p-DG cyclotomic nilHecke algebras

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Abstract

We categorify a tensor product of two Weyl modules for quantum $\mathfrak{sl}_2$ at a prime root of unity.

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1 Introduction

1.1 Motivation

Quantum groups for generic values of q are algebras defined over the ring $\mathbb{Z}[q, q^{-1}]$. The representation categories of these algebras are braided monoidal categories, which in turn lead to invariants of knots, links and tangles known as Witten-Reshetikhin-Turaev (WRT) invariants. In order to obtain a 3-manifold invariant, one needs to specialize q to a root of unity ζ. The corresponding quantum groups are algebras over the cyclotomic ring $\mathbb{Z}[\zeta]$. The representation theory of quantum groups at a root of unity plays a pivotal role in understanding 3-dimensional topological quantum field theory and 2-dimensional conformal field theory. Of particular importance is the concept of the *fusion product* of certain modules of quantum groups at roots of unity known as *tilting modules*.

Since the discovery of categorified quantum link invariants (see, e.g., [Kho00]), there have been substantial developments in categorification of the quantum algebraic invariants defined at a generic value of the quantum parameter. To this day, it remains a challenging problem to lift the 3-manifold invariants defined over $\mathbb{Z}[\zeta]$ into categorical invariants.

As a modest advancement in this direction, the subject of hopfological algebra was introduced in [Kho16] with the goal of categorifying the quantum WRT invariants at a prime root of unity. The generic ground algebra $\mathbb{Z}[q, q^{-1}]$ admits a straightforward categorification via chain complexes of $\mathbb{Z}$-graded vector spaces, the q-parameter coming from the grading shift on the categorical level. In the hopofological setting, we are forced into working with chain complexes of graded vector spaces whose differential satisfies $\partial^p \equiv 0$ over ground fields of characteristic $p > 0$. To test whether hopfological algebra can be utilized in categorically specializing WRT invariants to the cyclotomic ring[1] $\mathbb{O}_p$, one should look for p-nilpotent derivations on certain algebras categorifying quantum groups and their module categories for generic values of q.

The first example of the program outlined in [Kho16] was given in [KQ15] where the authors categorify one half of the small quantum $\mathfrak{sl}_2$ over $\mathbb{O}_p$ (denoted by $\dot{u}_{\mathbb{O}_p}(\mathfrak{sl}_2)$). The nilHecke algebra over a field of

[1]Here we usually consider a slightly larger ring $\mathbb{O}_p := \mathbb{Z}[q]/(\Psi_p(q^2))$ as the Grothendieck ring of p-complexes whose differentials have degree 2. The degree choice is made to match with previous representation theoretical constructions and does not cause essential differences from working over the genuine cyclotomic ring $\mathbb{Z}[\zeta_p]$.

characteristic p was equipped with a p-nilpotent derivation. It is proved that the direct sum of the compact derived categories of p-DG modules over all such p-DG nilHecke algebras categorify the positive or negative half of this small quantum group. One halves of quantum groups usually play special roles in (categorical) representation theory. One reason is that, as finite-dimensional representations of $\mathfrak{sl}_2$ can be realized as quotients of certain Verma modules, simple categorical representations of quantum $\mathfrak{sl}_2$ can be realized as specific categorical quotients of the nilHecke algebras known as the *cyclotomic quotients*. The story is similar at a prime root of unity. Simple $\mathfrak{sl}_2$-modules specialize at $\mathbb{O}_p$ to the (dual) Weyl modules. The Weyl modules of highest weight in the range $\{0, 1, \ldots, p-1\}$ are also categorified by the corresponding cyclotomic nilHecke algebras equipped with the natural quotient p-differential.

In [EQ16b, EQ16a], the one-half categorification in [KQ15] for $\mathfrak{sl}_2$ is further extended to a categorification of the entire quantum $\mathfrak{sl}_2$, for both the small version $\dot{u}_{\mathbb{O}_p}(\mathfrak{sl}_2)$ and an infinite dimensional version (the BLM form [BLM90]) $\dot{U}_{\mathbb{O}_p}(\mathfrak{sl}_2)$. This is realized by equipping Lauda's 2-category $\mathcal{U}$ and its Karoubi envelope $\dot{\mathcal{U}}$ with p-nilpotent differentials. Unlike at a generic q, there is some subtle hopfological behavior at a root of unity. The p-DG version of Lauda's category $\mathcal{U}$ is homotopy equivalent to its Karoubi envelope $\dot{\mathcal{U}}$ equipped with a compatible p-differential. But the homotopy equivalence fails to descend to the derived level. Rather, on the derived level, there is a fully-faithful embedding of the categorified small quantum group into the categorified BLM form. The two versions of quantum groups at a prime root of unity are also related by Lusztig's quantum Frobenius map, which is explained in [Qi17]. A similar feature also appears in the current work, as we will see in Section 6.4 and 9.3.

The next step is then to study the categorical representation theory of categorified quantum groups at a prime root of unity. Categorical representations, in particular, certain categorical tensor products of quantum groups have been proposed and studied by Webster in [Web17]. Recently, there has been minor progress on this front in [QS16], where a unique differential on the algebras defined by Webster, which is compatible with the categorical quantum group action, is determined. Furthermore, the second lowest weight space of $V_1^{\otimes m}$, where V_1 is the natural 2-dimensional representation of $\mathfrak{sl}_2$, has been categorified. This is done by equipping the quiver algebra Koszul dual to the zigzag algebra with a p-differential. The differential can be regarded as arising from the identification of the quiver algebra with the corresponding block of a Webster algebra. A braid group action is also exhibited in this case, providing some evidence of connections to quantum topology.

The current work will serve as the starting point of a series of works on categorifying tensor product representation of quantum $\mathfrak{sl}_2$ at a prime root of unity. Our goal is to set up a general framework for constructing such tensor product categorifications at a prime root of unity, as well as to gain better understanding of earlier works such as Webster [Web17] and Hu-Mathas [HM15]. In this work, we will focus on the particular case of categorifying a tensor product of two Weyl modules at a root of unity, since it is the foundation for constructing "*categorical fusion products*" of "*categorified tilting modules*". In a forthcoming paper, we will categorify the full representation $V_1^{\otimes m}$ and study the braid group invariants arising from these constructions.

1.2 Summary of contents

Fix two integers $r, s \in \mathbb{N}$ and let $l = r + s$. The main goal of this paper is to construct a categorification of the tensor product representation of two Weyl modules $V_r \otimes_{\mathbb{O}_p} V_s$, where V_r stands for the Weyl module of $\dot{U}_{\mathbb{O}_p}(\mathfrak{sl}_2)$ of rank $(r+1)$. We now give a brief preview of the contents of each section and how the construction is carried out.

In Section 2 and Section 3, we fix some notation and collect necessary background material on representation theory of quantum $\mathfrak{sl}_2$ at a prime root of unity and hopfological algebra of p-differential graded (p-DG) algebras. These sections are used as tool boxes for the remainder of the paper, and may be safely skipped on a first reading by the reader and referred back later when needed.

In Section 4, we develop some basic tools on constructing tensor-product categorifications. The basic question addressed here is that, starting from a collection of algebras categorifying a highest weight or lowest weight representation, how one can build tensor product categorifications directly out of these algebras. The question may be too general to admit a simple universal answer, but when the algebras are Frobenius, a framework for constructing potential tensor-product algebras is established in that section. In particular, we consider faithful (p-DG) modules of the Frobenius algebras, and take their (p-DG) endomorphism algebras as candidates for tensor-product categorifications. Faithfulness of the modules turns out to be a key condition to require, and it implies the double-centralizer property many other works have relied on (see, e.g. [Web17, HM15]). Under the assumption on the (p-DG) functors that they preseve the additive full subcategory generated by the faithful modules over the Frobenius algebra, the new (p-DG) module categories over the endomorphism algebras are acted on by some natural (p-DG) functors extending the initial ones. The basic idea goes back to the construction of some special indecomposable projective modules for a maximal singular block of category $\mathcal{O}(\mathfrak{gl}_m)$ given in [BFK99, Section 3.1.3], and is related to the work of Losev-Webster [LW15]. We will see instances of our construction in a sequel to this paper on categorifying the m-fold tensor $V_1^{\otimes m}$.

Section 5 is devoted to reviewing two seemingly different forms of the (p-DG) 2-categories $(\mathcal{U}, \partial)$, due respectively to Khovanov-Lauda [Lau10, KL10] and Rouquier [Rou08]. In a remarkable work of Brundan [Bru16], it is shown that the definitions of Khovanov-Lauda and Rouquier are equivalent. We then readily extend Brundan's theorem to the p-DG setting (Theorem 5.12).

Section 6 introduces the (p-DG) cyclotomic nilHecke algebra NH_n^l. This is a quotient of the nilHecke algebra NH_n of rank n by an ideal depending upon the natural number l. Taking the sum of the p-DG module categories $\oplus_{n=0}^{l}(\mathrm{NH}_n^l, \partial)\text{-mod}$ provides, imprecisely, a categorification of the Weyl module V_l. These algebras are symmetric Frobenius, and serve as the input data for the categorical framework of Section 4. In [EQ16b], the lowest weight $\mathfrak{sl}_2$-representation V_l is instead categorified via Rouquier's universal (p-DG) cyclotomic quotient. The version there is known to be Morita equivalent to the cyclotomic nilHecke constructions, but relies on certain implicit (p-DG) functors realizing the Morita equivalence. Here we strictify the functors acting on cyclotomic nilHecke algebras, and show that the p-DG 2-category in Rouquier's definition acts directly on the direct sum of p-DG nilHecke module categories. Here, a technical caveat is that we do not pass right away to derived categories of p-DG cyclotomic nilHecke algebras for categorifying V_l, but use the abelian category of p-DG NH_n^l-modules as an intermediate stepping stone.

Sections 7 and 8 constitute the technical heart of the current work. In Section 7, we recall the cellular structure of cyclotomic nilHecke algebras due to Hu-Mathas [HM15]. We utilize the cellular structure on NH_n^l to exhibit a natural collection of cyclic right p-DG modules over NH_n^l. A special collection of cyclic modules p-DG modules $e_\lambda G(\lambda)$ over NH_n^l are introduced, where λ's are parameterized by certain partitions of l into zeros and ones. The construction mimics those $G(\lambda)$'s appearing in Hu-Mathas [HM15], but is further truncated by an idempotent $e_\lambda \in \mathrm{NH}_n^l$. We then show explicitly that the generating functors $\mathcal{E}$ and $\mathcal{F}$ for the category $\mathcal{U}$ acting on $\oplus_{n=0}^{l}(\mathrm{NH}_n^l, \partial)\text{-mod}$ preserve the additive subcategory generated by these special modules $e_\lambda G(\lambda)$. This establishes the necessary conditions for the entire setup to fit into the framework of Section 4. The (p-DG) *two-tensor quiver Schur algebra* $S_n(r,s)$ is defined as (Definition 8.27)

$$S_n(r,s) := \mathrm{END}_{\mathrm{NH}_n^l}\left(\bigoplus_{\lambda} e_\lambda G(\lambda)\right),$$

where $(r,s) \in \mathbb{N}^2$ is the decomposition of l fixed earlier. By construction, the p-DG 2-category $(\mathcal{U}, \partial)$ acts on $\oplus_{n=0}^{l}(S_n(r,s), \partial)\text{-mod}$.

The two-tensor quiver Schur algebra is related to a special case of Webster's diagrammatic tensor algebra by performing a "thick" idempotent truncation, which is shown in Section 9. However, the two-tensor quiver Schur algebra has fewer generating idempotents than in Webster's definition, which allows an easier identification of Grothendieck groups in this particular case. While Webster's setup has certain advantages (for example functors for tangles are naturally defined) we focus on these subcategories of nilHecke algebras since it is a little easier to calculate the p-DG Grothendieck groups. We expect the categories defined in this work are Morita equivalent to Webster algebras but not p-DG Morita equivalent.

We then establish the following in Section 10:

Theorem (10.3)**.** There is an action of the derived p-DG Lauda category on $\oplus_{n=0}^{l}\mathcal{D}(S_n(r,s))$. The action induces an identification of the Grothendieck groups

$$K_0\Big(\bigoplus_{n=0}^{l}\mathcal{D}^c(S_n(r,s))\Big) \cong V_r \otimes_{\mathbb{O}_p} V_s$$

with the tensor product of the quantum $\mathfrak{sl}_2$ Weyl modules V_r and V_s at a primitive pth root of unity.

A comment on the p-DG stratified structure on the two-tensor quiver Schur algebra is briefly discussed in Section 10.2, and some further investigations are sketched out in the final Section 10.3.

1.3 Further comments

A more general m-tensor quiver Schur algebra can be defined when trying to categorify a more general tensor product $V_{r_1} \otimes_{\mathbb{O}_p} \cdots \otimes_{\mathbb{O}_p} V_{r_m}$. See Section 10.3 for a brief introduction. We will pursue further investigations in subsequent works. However, the case of just two tensor factors is relatively simpler. One reason is that the canonical basis of Lusztig [Lus93] has a very explicit description in the two-tensor factor case. For general m the canonical basis has only an inductive construction.

A categorification of tensor products of quantum $\mathfrak{sl}_2$ for generic q was given in [FKS06] by considering certain categories of Harish-Chandra bimodules. It would be interesting to understand the p-differential Lie theoretically.

Quiver Schur algebras were first introduced by Hu and Mathas [HM15] in the graded case. They were interested in putting a $\mathbb{Z}$-grading on the classical Schur algebras via an identification of cyclotomic Hecke algebras with cyclotomic KLR algebras due to Brundan and Kleshchev [BK09]. The quiver Schur algebras are close relatives of Webster's diagrammatic algebras in type A. Webster categorified tensor products of finite-dimensional irreducible representations for quantum groups at generic values of q [Web17] using his algebras. He also proved that special cases of the Webster algebras are graded Morita equivalent to the quiver Schur algebras of [HM15]. However, as the current work shows, in the presence of a p-differential, the Morita equivalence does not necessarily pass to the derived category level. Both perspectives of Hu-Mathas and Webster will exhibit specific features of categorified representation theory, as can be seen from this work and subsequent future works (e.g. on categorifying $V_1^{\otimes m}$ at a prime root of unity).

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2 The quantum group at prime roots of unity

In this section, we collect some basic facts and fix some notation on quantum groups at a prime root of unit. It will serve as the decategorified story of this work, and the interested reader may want to skip this part first and refer back to it later when needed.

Throughout, we assume $\mathbb{N}$ to be the set of non-negative integers (containing zero).

2.1 The small quantum $\mathfrak{sl}_2$

Let l be an odd natural number or 2, and take ζ_{2l} to be a primitive $2l$th root of unity. The quantum group $u_{\mathbb{Q}[\zeta_{2l}]}(\mathfrak{sl}_2)$, which we will denote simply by $u_{\mathbb{Q}[\zeta_{2l}]}$, is the $\mathbb{Q}[\zeta_{2l}]$-algebra generated by $E, F, K^{\pm 1}$ subject to relations:

(1) $KK^{-1} = K^{-1}K = 1$,

(2) $K^{\pm 1}E = \zeta_{2l}^{\pm 2} EK^{\pm 1}, \quad K^{\pm 1}F = \zeta_{2l}^{\mp 2} FK^{\pm 1}$,

(3) $EF - FE = \frac{K-K^{-1}}{\zeta_{2l}-\zeta_{2l}^{-1}}$,

(4) $E^l = F^l = 0$.

The quantum group is a Hopf algebra whose comultiplication map

$$\Delta \colon u_{\mathbb{Q}[\zeta_{2l}]} \longrightarrow u_{\mathbb{Q}[\zeta_{2l}]} \otimes_{\mathbb{Q}[\zeta_{2l}]} u_{\mathbb{Q}[\zeta_{2l}]}$$

is given on generators by

$$\Delta(E) = K^{-1} \otimes E + E \otimes 1, \qquad \Delta(F) = 1 \otimes F + F \otimes K, \qquad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}. \tag{2.1}$$

The comultiplication defined in equation (2.1) is related to the standard comultiplication (see for example [Kho97]) by the anti-isomorphism $E \mapsto F, F \mapsto E, K \mapsto K$.

For the purpose of categorification, it is more convenient to use the idempotented quantum group $\dot{u}_{\mathbb{Q}[\zeta_{2l}]}(\mathfrak{sl}_2)$, which we will denote simply by $\dot{u}_{\mathbb{Q}[\zeta_{2l}]}$. It is a non-unital $\mathbb{Q}[\zeta_{2l}]$-algebra generated by E, F and idempotents 1_m ($m \in \mathbb{Z}$), subject to relations:

(1) $1_m 1_n = \delta_{m,n} 1_m$,

(2) $E1_m = 1_{m+2}E, \quad F1_{m+2} = 1_m F$,

(3) $EF1_m - FE1_m = [m]1_m$,

(4) $E^l = F^l = 0$.

Here $[m] = \sum_{i=0}^{|m|-1} \zeta_{2l}^{1-|m|+2i}$ is the quantum integer specialized at ζ_{2l}.

The quantum group $\dot{u}_{\mathbb{Q}[\zeta_{2l}]}$ has an integral lattice subalgebra which we now recall. For any integer $n \in \{0, 1, \ldots, l-1\}$, let $E^{(n)} = \frac{E^n}{[n]!}$, and $F^{(n)} = \frac{F^n}{[n]!}$. The elements $E^{(n)}$, $F^{(n)}$ ($1 \leq n \leq l-1$), and 1_m ($m \in \mathbb{Z}$) generate an algebra over the ring of cyclotomic integers $\mathcal{O}_{2l} = \mathbb{Z}[\zeta_{2l}]$. Denote this integral form by $\dot{u}_{\zeta_{2l}}$.

Now let $l = p$ be prime. Introduce the auxiliary ring

$$\mathbb{O}_p = \mathbb{Z}[q]/(\Psi_p(q^2)), \tag{2.2}$$

where $\Psi_p(q)$ is the p-th cyclotomic polynomial. We can define in a similar fashion an integral form $u_{\mathbb{O}_p}$ and its dotted version $\dot{u}_{\mathbb{O}_p}$ for the small quantum $\mathfrak{sl}_2$ over $\mathbb{O}_p$. All the necessary changes only occur for the specialization of quantum numbers: instead of at the root of unity ζ_{2l}, we will set $[n]_{\mathbb{O}_p} := q^{n-1} + q^{n-3} + \cdots + q^{1-n}$ to be understood as an element of $\mathbb{O}_p$.

Definition 2.1. The $\mathbb{O}_p$-integral idempotented quantum algebra $\dot{u}_{\mathbb{O}_p}$ is generated by E, F and idempotents 1_m ($m \in \mathbb{Z}$), subject to the relations:

(1) $1_m 1_n = \delta_{m,n} 1_m$ for any $m, n \in \mathbb{Z}$,

(2) $E1_m = 1_{m+2}E, \quad F1_{m+2} = 1_m F$,

(3) $EF1_m - FE1_m = [m]_{\mathbb{O}_p} 1_m$,

(4) $E^p = F^p = 0$.

Let the lower half of $u_{\mathbb{O}_p}$ be the subalgebra generated by the $F^{(n)}$ ($0 \leq n \leq p-1$) and denote it by $u_{\mathbb{O}_p}^-$. Likewise, write the upper half as $u_{\mathbb{O}_p}^+$. For more details see [KQ15, Section 3.3].

2.2 The BLM integral form

We next recall the Beilinson-Lusztig-MacPherson [BLM90] (BLM) integral form of quantum $\mathfrak{sl}_2$ at a prime root of unity. The small quantum sits inside the BLM form as an (idempotented) Hopf algebra.

Definition 2.2. The non-unital associative quantum algebra $\dot{U}_{\mathbb{O}_p}(\mathfrak{sl}_2)$, or just $\dot{U}_{\mathbb{O}_p}$, is the $\mathbb{O}_p$-algebra generated by a family of orthogonal idempotents $\{1_n | n \in \mathbb{Z}\}$, a family of raising operator $E^{(a)}$ and a family of lowering operator $F^{(b)}$ ($a, b \in \mathbb{N}$), subject to the following relations.

(1) $1_m 1_n = \delta_{m,n} 1_m$ for any $m, n \in \mathbb{Z}$.

(2) $E^{(a)} 1_m = 1_{m+2a} E^{(a)}$, $F^{(a)} 1_{m+2a} = 1_m F^{(a)}$, for any $a \in \mathbb{N}$ and $m \in \mathbb{Z}$.

(3) For any $a, b \in \mathbb{N}$ and $\lambda \in \mathbb{Z}$,

$$E^{(a)}E^{(b)}1_m = \begin{bmatrix} a+b \\ a \end{bmatrix}_{\mathbb{O}_p} E^{(a+b)}, \qquad F^{(a)}F^{(b)}1_m = \begin{bmatrix} a+b \\ a \end{bmatrix}_{\mathbb{O}_p} F^{(a+b)}1_m. \tag{2.3}$$

(4) The divided power E-F relations, which state that

$$E^{(a)}F^{(b)}1_m = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a-b+n \\ j \end{bmatrix}_{\mathbb{O}_p} F^{(b-j)}E^{(a-j)}1_m, \tag{2.4a}$$

$$F^{(a)}E^{(b)}1_m = \sum_{j=0}^{\min(a,b)} \begin{bmatrix} a-b+n \\ j \end{bmatrix}_{\mathbb{O}_p} E^{(b-j)}F^{(a-j)}1_m. \tag{2.4b}$$

The elements $E^{(a)}1_m$, $F^{(a)}1_m$ will be referred to as the *divided power elements*.

The small version $\dot{u}_{\mathbb{O}_p}$ sits inside the BLM form $\dot{U}_{\mathbb{O}_p}$ by identifying $E^{(n)}1_\lambda$, $F^{(n)}1_\lambda$ $(1 \leq n \leq p-1)$ with the elements of the same name in $\dot{U}_{\mathbb{O}_p}$.

