$\mathcal{L}_2$-Gain of Double Integrators With Saturation Nonlinearity

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***Abstract*—This note uses quadratic surface Lyapunov functions (SuLFs) to efficiently check if a double integrator in feedback with a saturation nonlinearity has $\mathcal{L}_2$-gain less than $\gamma > 0$. We show that for many such systems, the $\mathcal{L}_2$-gain is nonconservative in the sense that this is approximately equal to the lower bound obtained by replacing the saturation with a constant gain of 1. These results allow the use of classical analysis tools like μ-analysis or integral quadratic constraints to analyze systems with double integrators and saturations, including servo systems like some mechanical systems, satellites, hard disks, compact disk players, etc.**

***Index Terms*—Double integrator, impact map, nonlinear robust analysis, quadratic surface Lyapunov function, saturation.**

I. Introduction

There are many control applications that can be modeled as a rational plant with a single integrator, a saturation nonlinearity, and a PI controller as shown in Fig. 1. One of the most simple is the position control of a body with a PI controller and a power limit actuator. In this case, the force $F = m\ddot{x} + k\dot{x}$, where m and k represents the mass of the body and the coefficient of friction, respectively. Typically, if the position $x(t)$ is to track some reference command $u(t)$, a PI controller is used. In this case, $P(s) = (ms + k)^{-1}$.

Not only systems satisfying the Newton's law $F = ma$ can be modeled as in Fig. 1. Many servo systems, including mechanical systems, are often modeled this way. A double integrator system may be used as a simple model for satellite control, modeling the relation between the angular position and velocity and the reaction jets. Other examples are the control of a hard-disk drive head, the laser beam of a compact disk, etc.

Analysis of saturation systems with double integrators has been done for many years. As explained in [9], in order to perform robustness analysis the system is typically transformed into one shown in Fig. 2, where the saturation is treated as an uncertainty. The problem with this approach is that it gives us a nominal plant that is marginally unstable, preventing us to apply some classical analysis tools such as the Popov criterion, μ-analysis, and integral quadratic constraints (IQCs).

An alternative is to encapsulate the unstable operator in an artificial feedback loop which defines a bounded operator. Robustness analysis can then be performed on the transformed system which consists of bounded operators. Assuming $P(s)$ is stable, this leaves us with the double integrator and the saturation to worry about. A possible loop transformation is shown in Fig. 3. In order to analyze the system, we must first check if Δ is a bounded operator. In this case, Δ is a double integrator in feedback interconnection with a saturation nonlinearity, where the output consists of signals from both the first and second integrator. The question of whether the system $\ddot{x} = sat(-k_1 x - k_2\dot{x} + u)$ has finite $\mathcal{L}_2$-gain from u to x, $\dot{x}$, or $\ddot{x}$, has been posted as an open problem [2]. It has been shown, meanwhile, that the $\mathcal{L}_2$-gain from u to x is infinite [10], and the $\mathcal{L}_2$-gain from u to $\dot{x}$ is also infinite [9]. This means the loop transformation in Fig. 3 does not result in a finite $\mathcal{L}_2$-gain operator Δ.

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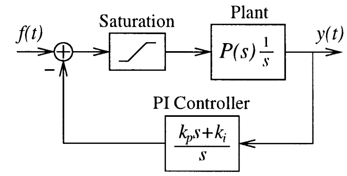

Fig. 1. PI position control system with power-limited actuator.

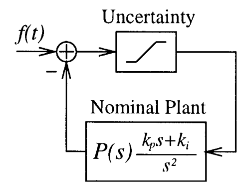

Fig. 2. Nominal system and uncertainty.

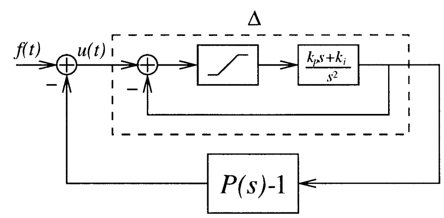

Fig. 3. Loop transformation with an *unstable* operator Δ.

In this note, we propose the loop decomposition shown in Fig. 4, where k_1, k_2, and $G(s)$ are functions of k_p, k_i, and $P(s)$, and $G(s)$ is stable (see Appendix-A for details). The loops of both systems in Figs. 1 and 4 are identical and analysis properties can be inferred from one to another and *vice versa*. The low-pass filter is used to exclude high frequency content from the feedback loop, as expected from real applications. In [3], it is shown that for $k_1 = k_2 = 1$ and $\alpha = 0$, the $\mathcal{L}_2$-gain of Δ is finite, but no upper bound of this gain is given. The goal of this note is, for given $k_1 > 0$, $k_2 \geq 0$, and $\alpha > 0$, to give sufficient conditions to: 1) check if the $\mathcal{L}_2$-gain of Δ is finite, and 2) find upper bounds on the $\mathcal{L}_2$-gain of Δ. We show that our method is *not conservative* for many values of k_1, k_2, and α since we are able to find upper bounds on the $\mathcal{L}_2$-gain of Δ that are approximately equal to lower bounds obtained when the saturation is replaced by a unity constant gain.[1] The method is based on constructing quadratic Lyapunov functions on the switching surface associated with the saturation system. The construction of such Lyapunov functions is done by solving a set of linear matrix inequalities (LMIs).

