A DETERMINISTIC ALGORITHM THAT ACHIEVES THE PMEPR OF $c \log n$ FOR MULTICARRIER SIGNALS

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ABSTRACT

Multicarrier signals often exhibit large peak to mean envelope power ratios (PMEPR) which can be problematic in practice. In this paper, we study adjusting the sign of each subcarrier in order to reduce the PMEPR of a multicarrier signal with n subcarriers. Considering that any randomly chosen codeword has PMEPR of $\log n$ with probability one and for large values of n [1], randomly choosing signs should lead to the PMEPR of $\log n$ in the probability sense. Based on the derandomization algorithm suggested in [2], we propose a deterministic and efficient algorithm to design signs such that the PMEPR of the resulting codeword is less than $c \log n$ for *any* n where c is a constant independent of n. By using a symmetric q-ary constellation, this algorithm in fact constructs a code with rate $1 - \log_q 2$, PMEPR of $c \log n$, and with simple encoding and decoding. We then present simulation results for our algorithm.

1. INTRODUCTION

Multicarrier modulation is one of the promising techniques for broadband communications due to its immunity to multipath fading and simplicity of the channel equalization. However, multicarrier signals suffer from high peak to mean envelope power ratio (PMEPR). For instance, for a signal with n subcarriers, PMEPR can be as high as n in the worst case.

Several methods have been proposed to reduce PMEPR including clipping, selective mapping, partial transmit sequence, coding, and using dummy carriers [3, 4, 5, 6]. The first three methods do not give any guarantee on the worst case PMEPR, but they improve the statistical properties of PMEPR at the price of adding redundancy to the codewords. On the other hand, coding methods can give a worst case guarantee on PMEPR at the price of a big rate hit for the existing codes. For example, Golay codes have PMEPR of 2 with a rate approaching to zero as n increases. In [5], codes are constructed with PMEPR of $O(\log^2 n)$, large minimum distance, and rate approaching to zero as n becomes large.

On the other hand, it has been shown that any random codeword chosen from a symmetric QAM or 2^m-PSK constellations, has PMEPR of $\log n$ with probability one asymptotically [1]. This in fact states that random methods should work well and give PMEPR of $\log n$ with very high probability and there is no rate hit by reducing the PMEPR to $\log n$, asymptotically. Recently, based on the elegant result of Spencer [7], the existence of codes with even constant PMEPR and high rates is also proved [8]. By choosing an optimum sign for each subcarrier, we can in fact achieve constant PMEPR for sufficiently large n.

In order to design the signs, we know that any randomly chosen sign will achieve PMEPR of $\log n$ in the probability sense and for large n [1]. However, randomized algorithms cannot give a guarantee on PMEPR. Therefore, we use the derandomization algorithm as [2] to design the signs that guarantee a PMEPR of $c \log n$ for any n where c is a constant independent of n. This algorithm in fact constructs a code family with rate $1 - \log_q 2$ and PMEPR of $c \log n$ by using a symmetric constellation with q alphabets.

Clearly the rate of the above code for BPSK scheme is zero, therefore, we consider using dummy carriers with optimum amplitude and phase to reduce the PMEPR for the BPSK case [4]. Based on Nehari's Theorem [9], we propose a computationally efficient method to find the values of the dummy carriers to minimize the maximum of the multicarrier signal. Even though the method does not minimize the ratio of the maximum to the average power, simulation results suggest that it is an effective method for the BPSK constellation.