2.3 Representations

Let V_l be the Weyl module for $\dot{U}_{\mathbb{O}_p}$. It has a basis $\{v_0, v_1, \ldots, v_l\}$ such that

$$1_n v_i = \delta_{n,l-2i} v_i \qquad F v_i = [i+1] v_{i+1} \qquad E v_i = [l-i+1] v_{i-1}. \tag{2.5}$$

The Weyl module V_l is irreducible when $l \leq p-1$.

On $V_r \otimes_{\mathbb{O}_p} V_s$ there is an action of $u_{\mathbb{O}_p}$ given via the comultiplication map Δ. The standard basis of $V_r \otimes V_s$ is given by $\{v_i \otimes v_j | 0 \leq i \leq r, 0 \leq j \leq s\}$.

The canonical basis (due independently to Kashiwara [Kas91] and Lusztig [Lus93]) is more natural from the categorical perspective we pursue here. For a detailed study of the canonical basis for quantum $\mathfrak{sl}_2$ see [Kho97]. The basis in Proposition 2.3 is related to that of [Kho97, Section 1.3.2] using the isomorphism $E \mapsto E, F \mapsto F, K \mapsto K^{-1}$, $q \mapsto q^{-1}$ and a map $V_l \to V_l$ where $v_i \mapsto v_{l-i}$.

Proposition 2.3. There is a basis $\{v_b \diamond v_d | 0 \leq b \leq r, 0 \leq d \leq s\}$ of $V_r \otimes_{\mathbb{O}_p} V_s$ which is given by

$$v_b \diamond v_d = \begin{cases} F^{(d)}(v_b \otimes v_0) = \sum_{j=0}^{d} q^{j(j+c)} \begin{bmatrix} b+j \\ j \end{bmatrix} v_{b+j} \otimes v_{d-j} & \text{if } b \leq c, \\ E^{(a)}(v_r \otimes v_d) = \sum_{j=0}^{a} q^{j(j+b)} \begin{bmatrix} c+j \\ j \end{bmatrix} v_{b+j} \otimes v_{d-j} & \text{if } b \geq c, \end{cases}$$

where $a = r - b$ and $c = s - d$.

Proof. One could easily verify the formulas for $v_b \diamond v_d$ and thus directly see that the set $\{v_b \diamond v_d | 0 \leq b \leq r, 0 \leq d \leq s\}$ is a basis by the comultiplication formulas in equation (2.6). The equation (2.6) in turn follows from (2.1) and induction.

$$\Delta(E^{(t)}) = \sum_{j=0}^{t} q^{-j(t-j)} E^{(t-j)} K^{-j} \otimes E^{(j)} \qquad \Delta(F^{(t)}) = \sum_{j=0}^{t} q^{-j(t-j)} F^{(t-j)} \otimes F^{(j)} K^{t-j}. \tag{2.6}$$

The result follows. □

Remark 2.4. Note that the elements of the canonical basis in Proposition 2.3 can be written as

$$v_b \diamond v_d = \begin{cases} F^{(d)}E^{(a)}(v_r \otimes v_0) & \text{if } b \leq c \\ E^{(a)}F^{(d)}(v_r \otimes v_0) & \text{if } b \geq c \end{cases}$$

where $a = r - b$ and $c = s - d$.

The description of the canonical basis in Proposition 2.3 makes it clear that the structure coefficients of the canonical basis under the operators E and F are elements of $\mathbb{N}[q, q^{-1}]$.

3 Elements of hopfological algebra

3.1 p-DG derived categories

As a matter of notation for the rest of the paper, the undecorated tensor product symbol $\otimes$ will always denote tensor product over the ground field $\Bbbk$. All of our algebras will be graded so $A\text{-mod}$ will denote the category of graded A-modules. We first recall some basic notions.

Definition 3.1. Let $\Bbbk$ be a field of positive characteristic p. A p-DG algebra A over $\Bbbk$ is a $\mathbb{Z}$-graded $\Bbbk$-algebra equipped with a degree-two[2] endomorphism ∂_A, such that, for any elements $a, b \in A$, we have

$$\partial_A^p(a) = 0, \qquad \partial_A(ab) = \partial_A(a)b + a\partial_A(b).$$

Compared with the usual DG case, the lack of the usual sign in the second equation above is because of the fact that the Hopf algebra $\Bbbk[\partial]/(\partial^p)$ is a genuine Hopf algebra, not a Hopf super-algebra.

As in the DG case, one has the notion of left and right p-DG modules.

Definition 3.2. Let (A, ∂_A) be a p-DG algebra. A left p-DG module (M, ∂_M) is a $\mathbb{Z}$-graded A-module endowed with a degree-two endomorphism ∂_M, such that, for any elements $a \in A$ and $m \in M$, we have

$$\partial_M^p(m) = 0, \qquad \partial_M(am) = \partial_A(a)m + a\partial_M(m).$$

Similarly, one has the notion of a right p-DG module.

It is readily checked that the category of left (right) p-DG modules, denoted $(A, \partial)\text{-mod}$ ($(A^{op}, \partial)\text{-mod}$), is abelian, with morphisms grading preserving A-module maps that also commute with differentials. When no confusion can be caused, we will drop all subscripts in differentials.

Definition 3.3. Let M and N be two p-DG modules. A morphism $f : M \longrightarrow N$ in $(A, \partial)\text{-mod}$ is called *null-homotopic* if there is an A-module map h of degree $2 - 2p$ such that

$$f = \sum_{i=0}^{p-1} \partial_N^i \circ h \circ \partial_M^{p-1-i}.$$

It is an easy exercise to check that null-homotopic morphisms form an ideal in $(A, \partial)\text{-mod}$. The resulting quotient category, denoted $\mathcal{H}(A)$, is called the *homotopy category* of left p-DG modules over A, and it is a triangulated category.

The simplest p-DG algebra is the ground field $\Bbbk$ equipped with the trivial differential, whose homotopy category is denoted $\mathcal{H}(\Bbbk)$[3]. In general, given any p-DG algebra A, one has a forgetful functor

$$\mathrm{For} : \mathcal{H}(A) \longrightarrow \mathcal{H}(\Bbbk) \tag{3.1}$$

by remembering only the underlying p-complex structure up to homotopy of any p-DG module over A. A morphism between two p-DG modules $f : M \longrightarrow N$ (or its image in the homotopy category) is called a *quasi-isomorphism* if $\mathrm{For}(f)$ is an isomorphism in $\mathcal{H}(\Bbbk)$. Denoting the class of quasi-isomorphisms in $\mathcal{H}$ by $\mathcal{Q}$, we define the p-DG derived category of A to be

$$\mathcal{D}(A) := \mathcal{H}(A)[\mathcal{Q}^{-1}], \tag{3.2}$$

the localization of $\mathcal{H}(A)$ at quasi-isomorphisms. By construction, $\mathcal{D}(A)$ is triangulated.

[2] In general one should define the degree of ∂_A to be one. We adopt this degree only to match earlier grading conventions in categorification. One may adjust the gradings of the algebras we consider so as to make the degree of ∂_A to be one, but we choose not to do so.

[3] This is usually known as the graded stable category of $\Bbbk[\partial]/(\partial^p)$.

3.2 Hopfological properties of p-DG modules

Many constructions in the usual homological algebra of DG-algebras translate over into the p-DG context without any trouble. For a starter, it is easy to see that the homotopy category of the ground field coincides with the derived category: $\mathcal{D}(\Bbbk) \cong \mathcal{H}(\Bbbk)$. We will see a few more illustrations of the similarities in what follows.

We first recall the following definitions.

Definition 3.4. Let A be a p-DG algebra, and K be a (left or right) p-DG module.

(1) The module K is said to satisfy *property P* if there exists an increasing, possibly infinite, exhaustive ∂_K-stable filtration $F^\bullet$, such that each subquotient $F^\bullet/F^{\bullet-1}$ is isomorphic to a direct sum of p-DG direct summands of A.

(2) The module K is called a *finite cell module*, if it satisfies property P, and as an A-module, it is finitely generated (necessarily projective by the property-P requirement).

Property-P modules are the analogues of projective modules in usual homological algebra. For instance, the morphism spaces from a property-P module to any p-DG module coincide in both the homotopy and derived categories.

It is a theorem [Qi14, Theorem 6.6] that there are always sufficiently many property-P modules: for any p-DG module M, there is a surjective quasi-isomorphism

$$\mathbf{p}(M) \longrightarrow M \tag{3.3}$$

of p-DG modules, with $\mathbf{p}(M)$ satisfying property P. We will usually refer to such a property-P replacement $\mathbf{p}(M)$ for M as a *bar resolution*. The proof of its existence is similar to that of the usual (simplicial) bar resolution for DG modules over DG algebras.

In a similar vein, finite cell modules play the role of finitely-generated projective modules in usual homological algebra.

3.3 p-DG functors

A p-DG bimodule ${}_AM_B$ over two p-DG algebras A and B is a p-DG module over $A \otimes B^{\mathrm{op}}$. One has the associated tensor and (graded) hom functors

$$M \otimes_B (-) : (B, \partial)\text{-mod} \longrightarrow (A, \partial)\text{-mod}, \quad X \mapsto M \otimes_B X, \tag{3.4}$$

$$\mathrm{HOM}_A(M, -) : (A, \partial)\text{-mod} \longrightarrow (B, \partial)\text{-mod}, \quad Y \mapsto \mathrm{HOM}_A(M, Y), \tag{3.5}$$

which form an adjoint pair of functors. In fact, we have the following enriched version of the adjunction.

Lemma 3.5. Let A, B be p-DG algebras and M a p-DG bimodule over $A \otimes B^{\mathrm{op}}$. Then, for any p-DG A-module Y and B-module X, there is an isomorphism of p-complexes

$$\mathrm{HOM}_A(M \otimes_B X, Y) \cong \mathrm{HOM}_B(X, \mathrm{HOM}_A(M, Y)).$$

Proof. See [Qi14, Lemma 8.5]. □

The functors descend to derived categories once appropriate property-P replacements are utilized. For instance, the derived tensor functor is given as the composition

$$M \otimes_B^{\mathbf{L}} (-) : \mathcal{D}(B) \longrightarrow \mathcal{D}(A), X \mapsto M \otimes_B \mathbf{p}_B(X) \tag{3.6}$$

where $\mathbf{p}_B(X)$ is a bar resolution for X as a p-DG module over B. Likewise, the derived HOM, denoted $\mathbf{R}\mathrm{HOM}$, is given by the functor

$$\mathbf{R}\mathrm{HOM}_A : \mathcal{D}(A) \longrightarrow \mathcal{D}(B), \quad Y \mapsto \mathrm{HOM}_A(\mathbf{p}_A(M), Y). \tag{3.7}$$

The functors form an adjoint pair, in the sense that

$$\mathrm{Hom}_{\mathcal{D}(A)}(M \otimes_B^{\mathbf{L}} X, Y) \cong \mathrm{Hom}_{\mathcal{D}(B)}(X, \mathbf{R}\mathrm{HOM}_A(M, Y)). \tag{3.8}$$

One useful application about such functors is the following theorem, whose proof can be found in [Qi14, Section 8].

Theorem 3.6. Let $f : M_1 \longrightarrow M_2$ be a quasi-isomorphism of p-DG bimodules. Then f descends to an isomorphism of the induced derived tensor product functors. □

3.4 Grothendieck groups

We next recall the notion of compact modules, which takes place in the derived category.

Definition 3.7. Let A be a p-DG algebra. A p-DG module M over A is called *compact* (in the derived category $\mathcal{D}(A)$) if and only if, for any family of p-DG modules N_i where i takes value in some index set I, the natural map

$$\bigoplus_{i\in I} \mathrm{Hom}_{\mathcal{D}(A)}(M, N_i) \longrightarrow \mathrm{Hom}_{\mathcal{D}(A)}(M, \bigoplus_{i\in I} N_i)$$

is an isomorphism of $\Bbbk$-vector spaces.

The strictly full subcategory of $\mathcal{D}(A)$ consisting of compact modules will be denoted by $\mathcal{D}^c(A)$. It is triangulated and will be referred to as the *compact derived category*.

As in the DG case, in order to avoid trivial cancellations in the Grothendieck group, one should restrict the class of objects used to define $K_0(A)$. It turns out that the correct condition is that of compactness. So we let $K_0(A) := K_0(\mathcal{D}^c(A))$. What we gain as dividend in the current situation is that, since $\mathcal{D}(A)$ is a "categorical module" over $\mathcal{D}(\Bbbk)$, the abelian group $K_0(A)$ naturally has a module structure over the auxiliary cyclotomic ring at a pth root of unity, which was defined in equation (2.2) of the previous section:

$$\mathbb{O}_p \cong K_0(\mathcal{D}^c(\Bbbk)). \tag{3.9}$$

The Grothendieck group $K_0(A)$ will be the primary algebraic invariant of the triangulated category $\mathcal{D}(A)$ that will interest us in this work.

A class of examples for which the Grothendieck group is relatively easy to compute is the following. The notion is introduce in [EQ16b, Section 2].

Definition 3.8. A p-DG algebra A is called *positive* if the following three conditions hold:

(1) A is supported on non-negative degrees: $A = \oplus_{k\in\mathbb{N}} A^k$, and it is finite dimensional in each degree.

(2) The homogeneous degree zero part A^0 is semisimple.

(3) The differential ∂_A acts trivially on A^0.

Theorem 3.9. Let A be a positive p-DG algebra. Then there is an isomorphism of Grothendieck groups

$$K_0(A) \cong K_0'(A) \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{O}_p,$$

where $K_0'(A)$ stands for the usual Grothendieck group of graded projective A-modules.

Proof. See [EQ16b, Corollary 2.18]. □

Remark 3.10 (Grading shift)**.** In what follows, we will abuse notation by writing $q^i M$, $i \in \mathbb{Z}$, for a p-DG module M to stand for M with grading shifted up by i. More generally, if $g(q) = a_0 + a_1 q + \cdots + a_m q^m \in \mathbb{N}[q, q^{-1}]$, then we will write

$$g(q)M := \bigoplus_{i=0}^{m} q^i M^{\oplus a_i}.$$

On the level of Grothendieck groups, the symbol of $g(q)M$ is equal to the multiplication of $g(q)$, regarded here as an element of $\mathbb{O}_p$, with the symbol of M.

Likewise, if $\mathcal{E}$ is a p-DG functor given by tensoring with the p-DG bimodule E over (A, B),

$$\mathcal{E} : (B, \partial)\text{-mod} \longrightarrow (A, \partial)\text{-mod}, \qquad M \mapsto E \otimes_B M,$$

then we will write $q^i\mathcal{E}$ as the functor represented by $q^i E$.

4 A double centralizer property

In this section, we analyze the double centralizer property in the context of categorical representation theory, and investigate the p-DG analogue. For two finite-dimensional algebras B and A, we will often have a (B, A)-bimodule M. Sometimes M will be denoted by ${}_BM, M_A$, or ${}_BM_A$ depending on what type of module we are emphasizing it to be.

4.1 Extension of categorical actions I

Consider the following situation. Let A be a finite-dimensional (graded) algebra over $\Bbbk$, and M be a finite-dimensional (graded) right A-module. Consider the (graded) algebra

$$B = \mathrm{END}_A(M).$$

Then M is a natural left B-module equipped with the action that, for any $b \in B$ and $m \in M$,

$$b \cdot m := b(m).$$

Since the left B-action commutes with the right A-action on it, M is in fact a (B, A)-bimodule. In this way one associates with M two natural functors on the left module categories

$$\mathcal{J} : A\text{-mod} \longrightarrow B\text{-mod}, \quad X \mapsto M \otimes_A X, \tag{4.1}$$

$$\mathcal{V} : B\text{-mod} \longrightarrow A\text{-mod}, \quad Y \mapsto \mathrm{HOM}_B(M, Y), \tag{4.2}$$

with $\mathcal{V}$ being right adjoint to $\mathcal{J}$, which also gives rise to a natural transformation of functors

$$\mathcal{I}d_{A\text{-mod}} \Rightarrow \mathcal{V} \circ \mathcal{J} : A\text{-mod} \longrightarrow A\text{-mod}.$$

Remark 4.1. In our context to follow, it will be too much to ask for both functors to be exact. Instead we will usually require $\mathcal{V}$ to be exact. In other words, M will usually be a projective left B-module. Under some additional assumptions, the functors $\mathcal{V}$ and $\mathcal{J}$ will play the roles of generalized *Soergel functor* and *projection functor* respectively.

Our next goal is to determine a condition under which categorical constructions on the A-module level will automatically translate into categorical constructions on the B-module level. To do so, let us consider the following situation. Let A_i, $(i = 1, 2, 3)$, be three finite-dimensional algebras, and M_i, $(i = 1, 2, 3)$, be left modules over respective A_i's. Suppose we now have functors between A_i-module categories given by bimodules ${}_{A_2}E_{1A_1}$, ${}_{A_3}E_{2A_2}$:

$$\mathcal{E}_1 : A_1\text{-mod} \longrightarrow A_2\text{-mod}, \quad X \mapsto E_1 \otimes_{A_1} X,$$

$$\mathcal{E}_2 : A_2\text{-mod} \longrightarrow A_3\text{-mod}, \quad Y \mapsto E_2 \otimes_{A_2} Y.$$

Together with the generalized Soergel and projection functors, we have a diagram:

$$\begin{array}{ccccc} A_1\text{-mod} & \xrightarrow{\mathcal{E}_1} & A_2\text{-mod} & \xrightarrow{\mathcal{E}_2} & A_3\text{-mod} \\ \mathcal{J}_1 \big\downarrow \big\uparrow \mathcal{V}_1 & & \mathcal{J}_2 \big\downarrow \big\uparrow \mathcal{V}_2 & & \mathcal{J}_3 \big\downarrow \big\uparrow \mathcal{V}_3 \\ B_1\text{-mod} & & B_2\text{-mod} & & B_3\text{-mod} \end{array} . \tag{4.3}$$

Composing functors gives rise to

$$\mathcal{E}'_1 := \mathcal{J}_2 \circ \mathcal{E}_1 \circ \mathcal{V}_1, \quad \mathcal{E}'_2 := \mathcal{J}_3 \circ \mathcal{E}_2 \circ \mathcal{V}_2$$

and their composition

$$\mathcal{E}'_2 \circ \mathcal{E}'_1 : B_1\text{-mod} \longrightarrow B_3\text{-mod}. \tag{4.4}$$

It is a natural question to ask whether this functor agrees with the composition

$$\mathcal{J}_3 \circ \mathcal{E}_2 \circ \mathcal{E}_1 \circ \mathcal{V}_1 : B_1\text{-mod} \longrightarrow B_3\text{-mod}. \tag{4.5}$$

The two functors will indeed agree if

$$\mathcal{I}d_{A_2\text{-mod}} \cong \mathcal{V}_2 \circ \mathcal{J}_2, \tag{4.6}$$

is equivalent to the identity functor on A_2-mod. In this case, any natural transformation of the functor

$$\mathcal{E}_2 \circ \mathcal{E}_1 = (E_2 \otimes_{A_2} E_1) \otimes_{A_1} (-) : A_1\text{-mod} \longrightarrow A_3\text{-mod},$$

which arises as an (A_3, A_1)-bimodule homomorphism

$$x : (E_1 \otimes_{A_2} E_2) \longrightarrow (E_1 \otimes_{A_2} E_2),$$

will also induce a natural transformation of the functor $\mathcal{E}_2' \circ \mathcal{E}_1'$.

Definition 4.2. Let A_i, $i \in I$, be a family of algebras, and for each i choose a right A_i-module M_i. Set $B_i = \mathrm{END}_{A_i}(M_i)$ so that each M_i is a (B_i, A_i)-bimodule. The categories of left modules A_i-mod and B_i-mod are connected via the generalized *Soergel functor* $\mathcal{V}_i$ and *projection functor* $\mathcal{J}_i$:

$$\mathcal{J}_i : A_i\text{-mod} \longrightarrow B_i\text{-mod}, \quad X \mapsto M_i \otimes_{A_i} X,$$

$$\mathcal{V}_i : B_i\text{-mod} \longrightarrow A_i\text{-mod}, \quad Y \mapsto \mathrm{HOM}_{B_i}(M_i, Y).$$

Suppose that, in addition, there are functors, one for each pair of $(i, j) \in I^2$, acting on A_i-module categories, which are given by bimodules ${}_{A_i}E_{ij}{}_{A_j}$:

$$\mathcal{E}_{ij} : A_j\text{-mod} \longrightarrow A_i\text{-mod}, \quad X \mapsto E_{ij} \otimes_{A_j} X.$$

We will say that *the categorical action on* $\oplus_{i \in I} A_i$-mod *by the bimodules* E_{ij}*'s extends to* $\oplus_{i \in I} B_i$-mod if the natural transformation of functors

$$\mathcal{J}_i \circ \mathcal{E}_{ij} \circ \mathcal{E}_{jk} \circ \mathcal{V}_k \Rightarrow \mathcal{J}_i \circ \mathcal{E}_{ij} \circ \mathcal{V}_j \circ \mathcal{J}_j \circ \mathcal{E}_{jk} \circ \mathcal{V}_k : B_k\text{-mod} \longrightarrow B_i\text{-mod}$$

is always an isomorphism, for any $i, j, k \in I$.

Let us now analyze when the condition (4.6) holds. This happens if and only if the adjunction map of functors

$$\mathcal{I}d_{A_2\text{-mod}} \Rightarrow \mathcal{V}_2 \circ \mathcal{J}_2$$

is an isomorphism. In other words, for any left A_2-module Y, there needs to be an isomorphism

$$Y \longrightarrow \mathrm{HOM}_{B_2}(M_2, M_2 \otimes_{A_2} Y).$$

Taking $Y = A_2$, we need the (A_2, A_2)-bimodule homomorphism

$$A_2 \longrightarrow \mathrm{HOM}_{B_2}(M_2, M_2) \quad a \mapsto (m \mapsto ma)$$

to be an isomorphism of bimodules.