This note is organized as follows. Section II contains the main result of the note and several illustrative examples. Section III proves the main result and Section IV gives conclusions. Finally, computational details can be found in the Appendix.

II. Main Results

A. Preliminaries

Let $\mathcal{L}_2$ denote the space of all functions $f\colon [0, \infty) \to \mathbb{R}$ which are square summable, i.e.,

$$\|f\|^2 = \int_0^\infty f^2(t)\,dt < \infty.$$

[1]In other words, a lower bound is obtained when the saturation $y = \mathrm{sat}(\xi)$ is replaced by a unit constant gain $y = \xi$.

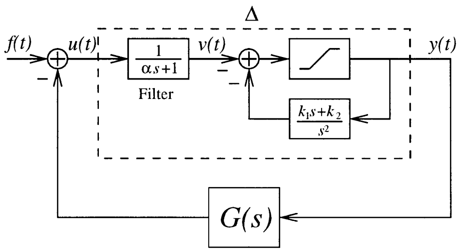

Fig. 4. Loop transformation with *stable* operators.

The extended space $\mathcal{L}_{2e}$ consists of all functions $f(t)$ which satisfy $P_T f(t) \in \mathcal{L}_2$, for all $T \geq 0$, where P_T is a truncation operator defined as $(P_T f)(t) = f(t)$ if $t \leq T$ and $(P_T f)(t) = 0$, otherwise.

We say that the $\mathcal{L}_2$-gain from input u to output y of some system is less than $\gamma \geq 0$ if

$$\int_0^T y^2(t)\,dt \leq \gamma \int_0^T u^2(t)\,dt \tag{1}$$

for all $T \geq 0$, and all $u \in \mathcal{L}_{2e}$. The $\mathcal{L}_2$-gain γ^* of the system from u to y is the infimum over all γ such that (1) is satisfied.

Consider the operator Δ in Fig. 4. For given $k_1,\ k_2,\ \alpha$, we are interested in finding an upper bound of the $\mathcal{L}_2$-gain of Δ. The following proposition gives an easy way to find a lower bound of the $\mathcal{L}_2$-gain of Δ. The proof, based on the fact that the saturation behaves linearly for small inputs, can be found in the Appendix-B.

Proposition 2.1: Consider the system Δ in Fig. 4. The $\mathcal{L}_2$-gain γ_L of the same system but with the saturation replaced by a constant gain of 1 is a lower bound of the $\mathcal{L}_2$-gain of Δ, i.e., $0 \leq \gamma_L \leq \gamma^*$.

Note that when the saturation is replaced by a constant gain of 1, the system becomes linear. Thus, γ_L is simply the square of the $\mathcal{H}_\infty$-norm of the linear system

$$\frac{Y(s)}{U(s)} = \frac{s^2}{(\alpha s+1)(s^2+k_1 s+k_2)}.$$

From this expression, we immediately see that it is necessary $k_1 > 0$, $k_2 \geq 0$, and $\alpha > 0$, or, otherwise, $\gamma_L = \infty$. When $k_2 = 0$ the original system is reduced to a single integrator which was studied in [8] and [11]. Hence, throughout the note we assume $k_2 > 0$. Note that the case of $k_2 = 0$ could also be analyzed using the same ideas described later [5]. The proof of the following proposition can be found in the Appendix-B.

Proposition 2.2: Consider the system Δ in Fig. 4. If there exists an $\alpha = \alpha_1 > 0$ such that the $\mathcal{L}_2$-gain of Δ is finite then the $\mathcal{L}_2$-gain is finite for any $\alpha > 0$.

A state-space representation of system Δ in Fig. 4 is

$$\begin{cases} \dot{x}_1 = k_2 x_2 \\ \dot{x}_2 = y \\ \dot{v} = -\dfrac{1}{\alpha}\,v + \dfrac{1}{\alpha}\,u \\ y = \operatorname{sat}(-x_1 - k_1 x_2 - v) \end{cases} \tag{2}$$

where $\operatorname{sat}(\cdot)$ denotes the standard saturation, defined as $\operatorname{sat}(\xi) = \operatorname{sign}(\xi)\min(|\xi|,\,1)$. Let $x = [x_1\, x_2\, v]'$ and $C = [1\, k_1\, 1]$. In the state–space, the system can be seen as a piecewise linear system, with three cells and two switching surfaces (see Fig. 5). The switching surfaces are

$$S = \{x \in \mathbb{R}^3 \colon Cx = 1\}$$

and $\underline{S} = -S$. When $Cx \geq 1$, $\dot{x}_2 = -1$, when $Cx \leq -1$, $\dot{x}_2 = 1$, and, finally, when $-1 \leq Cx \leq 1$, $\dot{x}_2 = -Cx$.

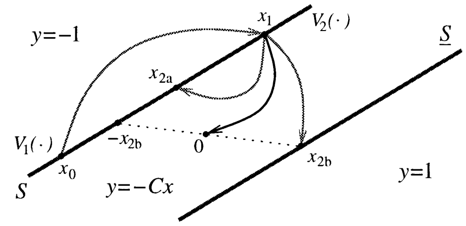

Fig. 5. Possible trajectories in the state–space.