2. DEFINITIONS

In this paper, we consider the normalized complex envelope of a multicarrier signal with n subcarriers as

$$s_C(\theta) = \sum_{i=1}^{n} c_i e^{j\theta i}, \qquad 0 \le \theta < 2\pi, \qquad (1)$$

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where $C = (c_1, \ldots, c_n)$ is the complex modulating vector with entries from a given complex constellation $\mathcal{Q}$ and where θ is the time axis. The admissible modulating vectors are called codewords and the ensemble of all possible codewords constitute the code $\mathcal{C}$. To quantify the variation of the signal, we define the peak to mean envelope power ratio (PMEPR) of each codeword C as,

$$\text{PMEPR}_{\mathcal{C}}(C) = \max_{0 \leq \theta < 2\pi} \frac{|s_C(\theta)|^2}{E\{\|C\|^2\}}. \tag{2}$$

Similarly, $\text{PMEPR}_{\mathcal{C}}$ is defined as the maximum of (2) over all codewords in $\mathcal{C}$. By a symmetric constellation, we mean that if there is a point A in the constellation, then $-A$ should be also in the constellation. It is worth noting that if the c_i's are randomly chosen from $\mathcal{Q}$, then $E\{\|C\|^2\} = nE_{av}$ where $E_{av} = E\{|c_i|^2\}$ is the average energy of $\mathcal{Q}$.

As it is clear from (2), PMEPR can be as high as n in the worst case, however, the probability of encountering such a high PMEPR is very low. Furthermore, it has been recently shown that for any given complex vector C and for sufficiently large n, there exists an optimum sign, ϵ_i, for each subcarrier such that $\text{PMEPR}(C_\epsilon) = \mathcal{K}^0$ where $C_\epsilon = (\epsilon_1 c_1, \ldots, \epsilon_n c_n)$ and $\mathcal{K}^0$ is a constant that only depends on the constellation type [8]. This result motivates the following problem:

Problem Statement: For any given complex vector $C = (c_1, \ldots, c_n)$, design the sign vector $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ where $\epsilon_i \in \{+1, -1\}$ such that

$$\min_{\epsilon} \max_{0 \leq \theta \leq 2\pi} \left| \sum_{i=1}^{n} \epsilon_i c_i e^{j\theta i} \right| \tag{3}$$

In Section 3, we propose an algorithm to design the sign vector to guarantee the PMEPR of $c \log n$ for any n where c is a constant independent of n throughout the paper.

3. DESIGN OF SIGNS TO REDUCE PMEPR

As mentioned in the previous section, there exists a sign vector that yields constant PMEPR for sufficiently large values of n. On the other hand, any random sign vector should have PMEPR of $\log n$ for large values of n with high probability, and therefore, random methods should work well in the probability sense. In what follows, we propose a deterministic and efficient algorithm which basically derandomizes the search for the sign vector ϵ and then we prove that our algorithm achieves PMEPR of $c \log n$ for any n (*not asymptotically*). We first use the following Lemma to change the problem in (3) to minimizing a finite number of linear forms.

Lemma 1. *Let $s_C^R(\theta)$ and $s_C^I(\theta)$ be the real and imaginary parts of $s_C(\theta)$, respectively. Also assume $k > 1$ such that kn is an interger and let $\theta_i = \frac{2\pi i}{kn}$ for $i = 1, \ldots, kn$. Then*

$$\max_{\theta} |s_C(\theta)| \leq \frac{1}{\cos \pi/2k} \sqrt{\max_{1 \leq i \leq kn} |s_C^R(\theta_i)|^2 + \max_{1 \leq i \leq kn} |s_C^I(\theta_i)|^2} \tag{4}$$

Proof: The lemma follows by using the result of [10] to bound the maximum of a real multicarrier signal with n subcarriers to the maximum of its kn uniform samples. ■

Lemma 1 reformulates the problem in (3) and instead of designing the vector ϵ to minimize the maximum of $|s_C(\theta)|$ over a continuous variable θ, we look for the optimum ϵ_i's to minimize $2kn$ linear forms corresponding to $s_C^R(\theta_i)$ and $s_C^I(\theta_i)$ for $i = 1, \ldots, n$ and defined as

$$\min_{\epsilon} \max_{1 \leq p \leq 2kn} \left| \sum_{i=1}^{n} \epsilon_i a_{pi} \right|, \tag{5}$$

where a_{pi} is defined as,

$$a_{pi} = \begin{cases} \text{Re}\{c_i e^{j\theta_p i}\} & 1 \leq p \leq kn \\ \text{Im}\{c_i e^{j\theta_p i}\} & kn+1 \leq p \leq 2kn, \end{cases} \tag{6}$$

and $\theta_p = \frac{2\pi p}{kn}$.

