinement of Algorithm 2 that further reduces the PMEPR at the cost of additional complexity.

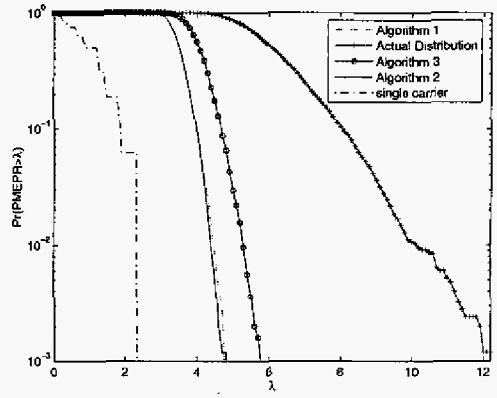

Fig. 3. A comparison of the PMEPR reduction using Algorithm 1, 2, and 3 for 5000 random codewords.

3. PRUNING BASED ALGORITHMS

As shown in Fig. 3, there is still a pretty large gap between the PMEPR of the multicarrier system (i.e., 4.8) and that of the single carrier systems (i.e., 2.3). More precisely, we would like to see whether we can efficiently find a better choice of the signs that further reduces the PMEPR and approaches the CCDF of the single carrier system. Here we consider two variations of algorithm 2.

Pruning Algorithm 1: In the first approach, we search over all the possible signs for the first m subcarriers and then we use Algorithm 2 to find the choice of the signs for the remaining $n - m$ signs. Finally we choose the sign vector (out of 2^{m-1} possible choices as $\epsilon_1 = 1$) that has the least PMEPR. This of course has the complexity of order $O(2^m n \log n)$ as it requires searching for the best vector by performing 2^m IFFTs with size n. Fig. 4 shows the performance of this scheme for different m's. It can be seen that the PMEPR has been reduced from 4.8 to 3.4 at the cost of additional computational complexity at the transmitter.

Pruning Algorithm 2: In the second approach, we consider the metric at the j'th stage to be $\|\sum_{i=1}^{j} a_i\epsilon_i\|_p^p$. Instead of just looking at the choice of sign that minimizes the metric at each stage, we keep the sign choices as long as the metric is less than some threshold value. One legitimate choice of the threshold would be the value of the metric by running Algorithm 2. In order to allow for more sign vectors, we may increase the threshold at each stage by some value (say η). At the end of the algorithm, we choose the best sign vector in terms of PMEPR. Fig. 4 shows the re-

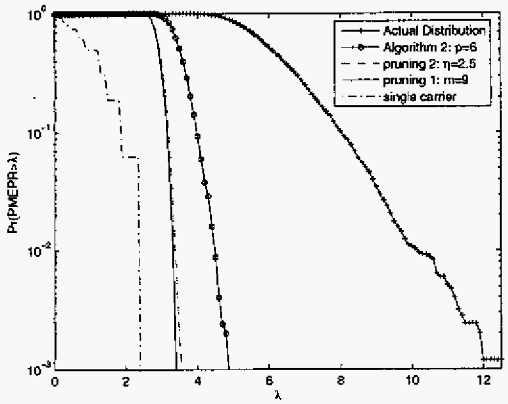

Fig. 4. Comparison of $\Pr(\text{PMEPR} > \lambda)$ for $n = 128$ using the pruning algorithms compared to Algorithm 2 with $p = 6$ for 5000 random codewords.

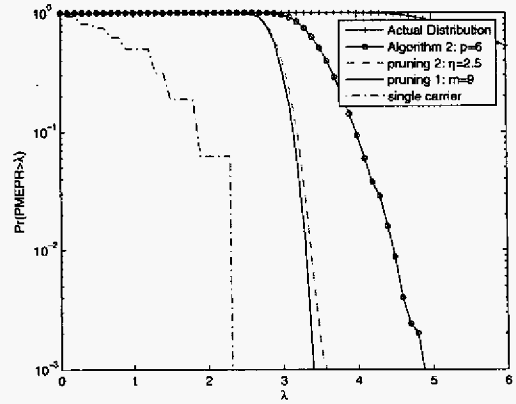

Fig. 5. Comparison of $\Pr(\text{PMEPR} > \lambda)$ for $n = 128$ using the pruning algorithms compared to Algorithm 2 with $p = 6$ for 5000 random codewords (re-scale of Fig. 4).

sulting PMEPR improvement for different values of η.

Fig. 5 is the re-scaled version of Fig. 4 to see better the difference in the CCDF of PMEPR for the pruning algorithms and Algorithm 1. Clearly, the PMPER is improved from 12.5 to 3.4 for the multicarrier system with 128 subcarriers and its PMEPR is just 1.6dB$= 10\log(3.4/2.3)$ worse than the single carrier system. This motivates further investigation for more effective algorithms to choose the sign vector with less complexity. Moreover, the question of how much further we can improve the PMEPR remains open.

4. CONCLUSION

We considered the problem of finding efficiently a sign vector to minimize the PMEPR of multicarrier signals. Solving this problem require a search over 2^n possible sign vectors. In this paper, we proposed a greedy algorithm based on p norm minimization and we showed that if p is properly chosen, this scheme can achieve a PMEPR of $c \log n$ for any n. We further decreased the PMEPR by expanding the search space and using our greedy algorithm. Our simulation results show that PMEPR can be significantly reduced at the cost of rate loss of one bit per subcarrier and modest increase in the computational complexity. In particular, for a multicarrier system with 128 subcarriers, simulation results show that the PMEPR can be reduced from 12.5 to 3.4.

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