B. Double Integrator

Assume that $k_2 > 0$. The following matrices will be needed in the main result. For some $T > 0$, let

$$W_a(T) = \begin{pmatrix} \dfrac{k_1}{k_2} - \dfrac{T}{2} \\ 0 \\ \dfrac{k_1}{k_2} + \dfrac{T}{2} \\ 0 \end{pmatrix} \qquad W_b(T) = \begin{pmatrix} -\dfrac{1}{k_2 T} & \dfrac{1}{k_2 T} \\ 1 & 0 \\ -\dfrac{1}{k_2 T} & \dfrac{1}{k_2 T} \\ 0 & 1 \end{pmatrix}$$

and

$$W_j(T) = \frac{2\gamma\alpha}{1 - e^{-(2T/\alpha)}} \begin{pmatrix} 1 \\ -e^{-(T/\alpha)} \end{pmatrix} \begin{pmatrix} 1 & -e^{-(T/\alpha)} \end{pmatrix}.$$

Define also

$$A = \begin{pmatrix} 0 & k_2 & 0 \\ -1 & -k_1 & -1 \\ 0 & 0 & -\dfrac{1}{\alpha} \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{\alpha} \end{pmatrix}$$

and

$$H = \begin{pmatrix} A & BB'/\gamma \\ -C'C & -A' \end{pmatrix} \qquad e^{HT} = \begin{pmatrix} e_{11}(T) & e_{12}(T) \\ e_{21}(T) & e_{22}(T) \end{pmatrix} \tag{3}$$

where each $e_{ij}(T)$ is a 3 by 3 matrix and

$$W_t(T) = \begin{bmatrix} e_{22}e_{12}^{-1} & \frac{1}{2}\left(e_{21} - e_{22}e_{12}^{-1}e_{11} - (e_{12}^{-1})'\right) \\ \frac{1}{2}\left(e_{21} - e_{22}e_{12}^{-1}e_{11} - (e_{12}^{-1})'\right)' & e_{12}^{-1}e_{11} \end{bmatrix}$$

where the notation $e_{ij} = e_{ij}(T)$ was used for simplification. Finally, define

$$W_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad W_2 = \begin{pmatrix} -k_1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -k_1 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad W_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}.$$

We are now ready for the main result of this note. In this result, we drop the argument (T) for simplification.

Theorem 2.1: Consider the system Δ in Fig. 4. Given $k_1,\ k_2,\ \alpha > 0$, let $\gamma \geq \gamma_L$. Also, let $p > 0$ be a 2 by 2 diagonal matrix and $g \in \mathbb{R}^2$. Define

$$P = \begin{pmatrix} p & 0 \\ 0 & -p \end{pmatrix} \quad G = \begin{pmatrix} g \\ -g \end{pmatrix} \quad \overline{G} = \begin{pmatrix} -g \\ -g \end{pmatrix}.$$

If

$$R_1(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_j - W_b' P W_b & -W_b'(PW_a + G) \\ -(W_a'P + G')W_b & -T - W_a'PW_a - 2W_a'G \end{pmatrix} > 0 \tag{4}$$

$$R_{2a}(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_2'W_tW_2 - P & W_2'W_tW_1 - G \\ W_1'W_tW_2 - G' & W_1'W_tW_1 \end{pmatrix} > 0 \tag{5}$$

$$R_{2b}(T) \stackrel{\text{def}}{=} \begin{pmatrix} W_2'W_tW_2 - P & W_2'W_tW_3 - \overline{G} \\ W_3'W_tW_2 - \overline{G}' & W_3'W_tW_3 \end{pmatrix} > 0 \tag{6}$$

for all $T > 0$, then the $\mathcal{L}_2$-gain of Δ is less or equal than γ.

The last theorem gives us a set of infinite-dimensional LMIs that, when satisfied, guarantee that Δ not only has finite $\mathcal{L}_2$-gain, but also that this is upper bounded by γ. This allows us to write an IQC of the form

$$\int_0^T y^2(t)\,dt \le \gamma \int_0^T u^2(t)\,dt \tag{7}$$

which, in turn, allows us to perform robustness and performance analysis on the system in Fig. 4 or, equivalently, on the original system in Fig. 1.

The method of proof is as follows. First, inequality (7) is satisfied if for every $u \in \mathcal{L}_{2e}$ there exists a Lyapunov function $V(\cdot)$ such that the solution $x(t)$ from the initial state $x(0) = 0$ satisfies

$$\int_{T_i}^{T_f} [\gamma u^2(t) - y^2(t)]\,dt \ge V(x(T_f)) - V(x(T_i)) \tag{8}$$

for all $0 \le T_i \le T_f$. To see this, let $T_i = 0$. Then, $V(x(0)) = 0$ and $V(x(T_f)) \ge 0$, since V is a Lyapunov function.

Fig. 5 shows possible trajectories of (2) starting at S. Depending on the control input u, a trajectory may enter the region where $y = -1$. Since $u \in \mathcal{L}_{2e}$, a switch must eventually occur at some point $x_1 \in S$. The control u may also be such that the trajectory enters the linear region where $y = -Cx$. In this case, there are three possibilities: the trajectory does not switch again and goes to zero as $t \to \infty$, it returns to S, or it intersects $\underline{S}$. Since the system is symmetric around the origin, for analysis purposes, any other trajectories can be reduced to the ones just described.