Lemma 4.3. Let M be a (B, A)-bimodule which is projective as a left B-module. Suppose there is an isomorphism of (A, A)-bimodules

$$A \cong \mathrm{HOM}_B(M, M).$$

Then the natural transformation of functors $\mathcal{I}d_{A\text{-mod}} \Rightarrow \mathcal{V} \circ \mathcal{J}$ is an isomorphism.

Proof. Let $A^m \longrightarrow A^n \longrightarrow Y \longrightarrow 0$ be a projective presentation of Y, where $m, n \in \mathbb{N} \cup \{\infty\}$. Since tensor product is right exact, we have, by applying $M \otimes_A (-)$ to the above, an exact sequence

$$M^m \longrightarrow M^n \longrightarrow M \otimes_A Y \longrightarrow 0.$$

Then, as M is projective over B, $\mathrm{HOM}_B(M, -)$ is exact, and we have the sequence

$$\mathrm{HOM}_B(M, M)^m \longrightarrow \mathrm{HOM}_B(M, M)^n \longrightarrow \mathrm{HOM}_B(M, M \otimes_A Y) \longrightarrow 0$$

is exact. Using the assumption $A \cong \mathrm{HOM}_B(M, M)$, we deduce, by the classical Five Lemma, that $Y \cong \mathrm{HOM}_B(M, M \otimes_A Y)$. The result follows. □

Together with the definition of the B-algebras, the discussion leads us to consider the *double centralizer condition*, which requires that, for a (B, A)-bimodule ${}_B M_A$, one has

$$B = \mathrm{END}_A(M_A), \qquad A = \mathrm{END}_B({}_B M). \tag{4.7}$$

We summarize the above discussion into the following theorem.

Theorem 4.4. Under the conditions of Definition 4.2, the categorical action on $\oplus_{i \in I} A_i\text{-mod}$ by the bimodules E_{ij}'s extends to a categorical action on $\oplus_{i \in I} B_i\text{-mod}$ if, for each $i \in I$, the double centralizer property

$$B_i = \mathrm{END}_{A_i}(M_{A_i}), \qquad A_i = \mathrm{END}_{B_i}({}_{B_i} M),$$

holds on the (B_i, A_i)-bimodules ${}_{B_i} M_{i\,A_i}$, and M_i is projective as a B_i-module. □

In general, we are not aware of the most natural conditions for a bimodule to satisfy (4.7). We will, however, move on to understand when the double centralizer condition holds for some particular class of algebras and bimodules.

4.2 Self-injective algebras

In this section, we will recall a class of examples for which the double centralizer property (4.7) holds. The result is well-known in the ungraded case, see, for instance, Curtis-Reiner [CR06]. For a more modern account that applies to some more general cases, see König-Slungård-Xi [KSX01]. We record the proofs here as well as some consequences for the sake of completeness.

Throughout this subsection, let A be a (graded) finite-dimensional *self-injective algebra*, i.e, the left or right regular module A is injective. We will take M to be a (graded) finitely-generated faithful right A-module.

Lemma 4.5. A direct sum of copies of M contains A as a direct summand.

Proof. Write A as a direct sum of indecomposable injective submodules. Each indecomposable injective $I_N \subset A$ is the injective envelope of its (simple) socle N. By the assumption, N, regarded as a submodule in the socle of A, embeds into M since it is simple and A acts faithfully on M. By injectivity of I_N, the embedding of N into M extends to an embedding of I_N into M. Hence I_N is a direct summand of M, and the result follows. □

Write $M_A = P_A \oplus N_A$, where P_A is the direct sum of projective-injective indecomposable summands of M_A, and N_A is a complementary module. By the previous lemma, every indecomposable projective summand of A figures in P_A as a direct summand and vice versa.

A main goal of this section is to establish the following double centralizer property (4.7) in this particular context, following [CR06].

Theorem 4.6. Let A be a finite-dimensional graded self-injective algebra, and M_A a faithful finitely-generated graded right A-module. Set $B = \mathrm{END}_A(M_A)$ so that M is a (B, A)-bimodule. Then there is a canonical isomorphism of graded algebras:

$$A = \mathrm{END}_B({}_B M), \quad a \mapsto (m \mapsto ma)$$

for any $a \in A$ and $m \in M$.

We first note the following Morita equivalence result, allowing us to exchange M_A with $M_A^{\oplus r}$ for any positive integer r.

Lemma 4.7. Let A be a ring and M_A be a right A-module. Then A and $\mathrm{END}_A(M)$ satisfy the double centralizer property if and only if A and $\mathrm{END}_A(M^{\oplus r})$ satisfy the double centralizer property for any positive integer r.

Proof. Let $B = \mathrm{END}_A(M)$, then $\mathrm{END}_A(M^{\oplus r})$ can be identified with an $r \times r$ matrix algebra $\mathrm{M}(r, B)$ with coefficients in B. These algebras are naturally Morita equivalent, and an equivalence is given by tensor product with the standard column module:

$$B\text{-mod} \longrightarrow \mathrm{M}(r, B)\text{-mod}, \quad {}_BN \mapsto B^{\oplus r} \otimes_B N$$

Under this equivalence, the module M is naturally sent to $M^{\oplus r}$. Therefore, if one of B or $M(r, B)$ satisfies the double centralizer property, so that $\mathrm{END}_B(M) \cong A$ or $\mathrm{END}_{\mathrm{M}(r,B)}(M^{\oplus r}) \cong A$, then so does the other, as the Morita equivalence preserves endomorphism spaces. □

Proof of Theorem 4.6. By Lemma 4.5, there is an $r \in \mathbb{N}_{>0}$ such that $M^{\oplus r}$ contains A as a direct summand. Lemma 4.7 allows us to replace M by $M^{\oplus r}$. Thus let us assume from the start that there is a decomposition $M_A \cong A \oplus N_A$. Then we may formally identify $B = \mathrm{END}_A(M_A)$ as a matrix algebra:

$$B \cong \begin{pmatrix} \mathrm{END}_A(A) & \mathrm{HOM}_A(A, N) \\ \mathrm{HOM}_A(N, A) & \mathrm{END}_A(N) \end{pmatrix} .$$

Let $e_A \in B$ be the idempotent corresponding to the identity of $\mathrm{END}_A(A)$, and likewise for e_N. Then for any $(a, n) \in A \oplus N$

$$e_A(a, n) = (a, 0) \qquad e_N(a, n) = (0, n).$$

Given any $x \in \mathrm{END}_B(M)$ and assuming $(a, n)x = (a_x, n_x)$, we have

$$(a, 0)x = (e_A(a, n))x = e_A((a, n)x) = (a_x, 0), \qquad (0, n)x = (e_N(a, n))x = e_N((a, n)x) = (0, n_x).$$

Therefore, any $x \in \mathrm{END}_B(M)$ acts on $A \oplus N$ componentwise. Furthermore, since the map $a \mapsto a_x$ is left linear over $\mathrm{END}_A(A) \cong A$, it follows that there is an $x_0 \in A$ such that $a_x = ax_0$ for all $a \in A$.

For any $n \in N$, let us define an element $b_n \in B$, sitting in the north-east corner $\mathrm{HOM}_A(A, N) \cong N$ of the matrix description, by

$$b_n(a, n') = (0, na)$$

for all $(a, n') \in A \oplus N$. Then we have,

$$(b_n(a, n'))x = b_n((a, n')x),$$

which, in turn, is equal to

$$(0, (na)_x) = (0, na_x) = (0, nax_0).$$

Taking $a = 1_A$ shows that $n_x = nx_0$. Thus, for all $(a, n) \in A \oplus N$, we have found an $x_0 \in A$ such that

$$(a, n)x = (ax_0, nx_0) = (a, n)x_0.$$

The desired claim now follows. □

Let A, B and M be as in Theorem 4.6. We have the following.

Corollary 4.8. The centers of the algebras A and B are canonically isomorphic.

Proof. Denote the center of A and B by $z(A)$ and $z(B)$ respectively.

The action of the center $z(A)$ on M commutes with that of A, and therefore $z(A)$ is a subalgebra of $B = \mathrm{END}_A(M)$, as the A, and thus the $z(A)$, action is faithful on M. As $z(A)$ is a commutative subalgebra of A, it commutes with the action of B, and therefore $z(A) \subset z(B)$. The reverse inclusion holds by interchanging the roles of A and B, and the result follows. □

To conclude this subsection, we will prove an exactness result about the Soergel functor $\mathcal{V}$.

Lemma 4.9. As a left module over B, the module M in Theorem 4.6 is projective and injective.

Proof. First, we show that M is a projective left B-module. By Lemma 4.5, there is an r such that $M^{\oplus r}$ contains A as a direct summand. Write $M^{\oplus r} = A \oplus M'$, where M' is some complementary module. Then

$$M^{\oplus r} = \mathrm{HOM}_A(A_A, M^{\oplus r}) \cong \mathrm{HOM}_A(eM^{\oplus r}, M^{\oplus r}) \cong \mathrm{HOM}_A(M^{\oplus r}, M^{\oplus r})e,$$

where e is the idempotent in $\mathrm{END}_A(M^{\oplus r})$ given as a composition of right A-modules

$$e : M^{\oplus r} \longrightarrow A \longrightarrow M^{\oplus r}.$$

It follows that $M^{\oplus r}$ is a projective left module over the ring $\mathrm{END}_A(M^{\oplus r})$. Since B and $\mathrm{END}_A(M^{\oplus r})$ are Morita equivalent, the claim follows.

If J_A is any indecomposable summand of A as a right module, then J_A is projective and injective as a right A-module. Now, as a right B-module,

$$B_B = \mathrm{HOM}_A(M_A, M_A) = \mathrm{HOM}_A(M_A, J_A) \oplus \mathrm{HOM}_A(M_A, J'_A),$$

where J'_A is some complementary module of J_A in M_A. It follows that $\mathrm{HOM}_A(M_A, J_A)$ is a projective right B-module. By taking the vector space dual, we have that

$$\mathrm{HOM}_A(M_A, J_A)^* = \mathrm{HOM}_{\Bbbk}(\mathrm{HOM}_A(M_A, J_A), \Bbbk)$$

is an injective left B-module. We would like to use this to show that ${}_BM$ is injective as a left B-module. Notice that the vector space dual of J_A, $J_A^* = \mathrm{HOM}_{\Bbbk}(J_A, \Bbbk)$, carries a natural left A-module structure, and is, in fact, an injective and projective left A-module. We have an isomorphism of left B-modules

$$\begin{aligned} {}_BM_A \otimes_A J_A^* &\cong ({}_BM_A \otimes_A J_A^*)^{**} \cong \mathrm{HOM}_{\Bbbk}({}_BM_A \otimes_A J_A^*, \Bbbk)^* \\ &\cong \mathrm{HOM}_A({}_BM_A, \mathrm{HOM}_{\Bbbk}(J_A^*, \Bbbk))^* \cong \mathrm{HOM}_A({}_BM_A, J_A)^*. \end{aligned}$$

It follows that ${}_BM_A \otimes_A J_A^*$ is an injective B-module. Furthermore, since J_A^* is an indecomposable summand of A as a left module, and A decomposes into a direct sum of such J_A^*'s with various multiplicities, we thus have that ${}_BM \cong {}_BM_A \otimes_A A$ is a direct sum of modules of the form ${}_BM_A \otimes_A J_A^*$. The injectivity of ${}_BM$ follows. □

The following consequence of the lemma is then immediate.

Corollary 4.10. The generalized Soergel functor

$$\mathcal{V} : B\text{-mod} \longrightarrow A\text{-mod}, \quad Y \mapsto \mathrm{HOM}_B(M, Y)$$

is exact. □

The functor $\mathcal{V}$, however, does not necessarily sends projective B-modules to projective A-modules, and thus does not descend to Grothendieck groups of finitely generated projective modules.

4.3 Frobenius algebras

We next would like to give a criterion for which the generalized Soergel functor $\mathcal{V} : B\text{-mod} \longrightarrow A\text{-mod}$ is always fully-faithful on projective modules. To do this, we use the following lemma.

Lemma 4.11. Let ${}_BM_A$ be a (B, A)-bimodule on which A and B are centralizers of each other. Then there is a canonical isomorphism

$$\mathrm{HOM}_B({}_BM, B) \cong \mathrm{HOM}_A(M_A, A)$$

as (A, B)-bimodules.

Proof. Let us introduce notations ${}^\vee M := \mathrm{HOM}_B({}_BM, B)$ (left dual) and $M^\vee := \mathrm{HOM}_A(M_A, A)$ (right dual). There are natural evaluation maps

$$M \otimes_A {}^\vee M \longrightarrow B, \quad x \otimes_A g \mapsto \langle x, g\rangle_B,$$

$$M^\vee \otimes_B M \longrightarrow A, \quad f \otimes_B x \mapsto \langle f, x\rangle_A,$$

which are respectively (B, B) and (A, A)-bimodule homomorphisms.

Consider the canonical map

$${}^\vee M \otimes_B M \longrightarrow \mathrm{END}_B(M), \quad g \otimes_B x \mapsto (y \mapsto \langle y, g\rangle_B x).$$

Together with the double commutant property $\mathrm{END}_B(M) \cong A$, we have obtained an (A, A)-module homomorphism

$${}^\vee M \otimes_B M \longrightarrow A, \quad g \otimes_B x \mapsto (y \mapsto \langle y, g\rangle_B x = y a_{g,x}), \tag{4.8}$$

where $a_{g,x} \in A$ is the unique element associated with $g \otimes_B x$.

Likewise, we have another canonical left (B, B)-module map

$$M \otimes_A M^\vee \longrightarrow B, \quad x \otimes_A f \mapsto b_{x,f}, \tag{4.9}$$

where $b_{x,f} \in B$ is the unique element such that

$$b_{x,f}y = x\langle f, y\rangle_A$$

for all $y \in M$.

By the tensor-hom adjunction, we have isomorphisms

$$\mathrm{HOM}_A({}^\vee M \otimes_B M, A) \cong \mathrm{HOM}_B({}^\vee M, \mathrm{HOM}_A(M, A)) = \mathrm{HOM}_B({}^\vee M, M^\vee),$$

$$\mathrm{HOM}_B(M \otimes_A M^\vee, B) \cong \mathrm{HOM}_A(M^\vee, \mathrm{HOM}_B(M, B)) = \mathrm{HOM}_A(M^\vee, {}^\vee M).$$

The images of the canonical maps (4.8) and (4.9) are uniquely determined by

$$\phi : {}^\vee M \longrightarrow M^\vee, \qquad y\langle \phi(g), x\rangle_A = \langle y, g\rangle_B x, \tag{4.10}$$

$$\psi : M^\vee \longrightarrow M^\vee, \qquad \langle x, \psi(f)\rangle_B y = x\langle f, y\rangle_A. \tag{4.11}$$

for any $x, y \in M$.

Now, given any $x, y \in M$, $g \in {}^\vee M$ and $f \in M^\vee$, we have

$$\langle x, \psi \circ \phi(g)\rangle_B y = x\langle \phi(g), y\rangle_A = \langle x, g\rangle_B y,$$

and likewise

$$y\langle \phi \circ \psi(f), x\rangle_A = \langle y, \psi(f)\rangle_B x = y\langle f, x\rangle_A.$$

By the faithfulness of the actions, we obtain

$$\langle x, \psi \circ \phi(g)\rangle_B = \langle x, g\rangle_B \qquad \langle \phi \circ \psi(f), x\rangle_A = \langle f, x\rangle_A.$$

Thus we have equalities $\psi \circ \phi(g) = g$ and $\phi \circ \psi(f) = f$. It follows that ψ and ϕ are inverse of each other, and the lemma follows. □

In what follows we will specialize to the case when A is a Frobenius algebra. Recall that a *Frobenius algebra* is a finite-dimensional algebra equipped with a non-degenerate trace map $\epsilon : A \longrightarrow \Bbbk$. The (graded) vector space dual $A^* = \mathrm{HOM}_\Bbbk(A, \Bbbk)$ is naturally an (A, A)-bimodule via the action

$$(a \cdot f \cdot b)(c) := f(bca), \quad f \in A^*, \quad a, b, c \in A. \tag{4.12}$$

The trace map induces an isomorphism $A \longrightarrow A^*$, $a \mapsto a \cdot \epsilon$, as left A-modules. The *Nakayama endomorphism* $\alpha : A \longrightarrow A$ is characterized by

$$\epsilon(ab) = \epsilon(\alpha(b)a), \tag{4.13}$$

for any $a, b \in A$. Using the Nakayama automorphism, we define a twisted bimodule structure on A^β by keeping the left regular module structure on A, while twisting the right regular A-module structure through $\beta = \alpha^{-1}$. Then the isomorphism ${}_AA \cong {}_AA^*$ can be strengthened to an (A,A)-bimodule isomorphism between A^β and A^*:

$$A^\beta \cong A^*, \quad a \mapsto a \cdot \epsilon.$$

This is true because, for any $a, b, c \in A$, we have

$$a \cdot 1 \cdot b = a\beta(b) \mapsto ((a\beta(b)) \cdot \epsilon)(c) = \epsilon(ca\beta(b)) = \epsilon(bca) = (a \cdot \epsilon \cdot b)(c).$$

Theorem 4.12. Let A be a Frobenius algebra, M_A be a faithful A module and $B = \mathrm{END}_A(M_A)$. Then the Soergel functor is fully-faithful on projective B-modules.

Proof. It suffices to prove the result for the left regular projective module B, since each indecomposable B-module is a direct summand of B.

By the previous lemma and the argument in the proof of Lemma 4.9, we have an identification of (A,B)-bimodules

$$\mathrm{HOM}_B({}_BM, B) \cong \mathrm{HOM}_A(M_A, A) \cong (M \otimes_A A^*)^*.$$

Since A is Frobenius, we have, by the above discussion, that $A^* \cong A^\beta$, where $\beta = \alpha^{-1}$ is the inverse Nakayama automorphism. Therefore, the isomorphism continues

$$\mathrm{HOM}_B({}_BM, B) \cong (M^\beta)^* = {}^\beta M^*,$$

where ${}^\beta M^*$ indicates that the left A-module structure on M^* is twisted by the inverse Nakayama automorphism β.

Notice that twisting the entire A-module category by an automorphism of A induces an automorphism of the A-module category, and thus it preserves HOM-spaces:

$$\begin{aligned}\mathrm{HOM}_A({}^\beta M_1, {}^\beta M_2) &\cong \mathrm{HOM}_A({}^\beta A \otimes_A M_1, {}^\beta M_2) \cong \mathrm{HOM}_A(M_1, \mathrm{HOM}_A({}^\beta A, {}^\beta M_2)) \\ &\cong \mathrm{HOM}_A(M_1, M_2).\end{aligned}$$

Using this equality, we have isomorphisms

$$\begin{aligned}\mathrm{HOM}_A({}^\beta M^*, {}^\beta M^*) &\cong \mathrm{HOM}_A(M^*, M^*) \cong \mathrm{HOM}_A(M^*, \mathrm{HOM}_\Bbbk(M, \Bbbk)) \\ &\cong \mathrm{HOM}_\Bbbk(M \otimes_A M^*, \Bbbk) \cong \mathrm{HOM}_A(M_A, M_A) \cong B,\end{aligned}$$

where the tensor-hom adjunctions are repeatedly used. The theorem now follows. ☐

4.4 Extension of categorical actions II

Let us consider, back in the setting of Theorem 4.4, the following question: under what conditions on the functor $\mathcal{E}_{ij}$ does the induced functor

$$\mathcal{E}'_{ij} : B_j\text{-mod} \xrightarrow{\mathcal{V}_j} A_j\text{-mod} \xrightarrow{\mathcal{E}_{ij}} A_i\text{-mod} \xrightarrow{\mathcal{J}_i} B_i\text{-mod}$$

descend to a map of Grothendieck groups of finitely-generated projective modules?

One immediate problem arises from the fact that the functor $\mathcal{J}_i$ is not always exact, and therefore the composition $\mathcal{E}'_{ij}$ will usually not be exact. To avoid this problem, we will, in this subsection, replace the functor $\mathcal{J}_i$ by a better behaved adjoint to the Soergel functor $\mathcal{V}_i$.

We start by observing that, in the setting of Section 4.1 that, if M is a (B,A)-bimodule that is projective as a B-module, then we have two incarnations of the Soergel functor:

$$\mathrm{HOM}_B(M, -) \cong {}^\vee M \otimes_B (-) : B\text{-mod} \longrightarrow A\text{-mod}, \tag{4.14}$$

provided that M is finitely-generated. Here ${}^\vee M := \mathrm{HOM}_B({}_BM, B)$ is an (A,B)-bimodule that is B-projective. This allows us to consider the right adjoint of $\mathcal{V} : B\text{-mod} \longrightarrow A\text{-mod}$, which is given by

$$\mathcal{I} : A\text{-mod} \longrightarrow B\text{-mod}, \quad X \mapsto \mathrm{HOM}_A({}^\vee M, X). \tag{4.15}$$

Lemma 4.13. Let A be a self-injective algebra, M_A be a finite-dimensional faithful right A-module, and $B = \mathrm{END}_A(M_A)$. Then (A, B) satisfies the double centralizer property on the left dual module ${}^\vee M = \mathrm{HOM}_B({}_BM, B)$.

Proof. By Lemma 4.9, ${}_BM$ is projective, and thus

$$\mathrm{HOM}_B({}^\vee M, B) \cong M.$$

We first compute

$$\begin{aligned}\mathrm{HOM}_A({}^\vee M, {}^\vee M) &= \mathrm{HOM}_A({}^\vee M, \mathrm{HOM}_B(M, B)) \cong \mathrm{HOM}_B(M \otimes_A {}^\vee M, B)\\ &\cong \mathrm{HOM}_A(M_A, \mathrm{HOM}_B({}^\vee M, B)) \cong \mathrm{HOM}_A(M, M) \cong B.\end{aligned}$$

On the other hand, since both M and ${}^\vee M$ are finitely-generated B-projective, we have

$$\begin{aligned}\mathrm{HOM}_B({}^\vee M, {}^\vee M) &= {}^\vee M \otimes_B \mathrm{HOM}_B({}^\vee M, B) \cong {}^\vee M \otimes_B M\\ &\cong \mathrm{HOM}_B(M, B) \otimes_B M \cong \mathrm{HOM}_B(M, M) \cong A.\end{aligned}$$

The lemma follows. ☐

Lemma 4.14. Let A be a self-injective algebra, and M_A a faithful right A-module. Then the composition of functors

$$\mathcal{V} \circ \mathcal{I} : A\text{-mod} \longrightarrow A\text{-mod}, \qquad X \mapsto \mathrm{HOM}_B(M, \mathrm{HOM}_A({}^\vee M, X))$$

is equivalent to the identity functor on A-mod.