In order to solve (5), we consider a more general setting for our problem. Let's consider the set of equiprobable vectors $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$, $\epsilon_i \in \{-1, +1\}$. Then, for any codeword C, we define A_p^λ as the event that the p'th linear form defined in (5) is greater than λ. Furthermore, assume λ is chosen such that $\sum_{i=1}^{2kn} \Pr\{A_i^\lambda\}$ is less than 1, and therefore, there exists a vector ϵ with the above property. We would like to efficiently find the vector ϵ, such that none of the bad events A_p^λ occur.

This problem has been considered in mathematics and is usually referred to as the derandomization of random algorithms [2]. In this approach, we assume that we can compute the conditional probability $\Pr\{A_p^\lambda | \epsilon_1, \ldots \epsilon_j\}$, and we find the ϵ_i's sequentially. At the j'th step, given $\epsilon_1, \ldots, \epsilon_{j-1}$, we choose $\epsilon_j \in \{+1, -1\}$ to minimize

$$\sum_{i=1}^{2kn} \Pr\{A_i^\lambda | \epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_j\}. \tag{7}$$

Due to the above recursive minimization, we have

$$\sum_{i=1}^{2kn} \Pr\{A_i^\lambda | \epsilon_1, \ldots, \epsilon_{j-1}\} \leq \sum_{i=1}^{2kn} \Pr\{A_i^\lambda\} < 1, \tag{8}$$

and finally, we will end up with

$$\sum_{i=1}^{2kn} \Pr\{A_i^\lambda | \epsilon_1, \ldots, \epsilon_n\} < 1. \tag{9}$$

Since there is no randomness in the events of (9) when all the ϵ_i's are determined, each $\Pr\{A_i^\lambda/\epsilon_1,\ldots,\epsilon_n\}$ is either one or zero. Therefore, Eq. (9) implies that all of the probabilities are zero, and consequently, the resulting vector ϵ guarantees that none of the events A_i^λ will occur.

The difficulty here is now in the efficient computation of the conditional probabilities. Instead of using the exact conditional probability functions, we can use the upper bounds

$$\Pr\{A_i^\lambda|\epsilon_1,\ldots,\epsilon_j\} \le F_i^\lambda(\epsilon_1,\ldots,\epsilon_j), \qquad (10)$$

if the upper bounds satisfy the following conditions:

$$\begin{array}{ll} i) & \sum_{i=1}^{2kn} F_i^\lambda < 1 \\ ii) & F_i^\lambda(\epsilon_1,\ldots,\epsilon_j) \ge \min_{\epsilon_j\in\{+1,-1\}} F_i^\lambda(\epsilon_1,\ldots,\epsilon_{j-1},\epsilon_j). \end{array} \qquad (11)$$

Obviously by the same reasoning used for the original algorithm, we can use the upper bounds to find the vector ϵ such that none of the events A_i^λ occur. Fortunately, Chernoff's bound does the work for us,

$$F_p^\lambda(\epsilon_1,\ldots,\epsilon_j) = 2e^{-\alpha\lambda}\cosh\left\{\alpha\sum_{r=1}^{j}\epsilon_r a_{pr}\right\}\prod_{r=j+1}^{n}\cosh\alpha a_{pr} \qquad (12)$$

for any $\alpha > 0$ and $1 \le p \le 2kn$. It can be then verified that Eq. (12) satisfies both conditions in (11). Now we return to our problem and peresent the following algorithm.