Second, define two Lyapunov functions V_1 and V_2 on the switching surface S. Condition (8) is satisfied if

$$\int_0^{T_1} [\gamma u_1^2(t) - y^2(t)]\,dt \ge V_2(x_1) - V_1(x_0) \tag{9}$$

$$\int_0^{T_{2a}} [\gamma u_{2a}^2(t) - y^2(t)]\,dt \ge V_1(x_{2a}) - V_2(x_1) \tag{10}$$

$$\int_0^{T_{2b}} [\gamma u_{2b}^2(t) - y^2(t)]\,dt \ge V_1(-x_{2b}) - V_2(x_1) \tag{11}$$

for all $x_0,\, x_1,\, x_{2a},\, -x_{2b} \in S$, and $T_1,\, T_{2a},\, T_{2b} > 0$, and where $u_1(t) \in \mathcal{L}_2$ is such that a trajectory starting at x_0 satisfies $x_1 = x(T_1)$ and $y = -1,\, t \in [0,\, T_1]$, and $u_i(t) \in \mathcal{L}_2,\, i = 2a,\, 2b$ is such that a trajectory starting at x_1 satisfies $x_i = x(T_i)$ and $y = -Cx,\, t \in [0,\, T_i]$.

Finally, under certain assumptions, the inputs $u_i, i = 1,\, 2a,\, 2b$, that minimize the integrals on the left side of the above inequalities can be explicitly found. If the Lyapunov functions are chosen to be quadratic functions, the results are conditions (4)–(6). The details of the proof can be found in Section III.

C. Examples

In order to solve an infinite-dimensional set of LMIs, there are some extra steps we need to take to make this solution computationally attractive. Obviously, it is not possible to solve the three quadratic inequalities for all $T > 0$. The idea is to find a finite sequence of times $\{T_i\}$ defined on some bounded set $\mathcal{T} = (0,\, T_+]$ such that it is sufficient (4)–(6) are satisfied in $\mathcal{T}$ to prove the desired result. It is then necessary to guarantee they are also satisfied in $T \in (T_+,\, \infty)$, and $T \in (T_i,\, T_{i+1})$ for all $T_i,\, T_{i+1} \in \mathcal{T}$. The latest can be guaranteed by estimating bounds on the derivative of each condition (4)–(6) between $T_i,\, T_{i+1} \in \mathcal{T}$ (see [4] for more details). Conditions to guarantee that (4)–(6) are also satisfied in $T \in (T_+,\, \infty)$, for some $0 < T_+ < \infty$, are given in Propositions C.1 and C.2.

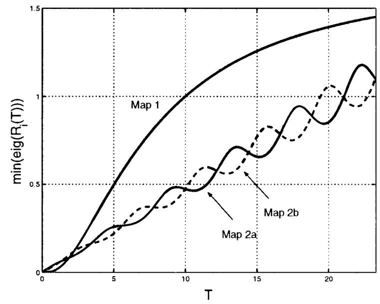

Fig. 6. Minimum eigenvalues of $R_i(T),\; i = 1,\, 2a,\, 2b$.

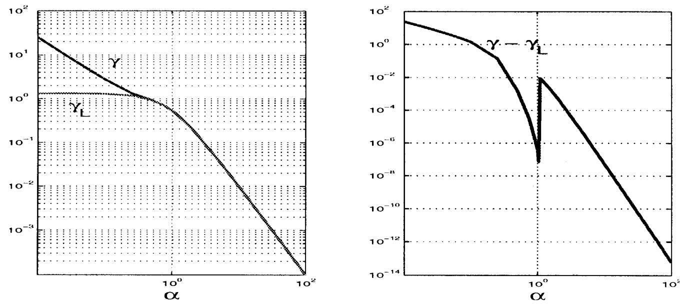

Fig. 7. γ and γ_L as a function of α (left) and $\gamma - \gamma_L$ (right).

The following examples were processed in Matlab code. The latest version of this software is available at [7]. Before presenting the examples, we briefly explain the Matlab function we developed. The user supplies $k_1 > 0, k_2 \ge 0$ (the case when $k_2 = 0$ results in the single integrator which will be dealt in the next section), and $\alpha > 0$. If all three conditions (4)–(6) are satisfied for all $T \in \mathcal{T}$, the function returns a graphic showing the minimum eigenvalues of each $R_i(T)$, which, obviously, must be positive for all $T \in \mathcal{T}$.

Example 2.1: Let $k_1 = 0.5, k_2 = 2$, and $\alpha = 2$. In this example, we find the smallest upper bound γ of the $\mathcal{L}_2$-gain of Δ in Fig. 4 using Theorem 2.1. A lower bound can be found by computing the linear gain, i.e., the $\mathcal{L}_2$-gain of Δ when the saturation nonlinearity is replaced by a constant gain of 1. Here, this is $\gamma_L = 0.889\,229\,7$. Using the software previously described, we found an upper bound of the $\mathcal{L}_2$-gain of Δ of $\gamma = 0.889\,229\,9$. Note that the difference between the upper and lower bound is smaller than 2×10^{-7}, i.e., the precision is less than 2.15×10^{-5}%.