Proof. Choose an injective presentation of X as a left A-module:

$$0 \longrightarrow X \longrightarrow A^m \longrightarrow A^n,$$

where $m, n \in \mathbb{N} \cup \{\infty\}$. Since $\mathrm{HOM}_A({}^\vee M, -)$ is left exact, we have that

$$0 \longrightarrow \mathrm{HOM}_A({}^\vee M, X) \longrightarrow \mathrm{HOM}_A({}^\vee M, A)^m \longrightarrow \mathrm{HOM}_A({}^\vee M, A)^n$$

is exact. By the previous lemma, since ${}^\vee M$ satisfies the double centralizer property, we have the left and right duals of ${}^\vee M$ coincide (Lemma 4.11):

$$\mathrm{HOM}_A({}^\vee M, A) \cong \mathrm{HOM}_B({}^\vee M, B).$$

Thus the above sequence becomes

$$0 \longrightarrow \mathrm{HOM}_A({}^\vee M, X) \longrightarrow M^m \longrightarrow M^n.$$

Applying the (exact) Soergel functor to the sequence gives us

$$0 \longrightarrow \mathrm{HOM}_B(M, \mathrm{HOM}_A({}^\vee M, X)) \longrightarrow A^m \longrightarrow A^n.$$

The result follows from the Five Lemma. ☐

We can now state a variation of Theorem 4.4. Let A_i, $i \in I$, be a family of self-injective algebras, and M_i, one for each $i \in I$, be faithful right modules over respective A_i's. Suppose there are also functors between A_i-module categories given by bimodules ${}_{A_i}E_{ij}{}_{A_j}$ $(i, j \in I)$

$$\mathcal{E}_{ij} : A_j\text{-mod} \longrightarrow A_i\text{-mod}, \quad X \mapsto E_{ij} \otimes_{A_j} X.$$

We have a diagram:

$$\begin{array}{ccccc} A_i\text{-mod} & \xrightarrow{\mathcal{E}_{ji}} & A_j\text{-mod} & \xrightarrow{\mathcal{E}_{kj}} & A_k\text{-mod} \\ {\scriptstyle \mathcal{I}_i}\downarrow\uparrow{\scriptstyle \mathcal{V}_i} & & {\scriptstyle \mathcal{I}_j}\downarrow\uparrow{\scriptstyle \mathcal{V}_j} & & {\scriptstyle \mathcal{I}_k}\downarrow\uparrow{\scriptstyle \mathcal{V}_k} \\ B_i\text{-mod} & & B_j\text{-mod} & & B_k\text{-mod} \end{array} . \tag{4.16}$$

Composing functors gives rise to

$$\mathcal{E}_{ij}^! := \mathcal{I}_i \circ \mathcal{E}_{ij} \circ \mathcal{V}_j : B_j\text{-mod} \longrightarrow B_i\text{-mod}.$$

Lemma 4.14 shows that the compositions are equal:

$$\mathcal{E}_{kj}^! \circ \mathcal{E}_{ji}^! = (\mathcal{E}_{kj} \circ \mathcal{E}_{ji})^! := \mathcal{I}_k \circ \mathcal{E}_{kj} \circ \mathcal{E}_{ji} \circ \mathcal{V}_i : B_i\text{-mod} \longrightarrow B_k\text{-mod}. \tag{4.17}$$

The same reasoning as for Theorem 4.4 proves the following.

Corollary 4.15. Under the above conditions, the categorical action on $\oplus_i A_i\text{-mod}$ by the functors $\mathcal{E}_{ij}$'s extends to a categorical action on $\oplus_i B_i\text{-mod}$ by the functors $\mathcal{E}_{ij}^!$'s. □

The functor $\mathcal{E}_{ij}^!$ behaves better on the class of projective B_i-modules, as we will see. In turn, the induced action on the Grothendieck groups of projective modules will be interesting in what follows.

Recall that the additive envelope of a module N in A-mod, denoted $(N)_A$, consists of direct summands of finite direct sums of N. The additive envelope is equivalent to finitely generated projective modules over $\mathrm{END}_A(N)$.

Let us consider a simplified situation as follows, which will be applied later. Suppose also that the family of self-injective algebras A_i are *symmetric* Frobenius algebras, i.e., the trace pairing $\epsilon : A_i \longrightarrow \Bbbk$ satisfies

$$\epsilon(ab) = \epsilon(ba)$$

for all $a, b \in A_i$. The Nakayama automorphism and its inverse are both equal to the identity map on A, and $A \cong A^*$ as an (A, A)-bimodule.

For a symmetric Frobenius algebra A, we have

$$\mathrm{HOM}_A(M_A, A) \cong (M \otimes_A A^*)^* \cong M^*.$$

Thus if M_A is a faithful A-module, we have an identification of (A, B)-bimodules (Lemma 4.11)

$$\mathrm{HOM}_B({}_BM, B) \cong \mathrm{HOM}_A(M_A, A) \cong M^*.$$

For the next theorem, assume we are in the setting of Corollary 4.15, and futhermore that A_i's are symmetric Frobenius. The non-symmetric cases can be treated similarly with appropriate twists by the Nakayama automorphisms inserted.

Theorem 4.16. If the functors $\mathcal{E}_{ij}$'s $(i, j \in I)$ preserve the additive envelopes of the left A_i-modules M_i^*, then the extended functors $\mathcal{E}_{ij}^!$ preserve finitely-generated projective modules in $\oplus_{j\in} B_j\text{-mod}$.

Proof. It suffices to show that $\mathcal{E}_{ij}^!(B_j)$ is a finitely-generated projective B_i-module. As in the proof of Theorem 4.12, we have

$$\mathcal{V}_j(B_j) = \mathrm{HOM}_{B_j}({}_{B_j}M_j, B_j) \cong B_j \otimes_{B_j} {}^\vee M_j \cong M_j^*.$$

Since $\mathcal{E}_{ij}$ sends M_j^* inside the additive envelope of M_i^*, it now suffices to show that $\mathcal{I}_i$ takes M_i^* to a finitely-generated projective B_i-module. But this is now automatic, since $M_i^* \cong {}^\vee M_i$ and

$$\mathcal{I}_i(M_i^*) \cong \mathrm{HOM}_{A_i}({}^\vee M_i, {}^\vee M_i) \cong B_i$$

via Lemma 4.13. The theorem follows. □

4.5 A p-DG context

In this subsection, we will establish the p-DG analogues of the results in the previous two subsections.

Let (A, ∂) be a p-DG algebra whose underlying algebra A is self-injective (Frobenius). In this case we will say that (A, ∂) is a *self-injective (Frobenius) p-DG algebra*.

We start with a simple case.

Lemma 4.17. Let A be a self-injective p-DG algebra, and M_A a right p-DG module containing A as a p-DG direct summand. Then ${}_BM$ is cofibrant as a left p-DG module over B.

Proof. Since M contains A as a summand, the action of A on M is faithful, so that the previous results apply. As in the proof of Lemma 4.9, we have

$$M_A \cong \mathrm{HOM}_A(A, M) \cong \mathrm{HOM}_A(eM, M) \cong \mathrm{END}_A(M)e,$$

where $e \in \mathrm{END}_A(M)$ is a p-DG idempotent such that $A \cong eM$ and $\partial(e) = 0$. Thus M is a p-DG summand of the p-DG algebra $B = \mathrm{END}_A(M)$, and the result follows. □

Our next goal is to develop a p-DG analogue of Theorem 4.16. To do this, let us consider an analogue of the additive envelope of a module in this situation.

Definition 4.18. Let A be a p-DG algebra, and X be a left (or right) p-DG module. The *filtered p-DG envelope of X*, consists of direct summands of left (or right) p-DG modules which have a finite filtration, whose subquotients are isomorphic to grading shifts of X as p-DG modules.

Consider now the following situation. Let A_i, $i \in I$, be a collection of symmetric Frobenius p-DG algebras, M_i be a collection of faithful p-DG module over A_i, and $B_i := \mathrm{END}_{A_i}(M_i)$ be the endomorphism p-DG algebras. In this case, the tensor-hom adjunctions are compatible with p-differentials (Lemma 3.5).

Lemma 4.19. Let X be a left p-DG module over A, and $B = \mathrm{END}_A(X)$ be the p-DG endomorphism algebra. Then

$$\mathrm{HOM}_A(X, -) : (A, \partial)\text{-mod} \longrightarrow (B, \partial)\text{-mod}$$

sends the p-DG filtered envelope of X to the p-DG filtered envelope of B. In particular, the image of the functor consists of cofibrant p-DG modules over B.

Proof. It is clear that, if Y is a filtered p-DG modules over A whose subquotients are isomorphic to X, then the image of Y under $\mathrm{HOM}_A(X, -)$ is a filtered p-DG module whose subquotients are isomorphic to grading shifts of B, and thus is a property-P module. Furthermore, the functor preserves p-DG summands. Hence it takes the additive envelope of X to the additive envelope of B, which consists of cofibrant modules. □

Example 4.20. We would like to emphasize that the lemma above does not require the module X to be cofibrant over A. This will play an important role in the story later in this paper. For instance, consider $A := \mathrm{END}_{\Bbbk}(U)$, where U is a finite-dimensional p-complex. Then $\mathrm{END}_A(U) \cong \Bbbk$ as a p-DG algebra. It is easy to check that the above functor

$$\mathrm{HOM}_A(U, -) : (A, \partial)\text{-mod} \longrightarrow (\Bbbk, \partial)\text{-mod}$$

sends any filtered p-DG module A of the form $U \otimes V$, where V is any p-complex, to the p-complex V itself, which is property-P over the ground field. Even when U is acyclic, in which case U is not cofibrant over A (otherwise $\mathrm{HOM}_A(U, U) = \Bbbk$ would compute the endomorphism space of U in the derived category), the image of the p-DG filtered envelope consists of cofibrant $(\Bbbk, \partial)$-modules.

We are now ready to state the p-DG analogue of Theorem 4.16.

Proposition 4.21. Suppose $\mathcal{E}_{ij} : (A_j, \partial)\text{-mod} \longrightarrow (A_i, \partial)\text{-mod}$, $i, j \in I$, are p-DG functors given by tensoring with p-DG bimodules E_{ij} over (A_i, A_j):

$$\mathcal{E}_{ij} : (A_j, \partial)\text{-mod} \longrightarrow (A_i, \partial)\text{-mod}, \quad N \mapsto E_{ij} \otimes_{A_j} N,$$

and the collection of functors $\mathcal{E}_{ij}$ sends the filtered p-DG envelope of M_j^* into that of M_i^*. Then the extended p-DG functors $\mathcal{E}_{ij}^!$ preserve compact cofibrant p-DG modules in $\oplus_{i \in I}(B_i, \partial)\text{-mod}$.

Proof. The proof follows in a similar fashion as in Theorem 4.16. One uses Lemma 4.19 to show that the compositions functor $\mathcal{E}_{ij}^!$ sends the cofibrant module B_j to a direct summand of a property-P module in (B_i, ∂)-mod. □

Finally, to conclude this section, let us record a special p-DG Morita equivalence result for later use.

Proposition 4.22. Let M_1 and M_2 be p-DG right modules over A, and let M_1' be in the filtered p-DG envelope of $M_1 \oplus M_2$, whose associated graded has the form $M_1 \oplus g(q)M_2$ for some $g(q) \in \mathbb{N}[q, q^{-1}]$. Set $B_1 := \mathrm{END}_A(M_1 \oplus M_2)$ and $B_2 := \mathrm{END}_A(M_1' \oplus M_2)$. Then there is a derived equivalence between $\mathcal{D}(B_1)$ and $\mathcal{D}(B_2)$.

Proof. Without the presence of the differential, the algebras B_1 and B_2 are clearly Morita equivalent to each other. The Morita equivalence is given by tensor product with the (B_1, B_2)-bimodule (resp. (B_2, B_1)-bimodule)

$$N := \mathrm{HOM}_A(M_1' \oplus M_2, M_1 \oplus M_2) \qquad (\text{resp. } N' := \mathrm{HOM}_A(M_1 \oplus M_2, M_1' \oplus M_2)\,).$$

It follows that B_1 and B_2 satisfy the double centralizer property on either of the bimodules.

Now the bimodules above carry natural p-DG structures. For the derived tensor product with these bimodules to descend to derived equivalences, it suffices to show that they are cofibrant as p-DG modules over both B_1 and B_2, though not necessarily cofibrant over $B_1 \otimes B_2^{\mathrm{op}}$ or $B_2 \otimes B_1^{\mathrm{op}}$ (see [Qi14, Proposition 8.8] for some more general conditions). We will only show that N is cofibrant, and the case of N' is entirely similar.

It is clear that $\mathrm{HOM}_A(M_1, M_1 \oplus M_2)$ and $\mathrm{HOM}_A(M_2, M_1 \oplus M_2)$ are cofibrant left p-DG modules over B_1, since they are clearly p-DG direct summands of $B_1 \cong \mathrm{HOM}_A(M_1 \oplus M_2, M_1 \oplus M_2)$. It follows that N is cofibrant over B_1 since it has a filtration whose subquotients are isomorphic to the previous direct summands. To show that it is cofibrant over B_2, we use that $\mathrm{HOM}_A(M_1' \oplus M_2, M_1')$ and $\mathrm{HOM}_A(M_1' \oplus M_2, M_2)$ are clearly cofibrant as direct summands of B_2. Now $\mathrm{HOM}_A(M_1' \oplus M_2, M_1')$ has a filtration whose associated graded has the form

$$\mathrm{HOM}_A(M_1' \oplus M_2, M_1) \oplus g(q)\mathrm{HOM}_A(M_1' \oplus M_2, M_2).$$

Since the last summands are cofibrant and so is the whole module, the cofibrance of $\mathrm{HOM}_A(M_1' \oplus M_2, M_1)$ follows from the usual "two-out-of-three" property. □

Remark 4.23. The proof above generalizes easily to the case when M_1' lies in the p-DG filtered envelope of $M_1 \oplus M_2$, and the corresponding M_1 summands do not form an acyclic p-DG sub or quotient module.

5 A categorification of quantum $\mathfrak{sl}_2$ at prime roots of unity

5.1 The p-DG 2-category $\mathcal{U}$

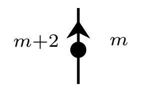
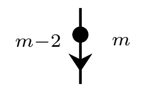
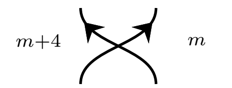
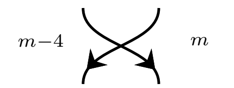
We begin this section by recalling the diagrammatic definition of the 2-category $\mathcal{U}$ introduced by Lauda in [Lau10]. Then we will recall a specific p-differential on $\mathcal{U}$ introduced in [EQ16b].

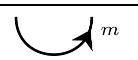
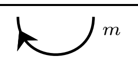
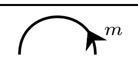
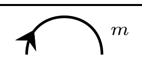
Definition 5.1. The 2-category $\mathcal{U}$ is an additive graded $\Bbbk$-linear category whose objects m are elements of the weight lattice of $\mathfrak{sl}_2$. The 1-morphisms are (direct sums of grading shifts of) composites of the generating 1-morphisms $\mathbb{1}_{m+2}\mathcal{E}\mathbb{1}_m$ and $\mathbb{1}_m\mathcal{F}\mathbb{1}_{m+2}$, for each $m \in \mathbb{Z}$. Each $\mathbb{1}_{m+2}\mathcal{E}\mathbb{1}_m$ will be drawn the same, regardless of the object m.

1-Morphism Generator	$m+2$  m	m  $m+2$
Name	$\mathbb{1}_{m+2}\mathcal{E}\mathbb{1}_m$	$\mathbb{1}_m\mathcal{F}\mathbb{1}_{m+2}$

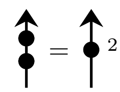
The weight of any region in a diagram is determined by the weight of any single region. When no region is labeled, the ambient weight is irrelevant.

The 2-morphisms will be generated by the following pictures.

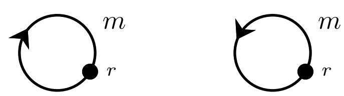
Generator				
Degree	2	2	-2	-2

Generator				
Degree	$1+m$	$1-m$	$1+m$	$1-m$

Before giving the full list of relations for $\mathcal{U}$, let us introduce some abbreviated notation. For a product of r dots on a single strand, we draw a single dot labeled by r. Here is the case when $r = 2$.



A *closed diagram* is a diagram without boundary, constructed from the generators above. The simplest non-trivial closed diagram is a *bubble*, which is a closed diagram without any other closed diagrams inside. Bubbles can be oriented clockwise or counter-clockwise.



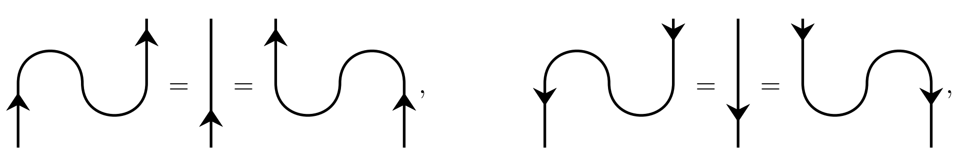
A simple calculation shows that the degree of a bubble with r dots in a region labeled m is $2(r+1-m)$ if the bubble is clockwise, and $2(r+1+m)$ if the bubble is counter-clockwise. Instead of keeping track of the number r of dots a bubble has, it will be much more illustrative to keep track of the degree of the bubble, which is in $2\mathbb{Z}$. We will use the following shorthand to refer to a bubble of degree $2k$.

This notation emphasizes the fact that bubbles have a life of their own, independent of their presentation in terms of caps, cups, and dots.

Note that m can be any integer, but $r \geq 0$ because it counts dots. Therefore, we can only construct a clockwise (resp. counter-clockwise) bubble of degree k when $k \geq 1-m$ (resp. $k \geq 1+m$). These are called *real bubbles*. Following Lauda, we also allow bubbles drawn as above with arbitrary $k \in \mathbb{Z}$. Bubbles with k outside of the appropriate range are not yet defined in terms of the generating maps; we call these *fake bubbles*. One can express any fake bubble in terms of real bubbles (see Remark 5.2).

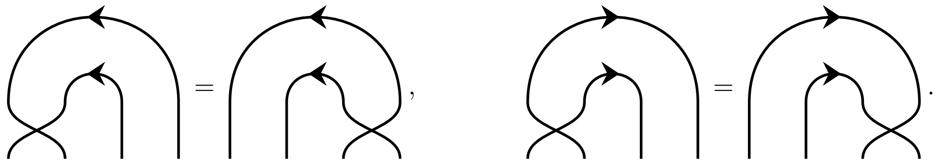
Now we list the relations. Whenever the region label is omitted, the relation applies to all ambient weights.

(1) **Biadjointness and cyclicity relations.** Relation 5.1a requires that the generating endomorphisms of $\mathcal{E}$ and $\mathcal{F}$ are biadjoint to each other:



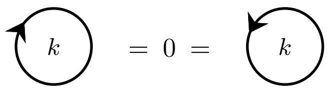
(5.1a)

(5.1b)



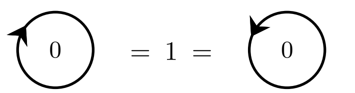
(5.1c)

(2) **Positivity and Normalization of bubbles.** Positivity states that all bubbles (real or fake) of negative degree should be zero.



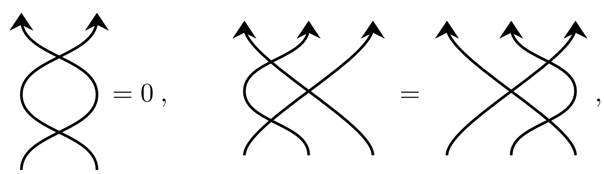
if $k < 0$. (5.2a)

Normalization states that degree 0 bubbles are equal to the empty diagram (i.e., the identity 2-morphism of the identity 1-morphism).

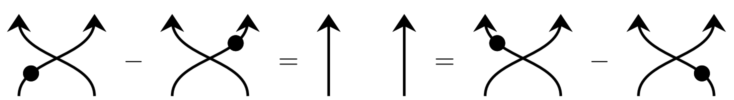


(5.2b)

(3) **NilHecke relations.** The upward pointing strands satisfy nilHecke relations. Note that, diagrammatically, far-away commuting elements become isotopy relations and are thus built in by default.

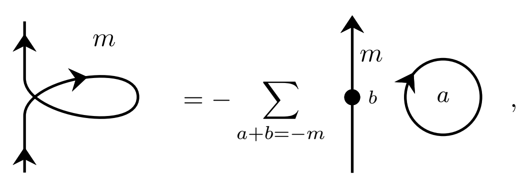


(5.3a)

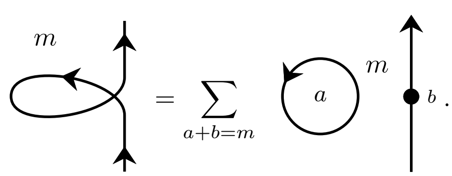


(5.3b)

(4) **Reduction to bubbles.** The following equalities hold for all $m \in \mathbb{Z}$.