Algorithm 1. *For any codeword $C = (c_1,\ldots,c_n)$, let a_{pi} be as in (6), k be as in Lemma 1, and $E_{max} = \max|c_i|^2$. Then $\epsilon_1 = 1$, and ϵ_j's are recursively determined as the minus sign of*

$$\sum_{p=1}^{2kn}\sinh\left\{\alpha^*\sum_{r=1}^{j-1}\epsilon_r a_{pr}\right\}\sinh(\alpha^* a_{pj})\prod_{r=j+1}^{n}\cosh\{\alpha^* a_{pr}\}.$$

for $j = 2,\ldots,n$, where $\alpha^* = \sqrt{\frac{2\log 4kn}{nE_{max}}}$. ■

The following Theorem gives the worst case guarantee on the PMEPR of the codeword $C_\epsilon = (\epsilon_1 c_1,\ldots,\epsilon_n c_n)$.

Theorem 1. *Let $C = (c_1,\ldots,c_n)$ be a given codeword where $|c_i| \le \sqrt{E_{max}}$ and $E_{av} = E\{|c_i|^2\}$. Also, let $C_\epsilon = (\epsilon_1 c_1,\ldots,\epsilon_n c_n)$ where $\epsilon_i \in \{+1,-1\}$ is determined according to Algorithm 1. Then* PMEPR *of the resulting codeword, C_ϵ, will be less than $\frac{4E_{max}}{\cos^2(\pi/2k)E_{av}}\log 4kn$ where k is as in Lemma 1.*

Proof: The proof relies on the derandomization method introduced before and using Chernoff bound for evaluating the conditional probability distribution. By properly choosing the optimum α and setting $\lambda = \sqrt{2nE_{max}\log 4kn}$ in (12), it can be shown that both conditions in (11) will be satisfied. Lemma 1 then completes the proof by relating PMEPR to the maximum of $2kn$ linear forms. ■

In order to get a better insight to the above result, we define the rate of a q-ary code family $\mathcal{C}$ as,

$$R = \frac{1}{n}\log_q|\mathcal{C}| \qquad (14)$$

where $|\mathcal{C}|$ is the cardinality of the set $\mathcal{C}$. In fact Theorem 1 implies that, by using the sign of each carrier to reduce PMEPR, we can construct a code with rate $1 - \log_q 2$ and PMEPR of $c\log n$ for any n. The rate and PMEPR of this code is much better than those of the previous codes proposed in [5] whose PMEPR is of $O(\log^2 n)$ and their rate is approaching to zero as n increases. It is also worth mentioning that finding optimum signs in the transmitted side can be done very efficiently, and the decoding is very simple since the sign of each subcarrier is just used for PMEPR reduction.

Since we cannot send information in the sign of subcarriers, the algorithm is not appropriate for the BPSK constellation. Therefore in the following subsection, we use the Nehari's Theorem to propose a computationally efficient method to reduce PMEPR for the BPSK case.

3.1. A Scheme for BPSK constellation

Using dummy carriers with optimum amplitude and phase to reduce PMEPR has been proposed previously in [4]. Recently the wellknown result of Nehari in functional analysis has been used to find a bound on the maximum reduction of the peak of a multicarrier signal by appending dummy carriers at the end of the signal [11]. The solution to the Nehari problem also suggests suboptimum values for phase and amplitude of dummy carriers which is computationally very efficient. The only problem with this approach is that the Nehari result minimizes the maximum of the multicarrier signal, however, in the PMEPR problem we are interested in minimizing the ratio of the maximum to the average power. In the mean time, these optimum values can still be considered for PMEPR reduction as a suboptimal solution. Here is the statement of the Nehari problem.