Fig. 6 shows the minimum eigenvalues of $R_i(T),\; i = 1,\, 2a,\, 2b$. For visualization purposes, the minimum eigenvalues of $R_{2a}(T)$ and $R_{2b}(T)$ were scaled by 2×10^6. ■

Example 2.2: Let $k_1 = k_2 = 1$. In this example, we find the smallest upper bound γ of the $\mathcal{L}_2$-gain of Δ for different values of $\alpha > 0$. The left-hand side of Fig. 7 shows the lower bound γ_L and the upper bound γ on the $\mathcal{L}_2$-gain of Δ. The right-hand side of Fig. 7 plots $\gamma - \gamma_L$. Logarithmic scales were used for better visualization.

From this figure, we can see that the difference between the upper and lower bound goes to zero as α goes to infinity. In fact, for $\alpha > 0.5$ the difference between γ and γ_L is less than 0.76%. For $\alpha > 5$ this difference is already smaller than 0.009% and less than 6×10^{-8}% for $\alpha > 100$.

If $\gamma \ge \gamma_L$ is chosen small enough, the Hamiltonian matrix H in (3) has pure imaginary eigenvalues. For $\alpha \ge 0.5$, it turns out that for *all* $\gamma > \gamma_L$ such that H has no pure imaginary eigenvalues, it was *always* possible to find $p,\ g$ such that conditions (4)–(6) are satisfied. In other words, numerically we found that for $\alpha \geq 0.5$ conditions (4)–(6) are satisfied if and only if H has no pure imaginary eigenvalues. Thus, for $\alpha \geq 0.5$, Fig. 7 also shows the smallest γ such that H does not have pure imaginary eigenvalues. For $\alpha < 0.5$, however, we encountered several numerical problems and γ tended to be higher than the smallest γ such that H has no pure imaginary eigenvalues.

Several questions can now be raised: is the gap between γ and γ_L increasing as α approaches zero due to numerical errors, conservatism of the method, or the fact that the $\mathcal{L}_2$-gain of the system is just larger than γ_L, and this gap increases as α approaches zero? Or is true that $\gamma = \gamma_L$ or $\gamma \approx \gamma_L$ for all $\alpha > 0$? Answers to such questions are currently under investigation.

Certainly, this example shows that our method is *not conservative*, except maybe for small values of α, since the upper and lower bounds of the $\mathcal{L}_2$-gain of Δ are almost identical. ■

III. Proof of Theorem 2.1

In this section, we show that if (4)–(6) are satisfied then so are conditions (9)–(11). However, before we do, consider (9) and (10). If $x_1 = x_0 = x_{2a} \in S$ and $T_1 = T_{2a} = 0$ then it results that the left side of both conditions is equal to zero, i.e.,

$$0 \geq V_2(x_0) - V_1(x_0)$$
$$0 \geq V_1(x_0) - V_2(x_0)$$

which means that $V_1(\cdot) = V_2(\cdot)$, i.e., the Lyapunov functions must be identical. So, from now on, we consider $V(\cdot) = V_1(\cdot) = V_2(\cdot)$.

A notion that will be usefully throughout the rest of the proof is the notion of an *impact map* [6], [4]. An impact map is simply a map from one switching surface to the next switching surface. There are three impact maps of interest associated with a saturation system (see Fig. 5). The fist impact map (impact map 1) takes points $x_0 \in S$ and maps them back to $x_1 \in S$ such that the trajectory stays in the region where $y = -1$. The second impact map (impact map $2a$) takes points from $x_1 \in S$ and also maps them back to $x_{2a} \in S$, but this time the trajectory stays in the region where $y = -Cx$. Finally, the third impact map (impact map $2b$) takes points from $x_1 \in S$ and maps them to $x_{2b} \in \underline{S}$ such that the trajectory stays in the region where $y = -Cx$.

Each of these impact maps is associated with each (9)–(11). We will start with impact map 1 and (9).

A. Impact Map 1

The first map we consider is the map that leaves S and returns to S and the trajectory remains in the region where $Cx \geq 1$. Here, $y = -1$ and therefore $\dot{x}_2 = -1$. Let $x_0,\ x_1 \in S$ and $T > 0$. For simplicity, write $x(0) = x_0 = [x_{10}\ x_{20}\ v_0]'$ and $x(T) = x_T = [x_{1T}\ x_{2T}\ v_T]'$. Note that, in this region, only the last state v is controllable. The first two states x_1 and x_2 do not depend on the input. Integrating, we get $x_2(t) = -t + x_{20}$ and, at $t = T$, $x_{2T} = -T + x_{20}$. This means that $\dot{x}_1(t) = -k_2 t + k_2 x_{20}$. Integrating, and evaluating at $t = T$, we get $x_{1T} = -k_2 T^2/2 + k_2 x_{20} T + x_{10}$. Since $x_0,\ x_1 \in S$, it is also true that $x_{10} + k_1 x_{20} + v_0 = 1$ and $x_{1T} + k_1 x_{2T} + v_T = 1$. This gives us four equations with six variables. Let the free variables be v_0 and v_T, and define

$$\begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix} \stackrel{\text{def}}{=} \begin{pmatrix} x_{2T} \\ v_T \\ x_{20} \\ v_0 \end{pmatrix} = W_a(T) + W_b(T) \begin{pmatrix} v_T \\ v_0 \end{pmatrix}.$$