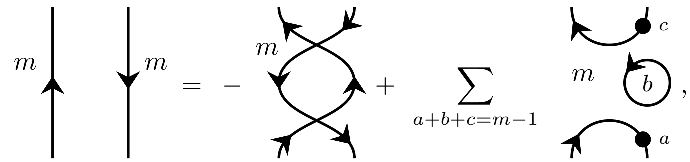
(5.4a)



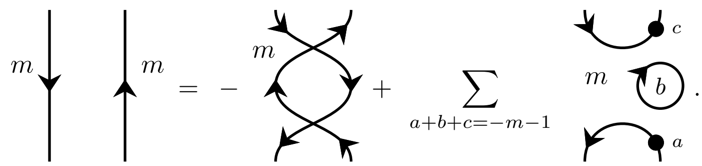
(5.4b)

These sums only take values for $a, b \geq 0$. Therefore, when $m \neq 0$, either the right curl or the left curl is zero.

(5) **Identity decomposition.** The following equations hold for all $m \in \mathbb{Z}$.



(5.5a)

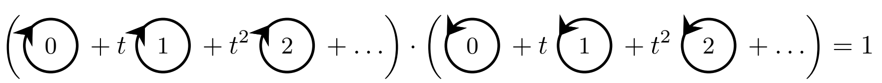


(5.5b)

The sum in the first equality vanishes for $m \leq 0$, and the sum in the second equality vanishes for $m \geq 0$.

The terms on the right hand side form a collection of orthogonal idempotents.

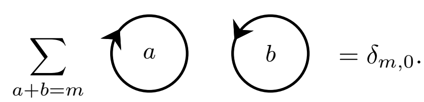
Remark 5.2 (Infinite Grassmannian relations)**.** This family of relations, which follows from the above defining relations, can be expressed most succinctly in terms of generating functions.



(5.6)

The cohomology ring of the "infinite dimensional Grassmannian" is the ring Λ of symmetric functions. Inside this ring, there is an analogous relation $\mathtt{e}(t)\mathtt{h}(t) = 1$, where $\mathtt{e}(t) = \sum_{i \geq 0} (-1)^i \mathtt{e}_i t^i$ is the total Chern class of the tautological bundle, and $\mathtt{h}(t) = \sum_{i \geq 0} \mathtt{h}_i t^i$ is the total Chern class of the dual bundle. Lauda has proved that the bubbles in a single region generate an algebra inside $\mathcal{U}$ isomorphic to Λ.

Looking at the homogeneous component of degree m, we have the following equation.

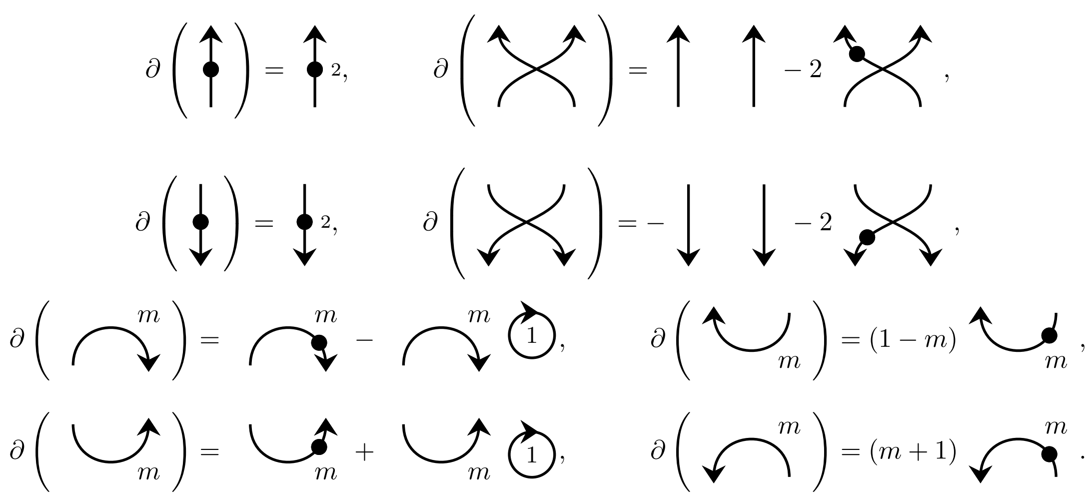


(5.7)

Because of the positivity of bubbles relation, this equation holds true for any $m \in \mathbb{Z}$, and the sum can be taken over all $a, b \in \mathbb{Z}$.

Using these equations one can express all (positive degree) counter-clockwise bubbles in terms of clockwise bubbles, and vice versa. Consequentially, all fake bubbles can be expressed in terms of real bubbles.

Definition 5.3. Let ∂ be the derivation defined on the 2-morphism generators of $\mathcal{U}$ as follows,



and extended to the entire $\mathcal{U}$ by the Leibniz rule.

Theorem 5.4. There is an isomorphism of $\mathbb{O}_p$-algebras

$$\dot{u}_{\mathbb{O}_p} \longrightarrow K_0(\mathcal{D}^c(\mathcal{U})),$$

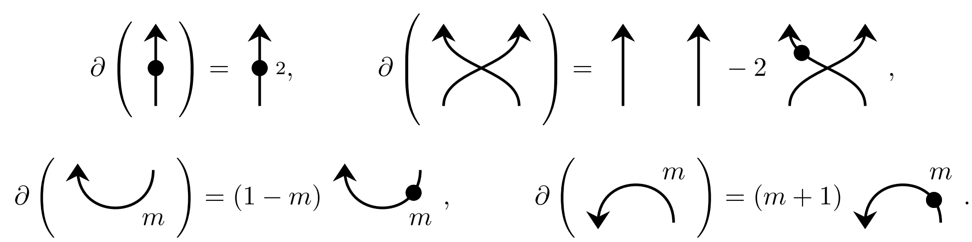
sending $1_{m+2}E1_m$ to $[\mathbb{1}_{m+2}\mathcal{E}\mathbb{1}_m]$ and 1_mF1_{m+2} to $[\mathbb{1}_m\mathcal{F}\mathbb{1}_{m+2}]$ for any weight $m \in \mathbb{Z}$.

Proof. This is [EQ16b, Theorem 6.11]. □

Remark 5.5. Let us also record some basic properties of the differential established in [EQ16b].

(1) In [EQ16b, Definition 4.6], a multi-parameter family of p-differentials is defined on $\mathcal{U}$ which preserve the defining relations of $\mathcal{U}$. This specific differential above is essentially the only one, up to conjugation by automorphisms of $\mathcal{U}$, that allows a *fantastic filtration* to exist on $(\mathcal{U}, \partial)$. The fantastic filtration in turn decategorifies to the quantum Serre relations for $\mathfrak{sl}_2$ at a prime root of unity.

(2) In the definition of the multi-parameter family of p-differentials, there are, though, certain redundancies set forth by the constraints on the parameters thereof (see [EQ16b, equation (4.8)]). Indeed, once one fixes the differential on the upward pointing nilHecke generators, clockwise cups and counter-clockwise caps, then the differential is uniquely specified on the entire $\mathcal{U}$. Therefore, in order to obtain the differential in Definition 5.3, it is enough to just specify



5.2 Thick calculus with the differential

We begin by recording some notation concerning symmetric polynomials. Let $\mathcal{P}(a,b)$ be the set of partitions which fit into an $a \times b$ box (height a, width b). An element $\mu = (\mu_1, \ldots, \mu_a) \in \mathcal{P}(a,b)$ with $\mu_1 \geq \cdots \geq \mu_a \geq 0$ gives rise to a *Schur polynomial* π_μ defined as follows:

$$\pi_\mu := \frac{\det(M_\mu)}{\prod_{1\leq i<j\leq a}(y_i - y_j)} \qquad (M_\mu)_{ij} := y_i^{a+\mu_j - j}. \tag{5.8}$$

The following are some special examples of Schur polynomials.

- If $\mu = (1^c)$ with $c \leq a$ then $\pi_\mu = \sum_{1\leq i_1 < \cdots < i_c \leq b} y_{i_1} \cdots y_{i_c}$ is the usual degree-c elementary symmetric function.
- If $\mu = (c)$ with $c \leq b$ then $\pi_\mu = \pi_\mu = \sum_{1\leq i_1 \leq \cdots \leq i_c \leq b} y_{i_1} \cdots y_{i_c}$ is the usual degree-c complete symmetric function.
- If $\mu = (c^a)$ with $c \leq b$ then $\pi_\mu = y_1^c \cdots y_a^c$.

For a partition $\mu \in \mathcal{P}(a,b)$ we will form a complementary partition $\hat{\mu} \in \mathcal{P}(b,a)$. First define the sequence

$$\mu^c = (b - \mu_a, \ldots, b - \mu_1).$$

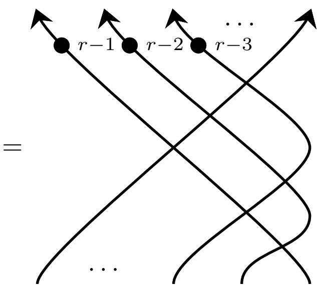
Then set

$$\hat{\mu} = (\hat{\mu}_1, \ldots, \hat{\mu}_b) \qquad \hat{\mu}_j = |\{i | \mu_i^c \geq j\}|$$

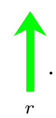
which we will refer to as the *complementary partition* of μ, with the underlying $\mathcal{P}(a,b)$ implicitly taken into account.

The 2-category $\mathcal{U}$ introduced in the previous section has a "thick enhancement" $\dot{\mathcal{U}}$ defined by Khovanov, Lauda, Mackaay and Stošić ([KLMS12]). The thick calculus $\dot{\mathcal{U}}$ is the Karoubi envelope of $\mathcal{U}$, and thus is Morita equivalent to $\mathcal{U}$.

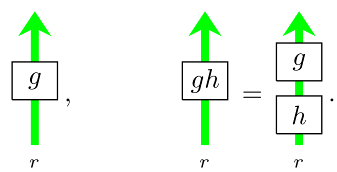
To construct $\dot{\mathcal{U}}$ it suffices to adjoin to $\mathcal{U}$ the image of the idempoents $e_r \in \mathrm{END}_{\mathcal{U}}(\mathcal{E}^r)$, one for each $r \in \mathbb{N}$, where e_r is diagrammatically given by

$e_r :=$  . (5.9)

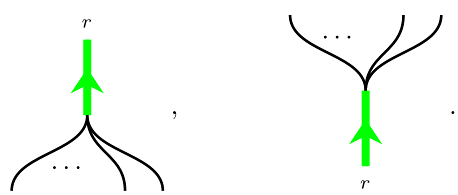
As in [KLMS12], we can depict the 1-morphism representing the pair $(\mathcal{E}^r, e_r)$ by an upward pointing "thick arrow" of thickness r (When the thickness equals one, the above 1-morphism agrees with the single black strand for $\mathcal{E}$.):



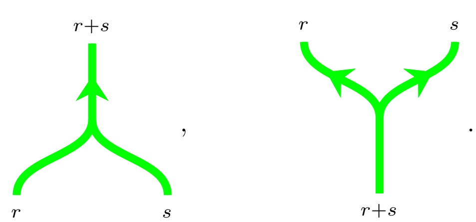
The newly introduced object of thickness r has symmetric polynomials in r variables as its endomorphism algebra. We express the multiplication of this endomorphism algebra elements by vertically concatenating pictures labeled by symmetric polynomials:



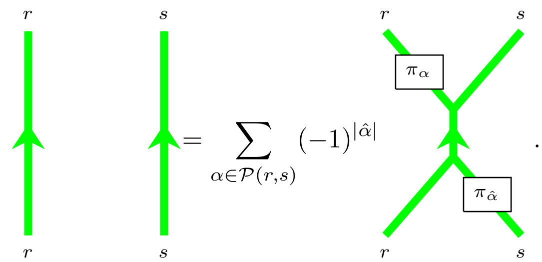
In general, the new object of thickness r, for each $r \in \mathbb{N}$, is called the *rth divided power* of $\mathcal{E}$, and is usually denoted as $\mathcal{E}^{(r)}$. The generating morphism from $\mathcal{E}^r$ to $\mathcal{E}^{(r)}$ and backwards are depicted as

 (5.10)

More generally, there are generating morphisms between $\mathcal{E}^{(r)}\mathcal{E}^{(s)}$ and $\mathcal{E}^{(r+s)}$ which are drawn respectively as



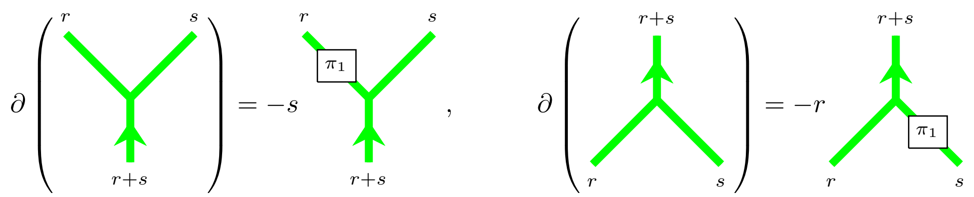
Such thick diagrams, carrying symmetric polynomials, satisfy certain diagrammatic identities which are consequences of relations in the 2-category $\mathcal{U}$. We refer the reader to [KLMS12] for the details. For instance, we have the following *identity decomposition relation*:



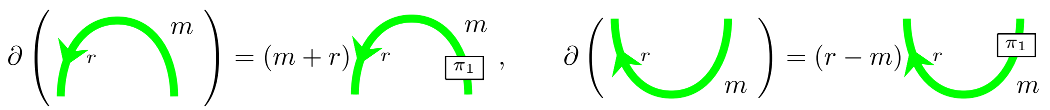
(5.11)

Here π_α stands for the Schur polynomial corresponding to a partition $\alpha \in \mathcal{P}(r,s)$, while $\hat{\alpha}$ stands for the complementary partition in $\mathcal{P}(s,r)$.

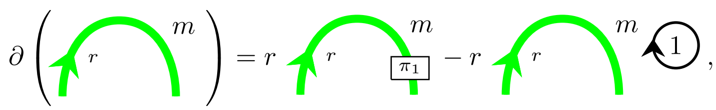
In [EQ16a], a p-differential is defined on $\dot{\mathcal{U}}$ extending that of $\mathcal{U}$:



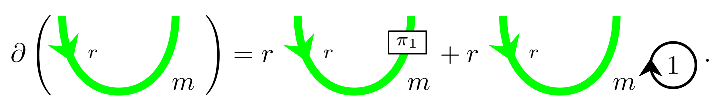
(5.12a)



(5.12b)

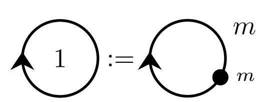


(5.12c)



(5.12d)

Here π_1 stands for the first elementary symmetric function in the number of variables labeled by the thickness of the strand, and the clockwise "bubble"



(5.12e)

agrees with the one defined in $\mathcal{U}$ in the previous subsection.

The thick cups and caps give rise to right and left adjoints of the 1-morphism $\mathcal{E}^{(r)}\mathbb{1}_m$ in $\dot{\mathcal{U}}$. Taking into account of degrees, they are given by

$$(\mathcal{E}^{(r)}\mathbb{1}_m)_R = \mathbb{1}_m\mathcal{F}^{(r)}\{-r(m+r)\}, \qquad (\mathcal{E}^{(r)}\mathbb{1}_m)_L = \mathbb{1}_m\mathcal{F}^{(r)}\{r(m+r)\}. \tag{5.13}$$

By construction, in the non-p-DG setting, $\dot{\mathcal{U}}$ is Morita equivalent to $\mathcal{U}$, and they both categorify quantum $\mathfrak{sl}_2$ at generic q values. However, unlike the abelian case, the p-DG derived categories are drastically different.

There is a natural embedding of p-DG 2-categories

$$\mathcal{J} : (\mathcal{U},\partial)\longrightarrow(\dot{\mathcal{U}},\partial),$$

which is given by tensor product with $\dot{\mathcal{U}}$ regarded as a $(\dot{\mathcal{U}},\mathcal{U})$-bimodule with a compatible differential. This functor, not surprisingly, induces an equivalence of abelian categories of p-DG modules, and further descends to an equivalence of the corresponding homotopy categories. However, under localization, it is no longer an equivalence, but instead is a fully-faithful embedding of derived categories:

$$\mathcal{J} : \mathcal{D}(\mathcal{U}) \longrightarrow \mathcal{D}(\dot{\mathcal{U}}).$$

The embedding is a categorical lifting of the fact that the small quantum $\mathfrak{sl}_2$ sits inside the BLM form (Definition 2.2). For related categorical results such as categorifying the quantum Frobenius map, see [Qi17].

Theorem 5.6. The derived embedding $\mathcal{J}$ categorifies the embedding of $\dot{u}_{\mathbb{O}_p}$ into the BLM form $\dot{U}_{\mathbb{O}_p}$ for quantum $\mathfrak{sl}_2$.

Proof. See [EQ16a]. □

Without the presence of ∂, the Stošić formula in [KLMS12] gives rise to a direct sum decomposition of 1-morphism $\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbb{1}_m$ for various $a, b \in \mathbb{N}$ and $m \in \mathbb{Z}$. In the p-DG setting, the direct sum decomposition is replaced by a fantastic filtration on the corresponding 1-morphisms.

Proposition 5.7. In the p-DG category $(\dot{\mathcal{U}}, \partial)$, the following categorical implications hold.

(i) The 1-morphisms in the collection

$$\{\mathcal{E}^{(a)}\mathcal{E}^{(b)}\mathbb{1}_m,\ \mathcal{F}^{(a)}\mathcal{F}^{(b)}\mathbb{1}_m | a, b \in \mathbb{N},\ m \in \mathbb{Z}\}$$

are equipped with a fantastic filtration, whose subquotients are isomorphic to grading shifts of $\mathcal{E}^{(a+b)}\mathbb{1}_m$ and $\mathcal{F}^{(a+b)}\mathbb{1}_m$ respectively. Consequently, the defining relation (2.3) for $\dot{U}_{\mathbb{O}_p}$ holds in the Grothendieck group of $\mathcal{D}(\dot{\mathcal{U}})$.

(ii) The 1-morphisms in the collection

$$\{\mathcal{E}^{(a)}\mathcal{F}^{(b)}\mathbb{1}_m,\ \mathcal{F}^{(a)}\mathcal{E}^{(b)}\mathbb{1}_m | a, b \in \mathbb{N},\ m \in \mathbb{Z}\}$$

admit natural ∂-stable fantastic filtrations. The direct summands in the Stǒsić formula consitute the associated graded pieces of the filtrations. Consequently, in the Grothendieck group of $\mathcal{D}(\dot{\mathcal{U}})$, the divided power E-F relations (equation (2.4a) and (2.4b)) hold.

Proof. See [EQ16a, Section 3 and Section 6]. □

5.3 The p-DG 2-category $\mathcal{U}^R$

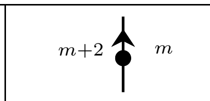
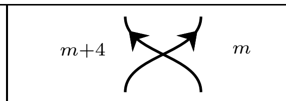
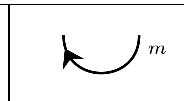
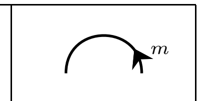
We now recall a similar version of the Khovanov-Lauda 2-category introduced by Rouquier in [Rou08] (for the special case of $\mathfrak{sl}_2$). The definition involves fewer generators and relations, and therefore is usually easier to examine on (potential) 2-representations.

Definition 5.8. The 2-category $\mathcal{U}^R$ is an additive graded $\Bbbk$-linear category whose objects m are elements of the weight lattice of $\mathfrak{sl}_2$. The 1-morphisms are (direct sums of grading shifts of) composites of the generating 1-morphisms $\mathbb{1}_{m+2}\mathcal{E}\mathbb{1}_m$ and $\mathbb{1}_m\mathcal{F}\mathbb{1}_{m+2}$, for each $m \in \mathbb{Z}$. Each $\mathbb{1}_{m+2}\mathcal{E}\mathbb{1}_m$ will be drawn the same, regardless of the object m.

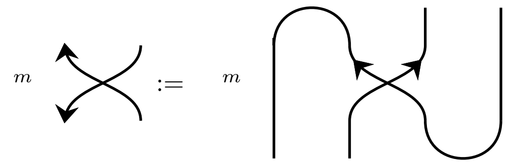
1-Morphism Generator	$m+2$  m	m  $m+2$
Name	$\mathbb{1}_{m+2}\mathcal{E}\mathbb{1}_m$	$\mathbb{1}_m\mathcal{F}\mathbb{1}_{m+2}$

The weight of any region in a diagram is determined by the weight of any single region. When no region is labeled, the ambient weight is irrelevant.

The 2-morphisms will be generated by the following pictures.

Generator				
Degree	2	-2	$1-m$	$1+m$

We will need the following 2-morphism defined in terms of the generating 2-morphisms.

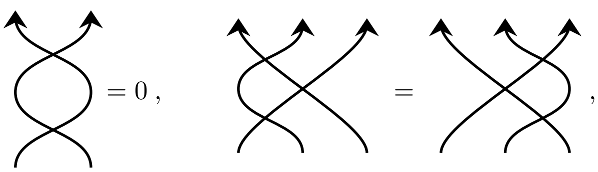


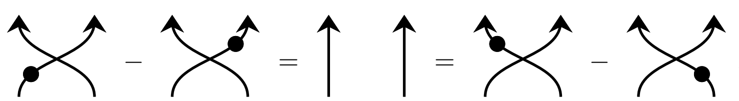
Now we list the relations. Whenever the region label is omitted, the relation applies to all ambient weights.

(1) **Adjointness relations.** Only one-sided adjunction is required between $\mathcal{E}$ and $\mathcal{F}$:

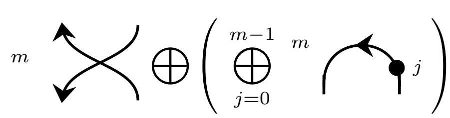
 (5.14a)

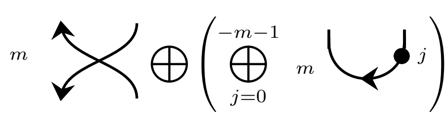
(2) **NilHecke relations.** The upward pointing strands satisfy nilHecke relations. Note that, diagrammatically, far-away commuting elements become isotopy relations and are thus built in by default.