Problem Statement: Let c_i's be given, $T(z) = \sum_{i=1}^{m} c_i z^i$ and $K(z) = \sum_{i=m+1}^{\infty} b_i z^i$, find the optimum values of b_i's and the best γ such that for all c_i's, we have

$$\|T(z) - K(z)\|_\infty \le \gamma \qquad (15)$$

It is shown in [9] that the best γ is the singular value of the Hankel matrix generated by the vector $(c_1,\ldots,c_m)$and, moreover, $K(z)$ can be computed efficiently [11]. For our problem, using an extra bandwidth of L, we simply truncate the $K(z)$ to its first L coefficients, and we numerically compute the resulting PMEPR. The average power can be also computed by taking the average of $\sum_{i=1}^{m}|c_i|^2 + \sum_{i=m+1}^{m+L}|b_i|^2$ over all vectors $(c_1,\ldots,c_m)$. The details of the computation of b_i's can be found in [11].

4. SIMULATION RESULTS

In this section we present simulation results for different constellations including BPSK, QPSK, and 16QAM and for $n = 128$. Fig. 1 shows the complementary cumulative distribution function of PMEPR, $\Pr\{\text{PMEPR} > a\}$, with and without using the optimum signs. Clearly the distribution function improves significantly and the resulting distribution after using signs is very abrupt. Considering (14), the rate of such a code for QPSK and 16QAM is $1/2$ and $3/4$, respectively.

Fig. 2 also compares the result of the sign algorithm for QPSK with the performance of the algorithm in section 3.1 for BPSK constellation. In the simulations, we use 64 and 32 dummy carriers which correspond to $1/2$ and $3/4$ rate coding. It also worth mentioning that the dummy carrier method requires different powers for different modulating vectors and also its performance deteriorates by using QPSK or higher order constellations.

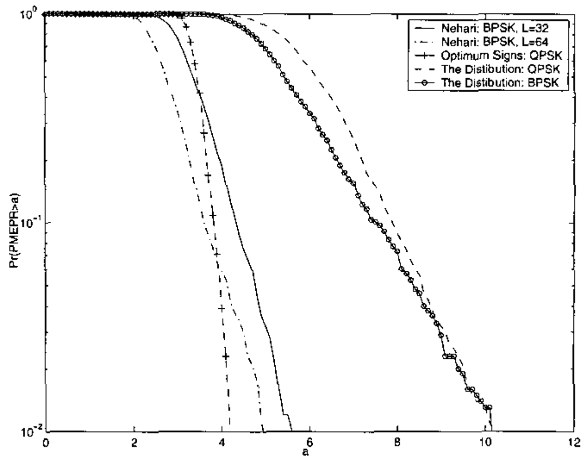

Fig. 2. PMEPR distribution for $n = 128$ and for the different scenarios.

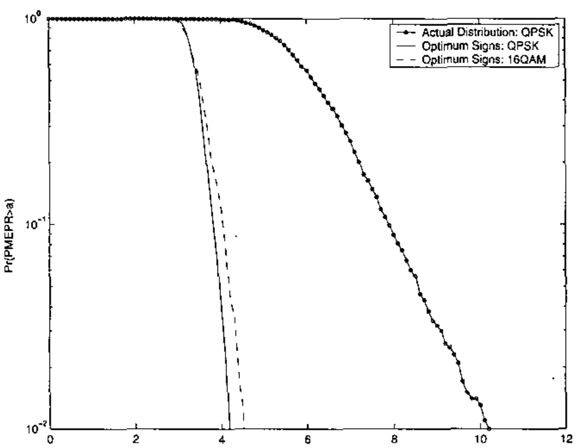

Fig. 1. PMEPR distribution for $n = 128$ befor and after using optimum signs.

5. CONCLUSION

We propose a deterministic and efficient algorithm to design optimum signs for each subcarrier of a multicarrier signal, modulated by any codeword, to reduce its PMEPR to $c \log n$. In other words, considering a symmetric q-ary constellation, we design a code with rate $1 - \log_q 2$ and PMEPR of $c \log n$ for any n with simple encoding and decoding. Numerical result shows that the use of signs makes the distribution of PMEPR very abrupt and the algorithm can significantly improve the PMEPR distribution for different constellation.

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