Next, we solve the following minimization problem:

$$J^* = \min_{u \in \mathcal{L}_2} \int_0^T \left(\gamma u^2(t) - y(t)\right)\, dt$$

subject to $\dot{v} = -(1/\alpha)v + (1/\alpha)u$, $v(0) = v_0$, $v(T) = v_T$, and u is such that $Cx(t) \geq 1$, $t \in [0,\ T]$. In order to find an explicit solution for u, we relax the problem by ignoring the fact that $Cx(t) \geq 1$, $t \in [0,\ T]$. The problem then becomes a standard $\mathcal{H}_2$ optimization problem whose solution can be found, for example, in [1] and [5]. In this case

$$J^* = \begin{pmatrix} v_T \\ v_0 \end{pmatrix}' W_j(T) \begin{pmatrix} v_T \\ v_0 \end{pmatrix} - T.$$

Define a quadratic surface Lyapunov function $V(\cdot)$ in S as $V(\Delta_i) = \Delta_i' p \Delta_i + 2\Delta_i' g$, where $p = p' > 0$. Hence

$$\begin{aligned} V(\Delta_1) - V(\Delta_0) &= \begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix}' P \begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix} + 2 \begin{pmatrix} \Delta_1 \\ \Delta_0 \end{pmatrix}' G \\ &= \begin{pmatrix} v_T \\ v_0 \end{pmatrix}' W_b' P W_b \begin{pmatrix} v_T \\ v_0 \end{pmatrix} + 2 \begin{pmatrix} v_T \\ v_0 \end{pmatrix}' \\ &\quad \cdot W_b'(PW_a + G) + W_a' P W_a + 2W_a' G \end{aligned}$$

where $W_a = W_a(T)$ and $W_b = W_b(T)$ were used for simplification. Finally, $J^* > V(\Delta_1) - V(\Delta_0)$ is equivalent to (4).

The reason why $p > 0$ is chosen a diagonal matrix versus a symmetric one comes from the following proposition. The proof can be found in [5].

Proposition 3.1: Let

$$p = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix} > 0.$$

If $p_3 \neq 0$, then (4) is never satisfied for large enough values of $T > 0$.

B. Impact Maps $2a$ and $2b$

The next map we consider is the map that leaves S and returns to S and the trajectory remains in the region where $-1 \leq Cx \leq 1$. This means $y = -Cx$, or $\dot{x}_2 = -Cx$. In this region, the system is linear given by $\dot{x} = Ax + Bu$. Let $x_1 = [x_{10}\ x_{20}\ v_0]'$, $x_{2a} = [x_{1T}\ x_{2T}\ v_T]' \in S$ and $T > 0$. Here, all the states are controllable and finding the optimal cost J^* follows the same way as previously shown or in [1] and [5], yielding

$$J^* = \begin{pmatrix} x_{2a} \\ x_1 \end{pmatrix}' W_t(T) \begin{pmatrix} x_{2a} \\ x_1 \end{pmatrix}.$$

Since $x_1,\ x_{2a} \in S$

$$\begin{pmatrix} x_{2a} \\ x_1 \end{pmatrix} = W_1 + W_2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}$$

where $\Delta_1 = [x_{20}\ v_0]'$ and $\Delta_{2a} = [x_{2T}\ v_T]'$. Hence

$$J^* = \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' W_2' W_t W_2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix} + 2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' W_2' W_t W_1 + W_1' W_t W_1.$$

On the other hand

$$V(\Delta_{2a}) - V(\Delta_1) = \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' P \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix} + 2 \begin{pmatrix} \Delta_{2a} \\ \Delta_1 \end{pmatrix}' G.$$

Finally, $J^* > V(\Delta_{2a}) - V(\Delta_1)$ is equivalent to (5).

The last impact map to consider is impact map $2b$ from S to $\underline{S}$ and where the trajectory remains in the same region as the previous map. The proof for this map is similar to the one from impact map $2a$. The only difference is that

$$\begin{pmatrix} x_{2b} \\ x_1 \end{pmatrix} = W_1 + W_3 \begin{pmatrix} \Delta_{2b} \\ \Delta_1 \end{pmatrix}$$

since $x_1 \in S$ and $x_{2b} \in \underline{S}$. This means that $-x_{2b} \in S$ resulting in $J^* > V(-\Delta_{2b}) - V(\Delta_1)$, which is equivalent to (6).

IV. Conclusion

This note gives conditions in the form of LMIs that, when satisfied, guarantee a system with a double integrator in feedback with a saturation nonlinearity has finite $\mathcal{L}_2$-gain. Moreover, for a large class of such systems, we showed that the linear $\mathcal{L}_2$-gain of the system, i.e., the $\mathcal{L}_2$-gain of the same system but with the saturation nonlinearity replaced by a constant gain of 1, is approximately equal to the $\mathcal{L}_2$-gain of the original system. These results allow the use of classical analysis tools like μ-analysis or IQCs to analyze systems with double integrators and saturations, including servo systems like some mechanical systems, satellites, hard-disks, compact disk players, etc.