 (5.15a)

 (5.15b)

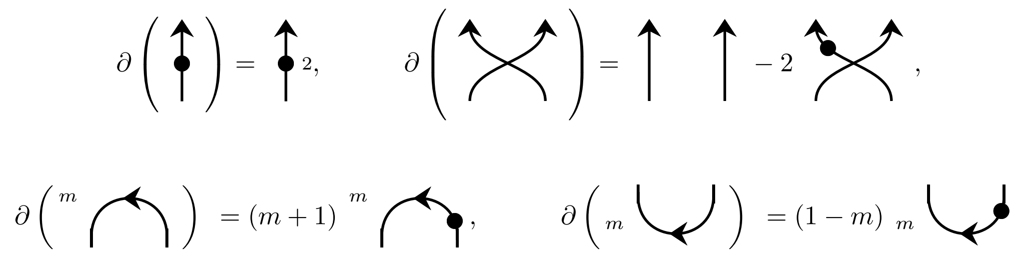
(3) **Isomorphism of** 1**-morphisms.** For all $m \in \mathbb{Z}$ there exist isomorphisms

 $: \mathcal{F}\mathcal{E}\mathbb{1}_m \longrightarrow \mathcal{E}\mathcal{F}\mathbb{1}_m \bigoplus \left(\bigoplus_{j=0}^{m-1} q^{m-1-2j}\mathbb{1}_m\right),$ (5.16a)

 $: \mathcal{F}\mathcal{E}\mathbb{1}_m \bigoplus \left(\bigoplus_{j=0}^{-m-1} q^{-m-1-2j}\mathbb{1}_m\right) \longrightarrow \mathcal{E}\mathcal{F}\mathbb{1}_m.$ (5.16b)

The sum in the first equality vanishes for $m \leq 0$, and the sum in the second equality vanishes for $m \geq 0$.

Definition 5.9. Let ∂ be the derivation defined on the 2-morphism generators of $\mathcal{U}^R$ as follows.



and extended to all of $\mathcal{U}^R$ by the Leibniz rule.

We have the following easily verified result.

Lemma 5.10. Over a field of characteristic $p > 0$, the above differential equips the 2-category $\mathcal{U}^R$ with the structure of a p-DG 2-category.

Proof. The proof is similar to that in [EQ16b]. One first shows that ∂ is compatible with all the relations of $\mathcal{U}^R$, and then shows that $\partial^p \equiv 0$ on the generators. ☐

5.4 An equivalence of 2-categories

In a remarkable work, Brundan [Bru16] establishes an equivalence between the two versions of the 2-categories defined by Khovanov-Lauda and Rouquier respectively.

Theorem 5.11. There is an equivalence of 2-categories $\mathcal{U} \cong \mathcal{U}^R$.

Proof. This is proven in [Bru16, Main Theorem]. ☐

Without much effort we extend this equivalence to the p-DG setting.

Theorem 5.12. There exists an equivalence of p-DG 2-categories $(\mathcal{U}, \partial) \cong (\mathcal{U}^R, \partial)$.

Proof. The Main Theorem of [Bru16] asserts an equivalence of 2-categories $\mathcal{U} \cong \mathcal{U}^R$. That is, there exist 2-morphisms

which are used to define the inverse isomorphisms of (5.16) and they satisfy the relations of the 2-morphisms of $\mathcal{U}$. By Remark 5.5, the derivation defined on the generators for $\mathcal{U}^R$ in Definition 5.9 determines the derivation on all of the other generators for $\mathcal{U}$ given in Definition 5.3. Thus there is an equivalence of p-DG 2-categories. ☐

6 The nilHecke algebra

6.1 Definitions

Recall that $\Bbbk$ is a field of characteristic $p > 0$. Let l and n be integers such that $l \geq n \geq 0$. Define the nilHecke algebra NH_n to be the $\Bbbk$-algebra generated by $y_1, \dots, y_n$ and $\psi_1, \dots, \psi_{n-1}$ with relations

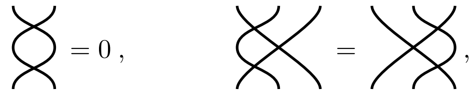
$$\begin{gathered} y_i y_j = y_j y_i, \quad y_i \psi_j = \psi_j y_i \ (i \neq j, j+1), \quad y_i \psi_i - \psi_i y_{i+1} = 1 = \psi_i y_i - y_{i+1} \psi_i, \\ \psi_i^2 = 0, \qquad \psi_i \psi_j = \psi_j \psi_i \ (|i-j| > 1), \qquad \psi_i \psi_{i+1} \psi_1 = \psi_{i+1} \psi_i \psi_{i+1}. \end{gathered} \tag{6.1}$$

The cyclotomic nilHecke algebra NH_n^l is the quotient of the nilHecke algebra NH_n by the *cyclotomic relation*

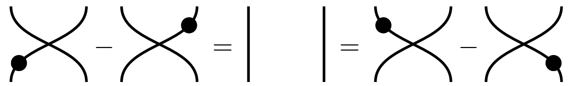
$$y_1^l = 0. \tag{6.2}$$

The (cyclotomic) nilHecke algebra is a graded algebra where the degree of y_i is 2 and the degree of ψ_i is -2.

The relations above translate into planar diagrammatic relations for the upward pointing strands in the 2-category $\mathcal{U}$ (see Section 5.1), with the orientation labels dropped:

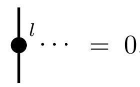


(6.3)



(6.4)

while the cyclotomic relation means that a black strand carrying l consecutive dots and appearing to the left of the rest of a diagram annihilates the entire picture:

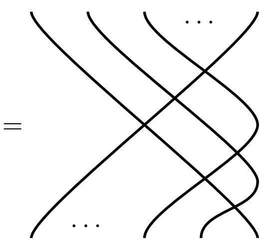


(6.5)

There is a graded anti-automorphism on the (cyclotomic) nilHecke algebras $*\colon \mathrm{NH}_n^l \to \mathrm{NH}_n^l$ defined by $\psi_i^* = \psi_i$ and $y_i^* = y_i$. Diagrammatically, it is interpreted as flipping a diagram upside down about a horizontal axis.

Let us recall some special elements of (cyclotomic) nilHecke algebras that correspond to symmetric group elements. Fix a reduced decomposition of $w \in S_n$, $w = s_{i_1} \cdots s_{i_r}$. This gives rise to an element $\psi_w = \psi_{i_1} \cdots \psi_{i_r} \in \mathrm{NH}_n^l$ which is independent of the expression for w by the second group of relations in (6.1). For instance, if $w_0 \in S_n$ is the usual longest element with respect to the usual Coxeter length function, then the corresponding (cyclotomic) nilHecke element is unambiguously depicted as the following n-stranded element:

$$\psi_{w_0} =$$

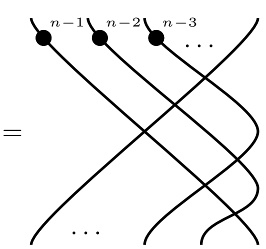


The element is symmetric with respect to the $*$ anti-automorphism.

6.2 Idempotents

Let w_0 be the longest element in the symmetric group S_n. This gives rise to an indecomposable idempotent $e_n \in \mathrm{NH}_n^l$

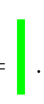
$$e_n := y_1^{n-1} \cdots y_n^0 \psi_{w_0} =$$



(6.6a)

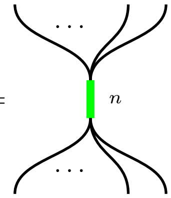
In the notation of Section 5.2, we will also depict the above idempotent as an unoriented thick strand

$$e_n =$$

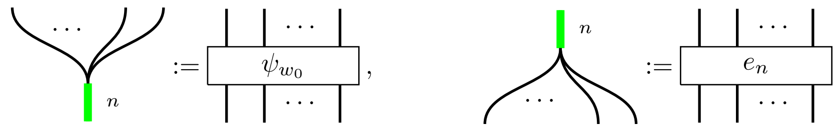


(6.6b)

Then one can show that

$\psi_{w_0} =$ 

The diagram in the middle is thick, indicating that it is a morphism factoring through the middle part represented by the image of the idempotent. In this notation, the diagram is the concatenation its two halves:

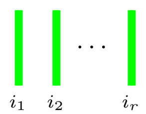


(6.7)

If $\mathbf{i} = (i_1, \ldots, i_r)$ be a tuple of natural numbers such that $i_1 + \cdots + i_r = n$. Then we set the idempotent

$$e_{\mathbf{i}} = e_{i_1} \otimes \cdots \otimes e_{i_r}. \tag{6.8}$$

This corresponds to putting the diagrams for e_{i_1},..., e_{i_r} side by side next to one another:



Rotating a diagram $180°$ turns e_n into a quasi-idempotent

$$e'_n := \psi_{w_0} \cdot y_1^0 \cdots y_n^{n-1}.$$

To obtain a genuine idempotent one needs to correct the element with the sign $(-1)^{n(n-1)/2}$.

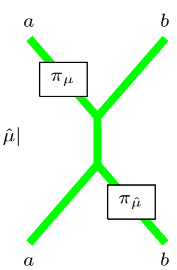
Let $a + b = n$. For $\mu \in P(a, b)$ we define a minimal idempotent $e^{\mu}_{(a,b)}$ of NH^l_n as follows. First set

$$\psi_{a,b} = (\psi_b \cdots \psi_{a+b-1}) \cdots (\psi_2 \cdots \psi_{a+1})(\psi_1 \cdots \psi_{a-1}).$$

Then we define

$$e^{\mu}_{(a,b)} = (-1)^{|\hat{\mu}|}(\pi_\mu(y_1, \cdots, y_a))(e_{(a,b)})(\psi_{a,b})(e_{a+b})(\pi_{\hat{\mu}}(y_{a+1}, \cdots, y_{a+b})). \tag{6.9}$$

The idempotent is diagrammatically depicted as (c.f. equation (5.11))

$(-1)^{|\hat{\mu}|}$ 

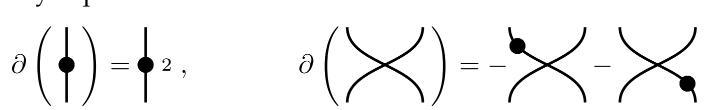
with it understood that we may let NH^l_n elements be multiplied from above and below of the element.

6.3 p-DG structure

The cyclotomic nilHecke algebra NH^l_n has a p-DG structure inherited from that of NH_n:

$$\partial(y_i) = y_i^2 \qquad \partial(\psi_i) = -y_i\psi_i - \psi_i y_{i+1}, \tag{6.10}$$

which is diagrammatically expressed as



Proposition 6.1. Assume $\mathbf{i} = (i_1, \ldots, i_r)$ with $i_1 + \cdots + i_r = n$. Then the projective NH_n^l-module $e_{\mathbf{i}}\mathrm{NH}_n^l$ is a right p-DG module over NH_n^l.

Proof. For simplicity we assume $r = 1$. The general case follows similarly. It follows from [EQ16a, Section 2.1] that

$$\partial(e_n) = -e_n \sum_{i=1}^{n} (i-1)y_i.$$

Thus $e_n\mathrm{NH}_n^l$ is stable under ∂. □

We will need later the easily verified fact below.

$$\partial(e_n') = -\sum_{i=1}^{n} (n-i)y_i e_n'. \tag{6.11}$$

Proposition 6.2. For $0 \le n \le p-1$, the p-DG module $e_n\mathrm{NH}_n^l$ is compact and cofibrant.

Proof. One repeats the arguments in the proof of [KQ15, Proposition 3.26] by making the following replacements

$$\mathrm{NH}_n \mapsto \mathrm{NH}_n^l, \qquad \mathcal{P}_n^+ \mapsto e_n\mathrm{NH}_n^l,$$
$$\Lambda_n \mapsto \mathrm{H}^*(\mathrm{Gr}(n,l)), \qquad \mathrm{NH}_n' \mapsto (\mathrm{NH}_n^l)',$$

where $\mathrm{H}^*(\mathrm{Gr}(n,l))$ is the cohomology of the Grassmannian of n-dimensional subspaces in $\mathbb{C}^l$ and $(\mathrm{NH}_n^l)'$ is the space of traceless $n! \times n!$ matrices over $\mathrm{H}^*(\mathrm{Gr}(n,l))$. □

6.4 A categorification of simples

We first review a categorification of the irreducible representation V_l of quantum $\mathfrak{sl}_2$ for a generic value of q using cyclotomic nilHecke algebras due to Kang and Kashiwara [KK12]. Then we enhance it to a categorification of cyclically generated modules over the small quantum group $u_{\mathbb{O}_p}$ and over the BLM quantum group $\dot{U}_{\mathbb{O}_p}$.

For any $a \in \mathbb{Z}_{\ge 0}$ there is an embedding $\mathrm{NH}_n^l \hookrightarrow \mathrm{NH}_{n+a}^l$ given by

$$y_i \mapsto y_i \quad (i = 1, \ldots, n), \qquad \psi_i \mapsto \psi_i \quad (i = 1, \ldots, n-1).$$

We use this embedding to produce functors between categories of nilHecke modules.

Definition 6.3. There is an induction functor

$$\mathfrak{F}^{(a)} : (\mathrm{NH}_n^l, \partial)\text{-mod} \longrightarrow (\mathrm{NH}_{n+a}^l, \partial)\text{-mod}$$

given by tensoring with the p-DG bimodule over $(\mathrm{NH}_n^l, \mathrm{NH}_{n+a}^l)$

$$e_{(1^n,a)}\mathrm{NH}_{n+a}^l, \qquad \partial(e_{(1^n,a)}) := -\sum_{i=1}^{a} (1-i)e_{(1^n,a)}y_{n+i}.$$

Notice that the differential action on the bimodule generator $e_{(1^n,a)}$ arises from the differential action on the idempotent e_a (Proposition 6.1).

There is a restriction functor

$$\mathfrak{E}^{(a)} : (\mathrm{NH}_{n+a}^l, \partial)\text{-mod} \longrightarrow (\mathrm{NH}_n^l, \partial)\text{-mod}$$

given by tensoring with the p-DG bimodule over $(\mathrm{NH}_{n+a}^l, \mathrm{NH}_n^l)$

$$\mathrm{NH}_{n+a}^l e_{(1^n,a)}^\star, \qquad \partial(e_{(1^n,a)}^\star) := \sum_{i=1}^{a} (2n+i-l)y_{n+i}e_{(1^n,a)}^\star.$$

Here the differential action on the bimodule generator $e_{(1^n,a)}^\star$ is twisted from the natural ∂-action on e_a' (see equation (6.11)) by the symmetric function $(a-l+2n)(y_{n+1} + \cdots + y_{n+a})$.

For simplicity, set $\mathfrak{F} = \mathfrak{F}^{(1)}$ and $\mathfrak{E} = \mathfrak{E}^{(1)}$.

Remark 6.4. In the absence of the p-DG structure, the functors $\mathfrak{E}$ and $\mathfrak{F}$ give rise to an action of Lauda's 2-category $\mathcal{U}$ on $\oplus_{n=0}^{l}\mathrm{NH}_n^l\text{-mod}$. See, for instance, [Rou08] and [KK12].

There is an adjunction map of the functors $\cap\colon \mathfrak{F}\mathfrak{E} \Rightarrow \mathrm{Id}$ given by the following bimodule homomorphism

$$\mathrm{NH}_n^l e^{\star}_{(1^{n-1},1)} \otimes_{\mathrm{NH}_{n-1}^l} e_{(1^{n-1},1)}\mathrm{NH}_n^l \longrightarrow \mathrm{NH}_n^l, \qquad \alpha\otimes\beta \mapsto \alpha\beta, \tag{6.12}$$

and similarly an adjunction map $\cup : \mathrm{Id} \Rightarrow \mathfrak{E}\mathfrak{F}$ arising from

$$\mathrm{NH}_n^l \longrightarrow \mathrm{NH}_{n+1}^l e^{\star}_{(1^n,1)}, \qquad \alpha \mapsto \alpha e^{\star}_{(1^n,1)}. \tag{6.13}$$

There is a "dot" natural transformation

$$Y\colon \mathfrak{E} \Rightarrow \mathfrak{E}, \qquad \left(\mathrm{NH}_n^l \longrightarrow \mathrm{NH}_n^l \qquad \alpha \mapsto \alpha y_n\right). \tag{6.14}$$

There is also a "crossing"

$$\Psi\colon \mathfrak{E}\mathfrak{E} \Rightarrow \mathfrak{E}\mathfrak{E}, \qquad \left(\mathrm{NH}_n^l \longrightarrow \mathrm{NH}_n^l \qquad \alpha \mapsto \alpha\psi_{n-1}\right). \tag{6.15}$$

Theorem 6.5. For any $l \in \mathbb{N}$, there is a 2-representation of the p-DG 2-category $(\mathcal{U},\partial)$ on $\oplus_{n=0}^{l}(\mathrm{NH}_n^l,\partial)\text{-mod}$ defined as follows.

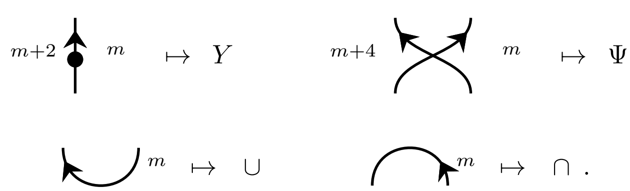
On 0-morphisms we have

$$m \mapsto \begin{cases} (\mathrm{NH}_n^l,\partial)\text{-mod} & \text{if } m = l-2n \\ 0 & \text{otherwise .} \end{cases}$$

On 1-morphisms we have

$$\mathcal{E}\mathbb{1}_m \mapsto \begin{cases} \mathfrak{E} & \text{if } m = l-2n \\ 0 & \text{otherwise} \end{cases} \qquad \mathcal{F}\mathbb{1}_m \mapsto \begin{cases} \mathfrak{F} & \text{if } m = l-2n \\ 0 & \text{otherwise .} \end{cases}$$

On 2-morphisms we have



Furthermore, for $0 \le l \le p-1$, there is an isomorphism $K_0(\oplus_{n=0}^{l}\mathcal{D}^c(\mathrm{NH}_n^l)) \cong V_l$ as (irreducible) modules over $\dot{u}_{\mathbb{O}_p}$.

Proof. By [KK12, Theorem 5.2] there are isomorphisms of functors

$$\rho\colon \mathfrak{E}\mathfrak{F} \bigoplus \left(\bigoplus_{r=0}^{l-2n-1} q^{l-2n-1-2r}\mathrm{Id}\right) \Rightarrow \mathfrak{F}\mathfrak{E}, \qquad \text{if } l-2n \geq 0$$

$$\rho\colon \mathfrak{E}\mathfrak{F} \Rightarrow \mathfrak{F}\mathfrak{E} \bigoplus \left(\bigoplus_{r=0}^{-l+2n-1} q^{l-2n+1+2r}\mathrm{Id}\right), \qquad \text{if } l-2n \leq 0.$$

which extend to a representation of Rouquier's category [Rou08] on $\oplus_{n=0}^{l}\mathrm{NH}_n^l\text{-mod}$. The main theorem of [Bru16] implies that this extends to a representation of the Lauda category $\mathcal{U}$ (ignoring ∂).

Now we check how the derivation acts on Y, Ψ, $\cap$, and $\cup$.

$$\begin{aligned}\partial(Y)(\alpha) &= \partial(Y(\alpha)) - Y(\partial(\alpha)) \\ &= \partial(\alpha y_n) - (\partial\alpha) y_n \\ &= (\partial\alpha) y_n + \alpha(\partial y_n) - (\partial\alpha) y_n \\ &= \alpha y_n^2.\end{aligned}$$

$$\begin{aligned}\partial(\Psi)(\alpha) &= \partial(\Psi\alpha) - \Psi(\partial\alpha) \\ &= \partial(\alpha\psi_{n-1}) - (\partial\alpha)\psi_{n-1} \\ &= (\partial\alpha)\psi_{n-1} - \alpha y_{n-1}\psi_{n-1} - \alpha\psi_{n-1} y_n - (\partial\alpha)\psi_{n-1} \\ &= \alpha(-y_{n-1}\psi_{n-1} - \psi_{n-1} y_n).\end{aligned}$$

$$\begin{aligned}\partial(\cap)(\alpha\otimes\beta) &= \partial(\cap(\alpha\otimes\beta)) - \cap(\partial(\alpha\otimes\beta)) \\ &= \partial(\alpha\beta) - \partial(\alpha)\beta - \alpha\partial(\beta) + (l-2n+1)\alpha y_n\beta \\ &= (l-2n+1)\alpha y_n \beta.\end{aligned}$$

$$\begin{aligned}\partial(\cup)(\alpha) &= \partial(\cup(\alpha)) - \cup(\partial(\alpha)) \\ &= \partial(\alpha) - (l-2n-1)\alpha y_{n+1} - \partial(\alpha) \\ &= -(l-2n-1)\alpha y_{n+1}.\end{aligned}$$

Thus we have

$$\partial(Y) = Y^2, \qquad \partial(\Psi) = -Y\Psi - \Psi Y, \qquad \partial(\cap) = (m+1)\cap Y, \qquad \partial(\cup) = (1-m)Y\cup \tag{6.16}$$

implying that the representation of $\mathcal{U}^R$ on $\oplus_{n=0}^{l}\mathrm{NH}_n^l\text{-mod}$ is actually a representation of the p-DG category $(\mathcal{U}^R, \partial)$. Theorem 5.12 implies that the action of $(\mathcal{U}^R, \partial)$ extends to an action of $(\mathcal{U}, \partial)$.

The isomorphism $K_0(\oplus_{n=0}^{l}\mathcal{D}^c(\mathrm{NH}_n^l)) \cong V_l$ is [EQ16b, Theorem 6.15]. □

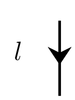
Remark 6.6. The restriction on l in the last part of the Theorem is essential when passing to derived categories $\mathcal{D}^c(\mathrm{NH}_n^l)$. The reason is that, for $n \geq p$, one can show in a similar fashion as in [KQ15, Section 3] that NH_n^l is always acyclic whenever $n \geq p$. Therefore, the sum $\oplus_{n=0}^{l}\mathcal{D}^c(\mathrm{NH}_n^l)$ only categorifies the subspace generated by the highest weight vector in the Weyl module V_l for $\dot{U}_{\mathbb{O}_p}$. For a categorification of the Weyl module V_l itself, see Section 9.3.

Remark 6.7. The biadjointness of the functors $\mathfrak{E}$ and $\mathfrak{F}$ follows from [Rou08, KK12] in conjunction with [Bru16]. Kashiwara [Kas12] showed directly that the functors $\mathfrak{E}$ and $\mathfrak{F}$ are biadjoint.

We would like to point out the connection between the construction above with the diagrammatical one in [EQ16b]. In [EQ16b, Section 6.3], it is exhibited that there is an action of the p-DG category $\mathcal{D}^c(\mathcal{U})$ on $\oplus_{n=0}^{l}\mathcal{D}^c(\mathrm{NH}_n^l)$. When $l \in \{0, \ldots, p-1\}$, the categorical action decategorifies to the small quantum group $\dot{u}_{\mathbb{O}_p}$-representation V_l.

Consider the *cyclotomic quotient category* $\mathcal{V}_l$ to be the quotient category of $\mathbb{1}_l\mathcal{U}$ by morphisms in the two-sided ideal which is right monoidally generated by

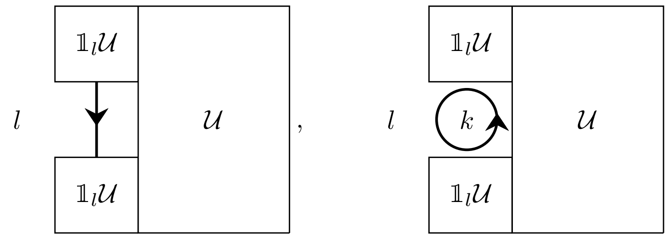
(1) Any morphism that contains the following subdiagram on the far left:



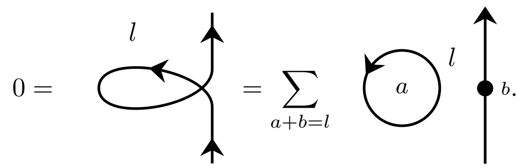
.

(2) All positive degree bubbles on the far left region labeled l.

Here by "two-sided" we mean concatenating diagrams vertically from top and bottom to those in the relations, while by "right monoidally generated" we mean composing pictures from $\mathcal{U}$ to the right of those generators. Schematically we depict elements in the ideal as follows.

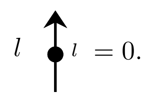


One implication of these relations is that



(6.17)

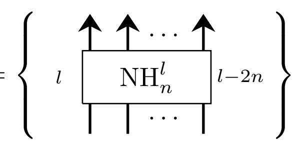
Moreover, every bubble with $a > 0$ is also in the ideal, so that



(6.18)

Therefore, it follows that the cyclotomic nilHecke algebra NH_n^l maps onto $\mathrm{END}_{\mathcal{V}_l}(\mathbb{1}_l\mathcal{E}^n\mathbb{1}_{l-2n})$. Further, Lauda [Lau10, Section 7] has proven that this map is an isomorphism:

$$\mathrm{NH}_n^l \cong \mathrm{END}_{\mathcal{V}_l}(\mathbb{1}_l\mathcal{E}^n\mathbb{1}_{l-2n}) =$$



(6.19)

Since the relation in defining $\mathcal{V}_l$ is clearly ∂-stable, $\mathcal{V}_l$ carries a natural quotient p-differential which is still denoted ∂. In this way, $(\mathcal{V}_l, \partial)$ is a p-DG module-category over $(\mathcal{U}, \partial)$. The isomorphism (6.19) is an isomorphism of p-DG algebras by Definition 5.3.

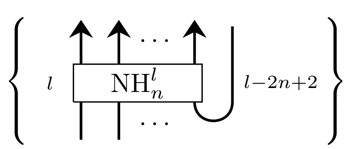
The assignment on 0-morphisms in Theorem 6.5 can now be seen as reading off the weights on the far right for the diagrams in equation (6.19).

To see the necessity of twisiting the differential on the restriction functor $\mathcal{E}$, it is readily seen that the restriction bimodule $\mathrm{NH}_n^l e^\star_{(1^{n-1},1)}$, regarded as a functor

$$\mathcal{E} = (-) \otimes_{\mathrm{NH}_n^l} \left(\mathrm{NH}_n^l e^\star_{(1^{n-1},1)}\right) : (\mathrm{NH}_n^l, \partial) \longrightarrow (\mathrm{NH}_{n-1}^l, \partial),$$

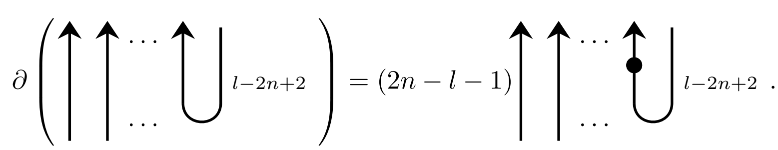
may be identified with the space of diagrams

$$\mathrm{NH}_n^l \cong$$



(6.20)

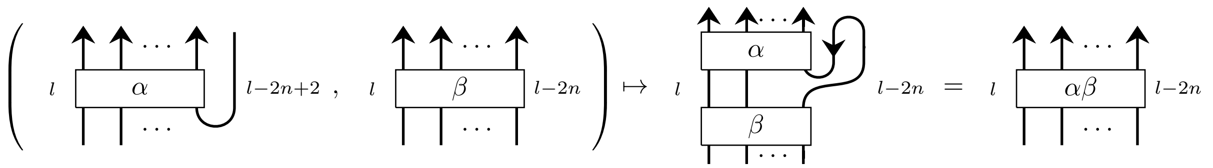
whose diagrammatic generator satisfies the differential formula



(6.21)

On the other hand, the induction functor has an obvious diagrammatic interpretation by identifying the bimodule ${}_{\mathrm{NH}^l_{n-1}}\left(e_{(1^{n-1},1)}\mathrm{NH}^l_n\right)_{\mathrm{NH}^l_n}$ as the space in equation (6.19), but the left NH^l_{n-1} acts only through the first $n-1$ strands on the top.

The bimodule homomorphism $\cap$ is now literally given by the "cap". More precisely, given elements $\alpha \in \mathrm{NH}^l_n e^{\star}_{(1^{n-1},1)}$ and $\beta \in e_{(1^{n-1},1)}\mathrm{NH}^l_n$, the bimodule homomorphism $\cap$ is diagrammatically given by



i.e., we "cap" off α by the element

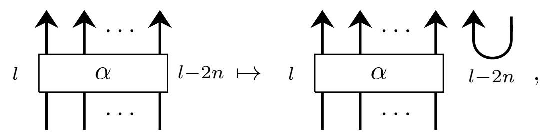
on the upper right corner, and multiply the resulting diagram with β. This is evidently the p-DG bimodule homomorphism

$$\mathrm{NH}^l_n e^{\star}_{(1^{n-1},1)} \otimes_{\mathrm{NH}^l_{n-1}} e_{(1^{n-1},1)}\mathrm{NH}^l_n \longrightarrow \mathrm{NH}^l_n, \qquad \alpha \otimes \beta \mapsto \alpha\beta$$

utilized in the statement of Theorem 6.5.

The cup bimodule homomorphism $\cup$ admits a similar diagrammatic description in $\mathcal{V}_l$ as well. It is given by

$$\mathrm{NH}^l_n \longrightarrow \mathrm{NH}^l_{n+1} e^{\star}_{(1^n,1)},$$



i.e., appending the corresponding cup to the far right of any diagram $\alpha \in \mathrm{NH}^l_n$. The differential action on the map is then a consequence of Definition 5.9.

7 Some cyclic modules

In this section, we will study a collection of combinatorially defined nilHecke modules introduced in the work of Hu-Mathas [HM10, HM15] under the action of p-differentials. These modules will be utilized to define an analogue of p-DG quiver Schur algebras in the current work.

7.1 Cellular structure

Definition 7.1. A NH^l_n-*multipartition* (or simply a *partition* for short) is an l-tuple $\mu = (\mu^1, \dots, \mu^l)$ such that $\mu^i \in \{0,1\}$ and $\mu^1 + \cdots + \mu^l = n$. We may also think of a partition as a sequence of empty slots and boxes.

The set of NH^l_n-partitions will be denoted by $\mathcal{P}^l_n$.

Example 7.2. As an example, the following partitions constitute the full list of all NH^3_2-multipartitions:

$$(\square, \square, \emptyset), \qquad (\square, \emptyset, \square), \qquad (\emptyset, \square, \square).$$

They correspond to the numerical notation of $(1,1,0)$, $(1,0,1)$ and $(0,1,1)$ respectively.

Definition 7.3. For two elements $\lambda, \mu \in \mathcal{P}^l_n$, declare $\lambda \geq \mu$ if

$$\lambda^1 + \cdots + \lambda^k \geq \mu^1 + \cdots + \mu^k$$

for $k = 1, \dots, l$. We will say $\lambda > \mu$ if $\lambda \geq \mu$ and $\lambda \neq \mu$. This defines a partial order on the set of partitions $\mathcal{P}^l_n$ called the *dominance order*.

Combinatorially, when regarded as partitions, $\lambda \geq \mu$ if μ can be obtained from λ by a sequence of moves which exchange a box and an empty space immediately right to the box. It is easily seen that there is always a unique partition that is minimal with respect to the dominance order, namely the one with all boxes on the right. We will denote the unique minimal element under this partial ordering by λ_0.

The following is an example of incomparable partitions:

$$(\,\emptyset\,,\,\Box\,,\,\Box\,,\,\emptyset\,)\,, \qquad (\,\Box\,,\,\emptyset\,,\,\emptyset\,,\,\Box\,)\,,$$

which are both greater than

$$(\,\emptyset\,,\,\Box\,,\,\emptyset\,,\,\Box\,)\,.$$

We next introduce the notion of tableaux and a partial order on them as well.

Definition 7.4. Given a partition $\mu \in \mathcal{P}^l_n$, suppose $\mu^{j_1} = \cdots = \mu^{j_n} = 1$ and $j_1 < \cdots < j_n$. A *tableau of shape* μ is a bijection

$$t\colon \{j_1, \ldots, j_n\} \longrightarrow \{1, \ldots, n\}.$$

Denote the set of μ-tableaux by $\mathrm{Tab}(\mu)$.

Given a tableau, we may think of it as a filling of its underlying partition labeled by the set of natural numbers $\{1, \ldots, n\}$. This, in turn, gives us a sequence of subtableaux in order of which the tableaux is built up by adding the at the kth step the box labeled by k ($1 \leq k \leq n$).

Example 7.5. For the partition $\mu := (\,\Box\,,\,\Box\,,\,\emptyset\,)$, we have its set of tableaux equal to

$$\mathrm{Tab}(\mu) = \left\{ (\,\boxed{1}\,,\,\boxed{2}\,,\,\emptyset\,)\,,\ (\,\boxed{2}\,,\,\boxed{1}\,,\,\emptyset) \right\}.$$

In these examples, the corresponding tableaux can be regarded as built up in two steps:

$$(\,\boxed{1}\,,\,\emptyset\,,\,\emptyset\,) \to (\,\boxed{1}\,,\,\boxed{2}\,,\,\emptyset\,)\,, \qquad (\,\emptyset\,,\,\boxed{1}\,,\,\emptyset\,) \to (\,\boxed{2}\,,\,\boxed{1}\,,\,\emptyset\,)\,.$$

Another example of the process can be read from

$$(\,\emptyset\,,\,\boxed{1}\,,\,\emptyset\,,\,\emptyset\,) \to (\,\emptyset\,,\,\boxed{1}\,,\,\boxed{2}\,,\,\emptyset\,) \to (\,\boxed{3}\,,\,\boxed{1}\,,\,\boxed{2}\,,\,\emptyset\,)\,.$$

Definition 7.6. (1) For a partition μ let t^μ be the tableau given by

$$t^\mu(j_k) = k \qquad\qquad k = 1, \ldots, n.$$

We will refer to the tableau as the *standard tableau* of shape μ.

(2) Any tableaux $t \in \mathrm{Tab}(\mu)$ can be obtained from t^μ by a unique permutation $w_t \in S_n$. We will call w_t the *permutation determined by* t.

Example 7.7. The tableaux $(\,\boxed{1}\,,\,\boxed{2}\,,\,\emptyset\,)$ is the standard one of its shape, while $(\,\boxed{2}\,,\,\boxed{1}\,,\,\emptyset\,)$ is non-standard. The corresponding permutations are the identity and non-identity element of the symmetric group S_2.

Definition 7.8. Given a tableau t, let $t_{\downarrow k}$ be the subtableau defined by

$$t_{\downarrow k}\colon t^{-1}\{1, \ldots, k\} \subset \{j_1, \ldots, j_n\} \to \{1, \ldots, k\}.$$

Note that $t_{\downarrow k}$ is the subtableau of t built in the first k steps, and it is a filling of a partition in the set $\mathcal{P}^l_k$. We are now ready to introduce a partial order on tableaux.

Definition 7.9. Let h be a μ-tableau and t a λ-tableau. We write $h \geq t$ if

$$h_{\downarrow k} \geq t_{\downarrow k}, \qquad\qquad \text{for all } k = 1, \ldots, n.$$

Moreover, if $h \geq t$ and $h \neq t$, we then write $h > t$.

Remark 7.10. We make several simple notes.

(1) By definition, we may think of a μ-tableau h to be greater than another λ-tableau t if each element in the filling of μ corresponding to h appears to the left of the same element in the filling of λ corresponding to t.

(2) One may easily verify that, $\mu \geq \lambda$ if and only if the standard filling t^{μ} of μ is greater than or equal to the standard filling t^{λ} of λ.

(3) It is clear from definition that $t^{\mu} \geq h$ for all μ-tableaux h.

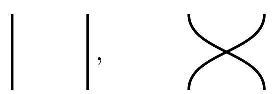
Definition 7.11. The degree of a μ-tableau t is defined by

$$deg(t) = nl - (j_1 + \cdots + j_n) - 2l(w_t),$$

where $l(w_t)$ is the length of the permutation w_t.

Remark 7.12. When $l = n$ the degree $deg(t) = \frac{n(n-1)}{2} - 2l(t)$.

We now consider a certain important elements in the cyclotomic nilHecke algebra. Recall from Definition 7.6 that if t is a tableau of shape μ, then w_t is a permutation, which in turn defines a nilHecke element $\psi_t := \psi_{w_t} \in \mathrm{NH}_n^l$ (see the end of Section 6.1). For instance, for the standard and non-standard tableaux in Example 7.7, we obtain their corresponding nilHecke elements :



Definition 7.13. Again suppose that for the partition μ that $\mu^{j_1} = \cdots = \mu^{j_n} = 1$ and $j_1 < \cdots < j_n$. For two μ-tableaux h and t let

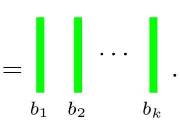
$$y^{\mu} := y_1^{l-j_1} \cdots y_n^{l-j_n}, \qquad \psi_{ht}^{\mu} := \psi_h^* y^{\mu} \psi_t. \tag{7.1}$$

When the composition μ is clear from context we will often abbreviate ψ_{ht}^{μ} by ψ_{ht}.

For later use, we also define an idempotent associated with $\mu \in \mathcal{P}_n$ as follows.

Definition 7.14. For $\mu = (0^{a_1}, 1^{b_1}, \ldots, 0^{a_k}, 1^{b_k}) \in \mathcal{P}_n^l$, we associate with it the sequence $\mathbf{b} := (b_1, \ldots, b_k)$ and define the idempotent $e_{\mu} \in \mathrm{NH}_n^l$ by (see equation (6.8))

$$e_{\mu} := e_{\mathbf{b}} =$$



Theorem 7.15. The set $\{\psi_{ht}^{\mu} | h, t \in \mathrm{Tab}(\mu), \mu \in \mathcal{P}_n^l\}$ is a graded cellular basis of NH_n^l. More precisely

(i) The degree of ψ_{ht} is the sum of the degrees of h and t.

(ii) For $\mu \in \mathcal{P}_n^l$ and $h, t \in \mathrm{Tab}(\mu)$, there are scalars $r_{tv}(x)$ which do not depend on h such that,

$$\psi_{ht} x = \sum_{v \in \mathrm{Tab}(\mu)} r_{tv}(x) \psi_{hv} \mod (\mathrm{NH}_n^l)^{>\mu},$$

where

$$(\mathrm{NH}_n^l)^{>\mu} = \Bbbk \langle \psi_{ab}^{\lambda} | \lambda > \mu, \text{ and } a, b \in \mathrm{Tab}(\lambda) \rangle.$$

(iii) The anti-automorphism $* \colon \mathrm{NH}_n^l \longrightarrow \mathrm{NH}_n^l$ sends ψ_{ht} to $\psi_{ht}^* = \psi_{th}$.

Proof. This can be found in [HM10, Theorem 5.8 and 6.11]. See also [Li14]. ☐

The next result shows that NH_n^l is a symmetric Frobenius algebra. The result in [HM10] is much more general than what we need in the current work. The special case of NH_n^l follows from the fact that it is isomorphic to an $n! \times n!$ matrix algebra over $\mathrm{H}^*(\mathrm{Gr}(n,l))$, and the algebra $\mathrm{H}^*(\mathrm{Gr}(n,l))$ is Frobenius with a trace that maps a generator in the top degree to 1 and everything else to 0. It would be nice to match it up with the graphical trace defined in [Web17].

Proposition 7.16. There is a non-degenerate homogeneous trace $\tau\colon \mathrm{NH}_n^l \longrightarrow \Bbbk$ of degree $-2n(l-n)$.

Proof. See [HM10, Theorem 6.17]. □

Proposition 7.17. The two-sided ideal $(\mathrm{NH}_n^l)^{>\mu}$ is preserved by the differential.

Proof. Let λ be a partition such that $\lambda > \mu$. Assume $\mu^{j_1} = \cdots = \mu^{j_n} = 1$ with $j_1 < \cdots < j_n$ and $\lambda^{k_1} = \cdots = \lambda^{k_n} = 1$ with $k_1 < \cdots < k_n$. Then $y^\lambda = y_1^{l-k_1} \cdots y_n^{l-k_n}$.

By definition, we have

$$\begin{aligned}\partial(\psi_h^* y_1^{l-k_1} \cdots y_n^{l-k_n} \psi_t) &= \sum_{i=1}^{n} (l+1-k_i)\psi_h^* y_i (y_1^{l-k_1} \cdots y_i^{l-k_i} \cdots y_n^{l-k_n})\psi_t + \partial(\psi_h^*) y^\lambda \psi_t + \psi_h^* y^\lambda \partial(\psi_t) \\ &= \sum_{i=1}^{n} (l+1-k_i)\psi_h^* y_i (y^\lambda)\psi_t + \partial(\psi_h^*) y^\lambda \psi_t + \psi_h^* y^\lambda \partial(\psi_t).\end{aligned}$$

Since y^λ is in the two-sided ideal $(\mathrm{NH}_n^l)^{>\mu}$, it is clear that each term in the summation above lies in $(\mathrm{NH}_n^l)^{>\mu}$. This finishes the proof. □

7.2 Specht modules

The modules over the cyclotomic nilHecke algebra which we are about to define are known as Specht modules. These modules have been considered for the classical cyclotomic Hecke algebras in earlier literature. Their graded lifts (as modules over graded KLR algebras) have been constructed by Brundan, Kleshchev, and Wang [BKW11]. Here we once again specialize their general KLR construction to nilHecke case.

Let $\mu \in \mathcal{P}_n^l$ and $t \in \mathrm{Tab}(\mu)$. One defines the right Specht module S_t^μ in terms of the cellular structure on NH_n^l defined in the previous section. Let S_t^μ be the submodule of $\mathrm{NH}_n^l/(\mathrm{NH}_n^l)^{>\mu}$ generated by the coset $\psi_{te}^\mu + (\mathrm{NH}_n^l)^{>\mu}$.

Proposition 7.18. The Specht module S_e^μ is a right p-DG module, where e the identity tableaux.

Proof. This follows from Proposition 7.17. □

Remark 7.19. To see that the above proposition is not true for an arbitrary tableaux consider the case $l = n = 2$. Then there is only one partition; call it μ. There are two tableaux: the identity e and the transposition s_1. Then S_e^μ has basis $\{y_1, y_1\psi_1\}$ and as in the proposition is obviously a p-DG module. However $S_{s_1}^\mu$ has a basis $\{\psi_1 y_1, \psi_1 y_1 \psi_1\}$ and is clearly not stable under ∂.

The module S_t^μ has a basis

$$\{\psi_{th}^\mu | h \in \mathrm{Tab}(\mu)\}.$$

Proposition 7.20. For any $t \in \mathrm{Tab}(\mu)$ there is an isomorphism of NH_n^l-modules $S_t^\mu \cong S_e^\mu\langle -2l(t)\rangle$.

Proof. This is obvious. There is an isomorphism that sends an element ψ_{th} to ψ_{eh}. □

Remark 7.21. Proposition 7.20 cannot be a statement of p-DG modules since there is only one preferred representative in the isomorphism class which actually even carries a p-DG structure.

Since S_e^μ is the preferred Specht module which carries a p-DG structure, we write $S^\mu := S_e^\mu$. We now give another realization of this Specht module. Let v^μ be a formal basis vector spanning a (p-DG) $\Bbbk[y_1,\ldots,y_n]$-module where $v^\mu y_i = 0$ for $i = 1,\ldots,n$. Define the module

$$\widetilde{S}^\mu = \Bbbk v^\mu \otimes_{\Bbbk[y_1,\ldots,y_n]} \mathrm{NH}_n^l.$$

Proposition 7.22. The NH_n^l-module $\widetilde{S}^\mu$ is isomorphic to the Specht module S^μ. Furthermore, this is an isomorphism of p-DG modules.