Appendix

A. Loop Transformation to Find Stable Operators

In this section, we show how to chose k_1, k_2, and $G(s)$ as functions of k_p, k_i, and $P(s)$ so that $G(s)$ is a proper stable system and the systems in Figs. 1 and 4 are equivalent, in the sense that both loops are identical. In other words, analysis properties can be inferred from one to another and vice versa. First, let $P(s)$ be written as

$$P(s) \stackrel{\text{def}}{=} \frac{n(s)}{d(s)} = \frac{\xi_m s^m + \cdots + \xi_1 s + \xi_0}{s^n + \beta_{n-1} s^{n-1} + \cdots + \beta_1 s + \beta_0}$$

where $m \le n$, $\xi_0 \ne 0$ or otherwise the system would have only one integrator, and also $\beta_0 \ne 0$ or otherwise the system would have three integrators and therefore be unstable (Sussmann and Yang [12] showed that a chain of n integrators, $n \ge 3$, cannot be stabilized by bounded linear feedback).

Proposition A.1: Let

$$k_1 = \frac{\xi_0}{\beta_0} k_p + \frac{1}{\beta_0}\left(\xi_1 - \xi_0 \frac{\beta_1}{\beta_0}\right) k_i \quad k_2 = \frac{\xi_0}{\beta_0} k_i$$

and the proper system

$$G(s) = \frac{\overline{n}(s)}{d(s)} (\alpha s + 1)$$

where the degree of $\overline{n}(s)$ is strictly less than the degree of $d(s)$. Then, the systems in Figs. 1 and 4 are have identical loops and analysis properties can be inferred from one to another and vice versa. Moreover, $G(s)$ is stable if and only if $P(s)$ is stable.

The proof, omitted here, is based on replacing the above equalities in the system in Fig. 4 and showing that this loop is indeed identical to the one in Fig. 1.

B. Proof of Propositions 2.1 and 2.2

Proof of Proposition 2.1: Consider the system Δ_L obtained from system Δ in Fig. 4 with the saturation replaced by a constant gain of 1, and let γ_L be the respective $\mathcal{L}_2$-gain. For simplicity, and without loss of generality, assume there exist a control input $u_L^* \in \mathcal{L}_2$ such that $\|y_L\|^2 = \gamma_L \|u_L^*\|^2$ (a similar argument can be applied if such $u_L^* \in \mathcal{L}_2$ does not exist by considering a sequence of $u_i \in \mathcal{L}_2$ resulting in γ_i arbitrarily close to γ_L). Since Δ_L is linear, u_L^* can be scaled such that $|y_L(t)| \le 1$. Hence, by applying such input u_L^* to Δ, we obtain $\|y\|^2 = \gamma_L \|u_L^*\|^2$ since the saturation never leaves the linear region. This means that γ_L is a lower bound of the $\mathcal{L}_2$-gain of Δ. ■

Proof of Proposition 2.2: Let $\gamma_1 < \infty$ be an upper bound of the $\mathcal{L}_2$-gain of Δ when $\alpha = \alpha_1 > 0$. Let now $\alpha > 0$ and consider the following subsystem:

$$\frac{U(s)}{\overline{U}(s)} = \frac{\alpha_1 s + 1}{\alpha s + 1}.$$

The $\mathcal{L}_2$-gain of this subsystem is $\overline{\gamma} = \max(1, \alpha_1/\alpha)$. Then, the $\mathcal{L}_2$-gain γ of Δ when $\alpha > 0$ ($\mathcal{L}_2$-gain from $\overline{u}$ to y) is upper bounded by $\overline{\gamma}\gamma_1$, i.e., $\gamma \le \overline{\gamma}\gamma_1 < \infty$. ■

C. Computational Details

In order to be able to solve for the parameters of the Lyapunov function, we need to first to solve several computational issues associated with (4)–(6) in Theorem 2.1. In particular, we need to guarantee the conditions are satisfied for large enough values of T. We start with impact map 1.

C.1 Impact Map 1: Proposition 3.1 establishes that $p > 0$ must be a diagonal matrix, i.e., $p = \operatorname{diag}(p_1, p_2)$, where $p_1, p_2 > 0$. Hence, after some manipulation, $R_1(T)$ reduces to the equation shown at the bottom of the page. From the main diagonal of $R_1(T)$, we see immediately that it is necessary that

$$2p_1 \frac{k_1}{k_2} + 2g_1 - 1 > 0 \text{ and } 0 < p_2 < 2\gamma\alpha.$$

The next proposition guarantees that if the inequalities in (12) are satisfied, then for any large enough $T > 0$ condition $R_1(T) > 0$ is always satisfied. The proof, omitted here, is based on showing that for large enough T all the eigenvalues of $R_1(T)$ are positive.

Proposition C.1: If both inequalities in (12) are satisfied then there exists a $T_{1+} \in [0, \infty)$ such that $R_1(T) > 0$ for all $T \ge T_{1+}$.

C.2 Impact Maps $2a$ and $2b$: The goal of this section is to give a similar result to proposition C.1 for impact maps $2a$ and $2b$. Let us start by decomposing the Hamiltonian matrix H in $H = V\Sigma U$ where

$$V = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} \quad \Sigma = \begin{pmatrix} D & 0 \\ 0 & -D \end{pmatrix} \quad U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

$$R_1(T) = \begin{pmatrix} \dfrac{2\gamma\alpha}{1 - e^{-(2T/\alpha)}} - p_2 & -\dfrac{2\gamma\alpha}{1 - e^{-(2T/\alpha)}} e^{-(T/\alpha)} & -\dfrac{1}{k_2} p_1 - g_2 \\ -\dfrac{2\gamma\alpha}{1 - e^{-(2T/\alpha)}} e^{-(T/\alpha)} & \dfrac{2\gamma\alpha}{1 - e^{-(2T/\alpha)}} e^{-(2T/\alpha)} + p_2 & \dfrac{1}{k_2} p_1 + g_2 \\ -\dfrac{1}{k_2} p_1 - g_2 & \dfrac{1}{k_2} p_1 + g_2 & T\left(2p_1 \dfrac{k_1}{k_2} + 2g_1 - 1\right) \end{pmatrix}$$

and D is such that all its eigenvalues are in the left-half plane. Define matrices M and m such that $W_2 = \mathrm{diag}(M, M)$ and $W_1 = (m' \quad m')'$. Define also

$$L_1 = \begin{pmatrix} M'V_{22}V_{12}^{-1}M & 0 \\ 0 & M'U_{22}^{-1}U_{21}M \end{pmatrix}$$

$$L_{2a} = \begin{pmatrix} M'V_{22}V_{12}^{-1}m \\ M'U_{22}^{-1}U_{21}m \end{pmatrix} \quad L_{2b} = \begin{pmatrix} -M'V_{22}V_{12}^{-1}m \\ M'U_{22}^{-1}U_{21}m \end{pmatrix}$$

and $L_3 = m'(V_{22}V_{12}^{-1} + U_{22}^{-1}U_{21})m$. Denote $G_a = G$ and $G_b = \overline{G}$. Then, after some manipulation of $R_{2a}(T)$ and $R_{2b}(T)$, we get

$$R_{2i\infty} = \lim_{T\to\infty} R_{2i}(T) = \begin{pmatrix} L_1 - P & L_{2i} - G_i \\ L_{2i}' - G_i' & L_3 \end{pmatrix}$$

for $i = a, b$. Then, the following proposition follows.

Proposition C.2: If $R_{2i\infty} > 0$, $i = a, b$, then there exist $T_{i+} \in [0, \infty)$ such that $R_{2i}(T) > 0$, for all $T \geq T_{i+}$, $i = a, b$.

Note that T_{1+}, T_{a+}, and T_{b+} in the last propositions can be found explicitly, although this is not done here.

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Smooth Output Feedback Stabilization of Planar Systems Without Controllable/Observable Linearization

Chunjiang Qian and Wei Lin

***Abstract*—This note considers the problem of global stabilization by output feedback for a family of planar systems whose Jacobian linearization is neither controllable nor observable. The problem cannot be dealt with by existing output feedback design methods—most of them are based on the separation principle. Under appropriate growth conditions, we propose an output feedback control scheme that does not rely on the separation principle and achieves global asymptotic stabilization. The novelty of our control scheme lies in the explicit design of a dynamic output compensator, which combines a nonlinear-gain observer design and the technique of adding a power integrator. As a consequence, an interesting global stabilization result by output feedback can be obtained for feedback linearizable systems in a triangular form, which turns out to be new even in the two-dimensional case.**

***Index Terms*—Global stabilization, nonseparation principle design, planar systems, smooth output feedback, uncontrollable/unobservable linearization.**

I. INTRODUCTION

In this note, we consider a single-input–single-output (SISO) planar system described by equations of the form

$$\begin{aligned} \dot{x}_1 &= x_2^p + \phi_1(x_1) \\ \dot{x}_2 &= u + \phi_2(x_1, x_2) \\ y &= x_1 \end{aligned} \tag{1.1}$$

where $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the control input and the system output, respectively, p is a positive *odd* integer. The functions ϕ_i, $i = 1, 2$, are continuously differentiable with $\phi_i(0) = 0$.

The objective of this note is twofold: to find a sufficient condition for the existence of a *smooth output* feedback control law

$$\begin{aligned} \dot{z} &= \eta(z, y), \qquad z \in \mathbb{R} \\ u &= u(z, y) \end{aligned} \tag{1.2}$$

rendering the closed-loop system (1.1) and (1.2) globally asymptotically stable (GAS), and to develop an explicit design method for the construction of the output dynamic compensator (1.2).

Without a doubt, global asymptotic stabilization by output feedback is one of the most fundamental problems in the area of nonlinear control. Compared with global stabilization by *state* feedback, output feedback stabilization in the large is much more challenging, because the separation principle does not hold usually for nonlinear systems. Over the past decade, a number of researchers have studied this difficult problem and obtained some interesting results. For example, for a class of detectable affine systems [4], [9] or nonaffine systems with dissipative free dynamics [10], global stabilization via output feedback

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