Proof. This is obvious. □

Proposition 7.23. The Specht module S^μ is irreducible. In particular, up to isomorphism and grading shifts, the isomorphism class of the p-DG module S^μ is independent of the partition $\mu \in \mathcal{P}_n^l$.

Proof. By Proposition 7.22, $S^\mu \cong \Bbbk v^\mu \otimes_{\Bbbk[y_1,\dots,y_n]} \mathrm{NH}_n^l$ where $\Bbbk v^\mu$ is the trivial module over the polynomial algebra $\Bbbk[y_1,\dots,y_n]$. This shows that the Specht module has a basis $\{v^\mu \otimes \psi_w | w \in S_n\}$.

It is now straightforward to show directly that the Specht module is irreducible. Indirectly, the cyclotomic nilHecke algebra is isomorphic to an $n! \times n!$ matrix algebra over $\mathrm{H}^*(\mathrm{Gr}(n,l))$. Thus it has a unique simple module of dimension $n!$. Since the basis for the Specht module given above has $n!$ elements, it follows that S^μ must be irreducible. □

7.3 The modules $G(\lambda)$

Once again, fix a partition λ with $\lambda^{j_1} = \cdots = \lambda^{j_n} = 1$ and $j_1 < \cdots < j_n$. Recall that

$$y^\lambda = y_1^{l-j_1} \cdots y_n^{l-j_n}.$$

As in [HM15] we define for each partition a cyclic module.

Definition 7.24. Let $\lambda \in \mathcal{P}_n^l$ be a partition. Define the module

$$G(\lambda) := q^{-nl+j_1+\cdots+j_n} y^\lambda \mathrm{NH}_n^l.$$

The next definition is used for describing a basis of $G(\lambda)$.

Definition 7.25. Let λ and μ be two partitions. We define a subset of μ-tableaux by

$$\mathrm{Tab}^\lambda(\mu) := \{t \in \mathrm{Tab}(\mu) | t \geq t^\lambda\}.$$

Proposition 7.26. The module $G(\lambda)$ has a basis

$$\{\psi_{th} | t \in \mathrm{Tab}^\lambda(\mu), h \in \mathrm{Tab}(\mu), \mu \in \mathcal{P}_n^l\}.$$

Proof. See [HM15, Theorem 4.9]. □

Since $\partial(y^\lambda) = y^\lambda a$ for some $a \in \Bbbk[y_1,\dots,y_n]$, the module $G(\lambda)$ is naturally a right p-DG module.

Remark 7.27. If $\lambda = (1^n 0^{l-n})$ then $G(\lambda) \cong q^{-nl+j_1+\cdots+j_n} S^\lambda$ as p-DG modules. By Proposition 7.26, since λ is the maximal partition in the dominance order, the p-DG module $G(\lambda)$ has a basis labeled by $\mathrm{Tab}(\lambda)$. Thus it has dimension $n!$ and must be the Specht module up to grading shift. The claim now follows from the uniqueness of the p-DG structure on Specht modules (Proposition 7.23).

Proposition 7.28. The module $G(\lambda)$ has a filtration whose subsequent quotients are isomorphic to Specht modules. More specifically let $\cup_\mu \mathrm{Tab}^\lambda(\mu) = \{w_1,\dots,w_m\}$ be such that $w_i \geq w_j$ whenever $i \leq j$. Suppose the tableau w_i corresponds to a partition ν_i. Then $G(\lambda)$ has a filtration

$$G(\lambda) = G_m \supset G_{m-1} \supset \cdots \supset G_0 = 0$$

such that

$$G_i/G_{i-1} \cong q^{l(w_i)-nl+(j_1+\cdots+j_n)} S^{\nu_i}.$$

Proof. This is [HM15, Corollary 4.11]. □

Proposition 7.29. The Specht filtration of $G(\lambda)$ in Proposition 7.28 is a filtration of p-DG modules.

Proof. Following [HM15, Corollary 4.11], let $\cup_\mu \mathrm{Tab}^\lambda(\mu) = \{w_1, \ldots, w_m\}$. Suppose that $w_r \in \mathrm{Tab}(\nu_r)$. Define

$$G_i = \Bbbk\{\psi^*_{w_j} y^{\nu_j} \psi_t | j \le i, t \in \mathrm{Tab}(\nu_j)\}.$$

Now we compute the derivation on a basis vector $\psi^*_{w_j} y^{\nu_j} \psi_t$ from above:

$$\partial(\psi^*_{w_j} y^{\nu_j} \psi_t) = \partial(\psi^*_{w_j}) y^{\nu_j} \psi_t + \psi^*_{w_j} \partial(y^{\nu_j}) \psi_t + \psi^*_{w_j} y^{\nu_j} \partial(\psi_t). \tag{7.2}$$

Now we analyze each term in (7.2).

Note that $\psi^*_{w_j} y^{\nu_j} \in G_i$. By [HM15, Corollary 4.11], G_i is a right NH^l_n-module. Since $\partial(\psi_t) \in \mathrm{NH}^l_n$ it follows that $\psi^*_{w_j} y^{\nu_j} \partial(\psi_t) \in G_i$.

Next,

$$\psi^*_{w_j} \partial(y^{\nu_j}) \psi_t = \sum_k d_k \psi^*_{w_j} y^{\nu_j} y_k \psi_t$$

for some constants $d_k \in \mathbb{F}_p$. Once again, since $\psi^*_{w_j} y^{\nu_j} \in G_i$ and $y_k \psi_t \in \mathrm{NH}^l_n$ it follows that

$$\sum_k d_k \psi^*_{w_j} y^{\nu_j} y_k \psi_t \in G_i.$$

Finally we consider $\partial(\psi^*_{w_j})$. Let $\psi^*_{w_j} = \psi_{i_1} \cdots \psi_{i_r}$. Then it is easy to see that

$$\partial(\psi^*_{w_j}) = \sum_m c_m \psi_{i_1} \cdots \hat{\psi}_{i_m} \cdots \psi_{i_r} + \sum_m d_m \psi^*_{w_j} y_m$$

for constants c_m and d_m where $\hat{\psi}_{i_m}$ means omit the factor ψ_{i_m}. If non-zero, the expression $\psi_{i_1} \cdots \hat{\psi}_{i_m} \cdots \psi_{i_r}$ is greater than or equal to ψ_{w_j}. Thus

$$\sum_m c_m \psi_{i_1} \cdots \hat{\psi}_{i_m} \cdots \psi_{i_r} \in G_i.$$

By the arguments from earlier it follows that

$$\sum_m d_m \psi^*_{w_j} y_m \in G_i.$$

This finishes the proof of the proposition. □

Proposition 7.30. There is a non-degenerate bilinear form on $G(\lambda)$. In particular, the graded dual of $G(\lambda)$ is isomorphic to itself.

Proof. This is proved in [HM15, Theorem 4.14]. It utilizes the Frobenius structure of NH^l_n. □

Recall from Definition 7.14 the idempotent e_λ associated with a partition $\lambda \in \mathcal{P}^l_n$.

Proposition 7.31. Suppose $\lambda = (0^{a_1} 1^{b_1} \ldots 0^{a_r} 1^{b_r}) \in \mathcal{P}^l_n$. Let $\sum_{i=1}^r (a_i + b_i) = l$ and $\sum_{i=1}^r b_i = n$. Then the right module $e_\lambda y^\lambda \mathrm{NH}^l_n$ is a p-DG submodule of $G(\lambda)$.

Proof. For notational convenience set $a_0 = b_0 = 0$. By definition

$$\begin{aligned} y^\lambda &= \prod_{i=0}^{r-1} y_{b_0+\cdots+b_i+1}^{l-(a_1+b_1+\cdots+a_i+b_i+a_{i+1}+1)} \cdots y_{b_0+\cdots+b_{i+1}}^{l-(a_1+b_1+\cdots+a_{i+1}+b_{i+1})} \\ &= \prod_{i=0}^{r-1} y_{b_0+\cdots+b_i+1}^{l-(a_1+b_1+\cdots+a_{i+1}+b_{i+1})} \cdots y_{b_0+\cdots+b_{i+1}}^{l-(a_1+b_1+\cdots+a_{i+1}+b_{i+1})} y^\gamma \end{aligned}$$

where $y^\gamma = y_{b_0+\cdots+b_i+1}^{b_{i+1}-1} \cdots y_{b_0+\cdots+b_{i+1}}^{0}$.

The idempotent e_λ may be written as

$$e_\lambda = \prod_{i=0}^{r-1} (y_{b_0+\cdots+b_i+1}^{b_{i+1}-1} \cdots y_{b_0+\cdots+b_{i+1}}^{0}) \psi_w,$$

where $w \in S_{b_1} \times \cdots \times S_{b_r}$ is the longest element in the parabolic subgroup.

Now we are able to write

$$e_\lambda y^\lambda = y^\lambda \psi_w y^\gamma.$$

Thus $e_\lambda y^\lambda \mathrm{NH}_n^l$ is a submodule of $G(\lambda)$.

It is known that e_λ is a ∂-stable idempotent. Thus $e_\lambda y^\lambda$ generates a p-DG ideal. Therefore $e_\lambda y^\lambda \mathrm{NH}_n^l$ is a p-DG submodule of $G(\lambda)$. □

8 Two-tensor quiver Schur algebra

8.1 Quiver Schur algebra

In this section we recall the definition of (p-DG) quiver Schur algebra for $\mathfrak{sl}_2$ in the sense of Hu and Mathas [HM15]. Although it will only play an auxiliary role for the current work, it will be studied in more detail in a sequel to this paper.

Definition 8.1. The *(graded) quiver Schur algebra* is by definition

$$S_n(l) := \mathrm{END}_{\mathrm{NH}_n^l} \left(\bigoplus_{\lambda \in \mathcal{P}_n^l} G(\lambda) \right). \tag{8.1}$$

The algebra $S_n(l)$ inherits a p-DG structure from the p-DG structure on NH_n^l. If $f \in S_n(l)$ then set

$$(\partial f)(x) = \partial(f(x)) - f(\partial x).$$

We now recall a basis of this algebra from [HM15, Section 4.2]. For $t \in \mathrm{Tab}^\mu(\lambda), h \in \mathrm{Tab}^\nu(\lambda)$, there is a map

$$\Psi_{th}^{\mu\nu} \colon G(\nu) \longrightarrow G(\mu)$$

defined by

$$\Psi_{th}^{\mu\nu}(y^\nu) = \psi_{th}^\lambda.$$

Note that there is a hidden dependence of $\Psi_{th}^{\mu\nu}$ on λ because t and h are λ-tableaux for some λ. When we want to stress the dependence on λ we write $\Psi_{th\lambda}^{\mu\nu}$ instead.

Definition 8.2. (1) Let $S_n^{>\lambda}(l)$ be the $\Bbbk$-vector space spanned by $\{\Psi_{th}^{\mu\nu}\}$ for $t \in \mathrm{Tab}^\mu(\gamma)$ and $h \in \mathrm{Tab}^\nu(\gamma)$ with $\gamma > \lambda$.

(2) Let $S_n^{\geq\lambda}(l)$ be the $\Bbbk$-vector space spanned by $\{\Psi_{th}^{\mu\nu}\}$ for $t \in \mathrm{Tab}^\mu(\gamma)$ and $h \in \mathrm{Tab}^\nu(\gamma)$ with $\gamma \geq \lambda$.

Proposition 8.3. The algebra $S_n(l)$ is a graded cellular algebra with cellular basis

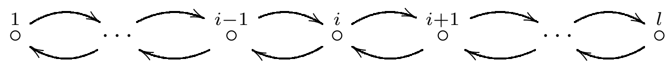
$$\{\Psi_{th}^{\mu\nu} | t \in \mathrm{Tab}^\mu(\lambda),\ h \in \mathrm{Tab}^\nu(\lambda),\ \lambda \in \mathcal{P}_n^l\}.$$

Proof. This is [HM15, Theorem 4.19]. □

The modules $G(\lambda)$ are generally decomposable. For each $\lambda \in \mathcal{P}_n^l$ there exists a unique module $Y(\lambda)$ such that $Y(\lambda)$ is a summand of $G(\lambda)$ of multiplicity one and does not appear in a decomposition of $G(\mu)$ for any $\mu > \lambda$. It is difficult to explicitly construct each $Y(\lambda)$. As a consequence it is not clear that $Y(\lambda)$ is a p-DG module. This prevents us from understanding the Grothendieck group of compact p-DG $S_n(l)$-modules in a straightforward way.

Example 8.4. When $n = 1$, the quiver Schur algebra $S_1(l)$ is isomorphic to $A_l^!$ which is the algebra Koszul dual to the zigzag algebra. This algebra has been studied with its p-DG structure in [QS16]. It is shown that there is a braid group action on the derived category of compact modules.

Let l be a natural number greater than or equal to two, and Q_l be the following quiver:



(8.2)

Let $\Bbbk Q_l$ be the path algebra associated to Q_l over the ground field. We use, for instance, the symbol $(i|j|k)$, where i, j, k are vertices of the quiver Q_l, to denote the path which starts at a vertex i, then goes through j (necessarily $j = i \pm 1$) and ends at k. The composition of paths is given by

$$(i_i|i_2|\cdots|i_r) \cdot (j_1|j_2|\cdots|j_s) = \begin{cases} (i_i|i_2|\cdots|i_r|j_2|\cdots|j_s) & \text{if } i_r = j_1, \\ 0 & \text{otherwise,} \end{cases} \tag{8.3}$$

where $i_1, \dots, i_r$ and $j_1, \dots, j_s$ are sequences of neighboring vertices in Q_l.

The algebra $A_l^!$ is the quotient of the path algebra $\Bbbk Q_l$ by the relations

$$(i|i-1|i) = (i|i+1|i) \ \ (i = 2, \dots, l-1), \qquad (1|2|1) = 0.$$

The identification of $A_l^!$ with $S_1(l)$ is obtained by associating $(i+1|i)$ to the morphism from $G(0^{i-1}10^{l-i})$ to $G(0^i 10^{l-i-1})$ given by mapping y_1^{l-i} to y_1^{l-i}. The element $(i|i+1)$ corresponds to the morphism from $G(0^i10^{l-i-1})$ to $G(0^{i-1}10^{l-i})$ given by mapping y_1^{l-i-1} to y_1^{l-i}.

This realization of $A_l^!$ endows the algebra with a p-DG structure. On NH_1^l there is a differential ∂ given by $\partial(y_1) = y_1^2$. It is clear that each $G(0^{i-1}10^{l-i})$ is a p-DG submodule of NH_1^l. This gives $A_l^!$ the structure of a p-DG algebra where the differential is given by

$$\partial(i|i+1) = (i|i+1|i|i+1), \qquad \partial(i|i-1) = 0.$$

Example 8.5. The quiver Schur algebra $S_2(3)$ is the first example of a quiver Schur algebra which is not basic. There are three partitions in this case. Let

$$\lambda = (\,\square\,,\,\square\,,\,\emptyset\,) \qquad \mu = (\,\square\,,\,\emptyset\,,\,\square\,) \qquad \zeta = (\,\emptyset\,,\,\square\,,\,\square\,).$$

Then we have

$$G(\lambda) = y_1^2 y_2 \mathrm{NH}_2^3.$$

It is spanned by

$$\{\psi_{e,e}^\lambda = y_1^2 y_2, \psi_{e,s_1}^\lambda = y_1^2 y_2 \psi_1\}.$$

It is clear that (up to a shift) $G(\lambda) \cong S^\lambda$, and so it is simple implying that $Y(\lambda) = G(\lambda)$.

Next consider

$$G(\mu) = y_1^2 \mathrm{NH}_2^3.$$

It is spanned by

$$\{\psi_{e,e}^\lambda = y_1^2 y_2, \psi_{e,s_1}^\lambda = y_1^2 y_2 \psi_1, \psi_{e,e}^\mu = y_1^2, \psi_{e,s_1}^\mu = y_1^2 \psi_1\}.$$

It is easy to check that the endomorphism algebra of $G(\mu)$ is non-negatively graded and its degree zero piece is one-dimensional. Thus $Y(\mu) = G(\mu)$.

The remaining cyclic module is

$$G(\zeta) = y_1 \mathrm{NH}_2^3.$$

It is spanned by

$$\{\psi_{e,e}^\lambda = y_1^2 y_2, \psi_{e,s_1}^\lambda = y_1^2 y_2 \psi_1, \psi_{s_1,e}^\lambda = \psi_1 y_1^2 y_2, \psi_{s_1,s_1}^\lambda = \psi_1 y_1^2 y_2 \psi_1,$$
$$\psi_{e,e}^\mu = y_1^2, \psi_{e,s_1}^\mu = y_1^2 \psi_1, \psi_{e,e}^\zeta = y_1, \psi_{e,s_1}^\zeta = y_1 \psi_1\}.$$

There is a decomposition

$$G(\zeta) \cong e_2 G(\zeta) \oplus (1-e_2)G(\zeta).$$

It is readily checked that $e_2G(\zeta) \cong Y(\zeta)$ is the indecomposable projective module over NH_2^3 and is closed under the p-differential action. While $e_2G(\zeta)$ is a p-DG submodule, this decomposition does not respect ∂ so we merely have a short exact sequence of p-DG modules

$$0 \longrightarrow e_2G(\zeta) \longrightarrow G(\zeta) \longrightarrow (1-e_2)G(\zeta) \longrightarrow 0.$$

One can also identify $(1-e_2)G(\zeta)$ with the Specht module $G(\lambda) \cong Y(\lambda)$.

Due to the decomposability of $G(\zeta)$, the endomorphism algebras $\mathrm{END}_{\mathrm{NH}_2^3}(G(\lambda) \oplus G(\mu) \oplus G(\zeta))$ and $\mathrm{END}_{\mathrm{NH}_2^3}(Y(\lambda) \oplus Y(\mu) \oplus Y(\zeta))$ are not isomorphic, but they are Morita equivalent.

It is straightforward to show that $\mathrm{END}_{\mathrm{NH}_2^3}(Y(\lambda) \oplus Y(\mu) \oplus Y(\zeta)) \cong A_3^!$ from Example 8.4 where $(2|1)$ corresponds to the homomorphism $\Psi_{ee\lambda}^{\mu\lambda}$. The element $(1|2)$ corresponds to the homomorphism $\Psi_{ee\lambda}^{\lambda\mu}$. The path $(3|2)$ corresponds to the composition of the homomorphism $\Psi_{ee\mu}^{\zeta\mu}$ with projection onto the summand $Y(\zeta)$. Finally the element $(2|3)$ corresponds to inclusion of $Y(\zeta)$ into $G(\zeta)$ composed with $\Psi_{ee\mu}^{\mu\zeta}$.

In the next subsection we will consider a collection of λ's where we have a full understanding of $Y(\lambda)$'s, leading in turn to a categorification of $V_r \otimes_{\mathbb{O}_p} V_s$ at a prime root of unity.

8.2 A category of nilHecke modules

In this section we fix $l, n \in \mathbb{N}$ and two other natural numbers r, s such that $r + s = l$. We will abbreviate $\mathrm{HOM}_{\mathrm{NH}_n^l}$ simply as HOM for the ease of notation.

Definition 8.6. Let $\mathcal{P}_n^{r,s}$ be the subset of all partitions $\lambda \in \mathcal{P}_n^l$ of the form $\lambda = (0^a 1^b 0^c 1^d)$ with

$$r+s=l, \quad a+b=r, \quad c+d=s, \quad b+d=n.$$

We will think of such a sequence as a partition

$$(\underbrace{\emptyset, \dots, \emptyset}_{a}, \underbrace{\square, \dots, \square}_{b} \mid \underbrace{\emptyset, \dots, \emptyset}_{c}, \underbrace{\square, \dots, \square}_{d}).$$

The collection of partitions is a subset of $\mathcal{P}_n^l$, and thus inherits the partial order on $\mathcal{P}_n^l$ (Definition 7.3). Notice that the unique minimal element $\lambda_0 \in \mathcal{P}_n^l$ always lies inside $\mathcal{P}_n^{r,s}$, and is also the minimal element here. It is the element of $\mathcal{P}_n^{r,s}$ where the entries labeled by 1 are as far to the right as possible. Recall that if $\lambda = (0^a 1^b 0^c 1^d) \in \mathcal{P}_n^{r,s}$, we have (Definitions 7.13 and 7.14)

$$y^\lambda = y_1^{b+c+d-1} y_2^{b+c+d-2} \cdots y_b^{c+d} y_{b+1}^{d-1} y_{b+2}^{d-2} \cdots y_{b+d}^0, \qquad e_\lambda = e_{(b,d)} = e_b \otimes e_d. \tag{8.4}$$

Definition 8.7. Let $\mathcal{NH}_n^{r,s}$ be the filtered p-DG envelope (Definition 4.18) inside $(\mathrm{NH}_n^l, \partial)\text{-mod}$ generated by the collections of p-DG modules $e_\lambda G(\lambda)$ with $\lambda = (0^a 1^b 0^c 1^d) \in \mathcal{P}_n^{r,s}$.

The definition is equivalent to the following more (p-DG) ring-theoretic description.

Definition 8.8. Let n, l be two natural numbers, and $r, s \in \mathbb{N}$ such that $r + s = l$. We define the *two-tensor quiver Schur algebra* to be

$$S_n(r,s) := \mathrm{END}_{\mathrm{NH}_n^l}\left(\bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} e_\lambda G(\lambda) \right).$$

The algebras are equipped with the induced p-differentials on $e_\lambda G(\lambda)$'s, and is therefore a p-DG algebra.

Remark 8.9. We collect some simple facts about the definitions here.

(1) It is clear, by construction, that the objects of $\mathcal{NH}_n^{r,s}$ consist of finitely-generated, cofibrant left p-DG modules over $S_n(r,s)$. We will thus use both descriptions for the convenience of the context.

(2) In the definition of the category $\mathcal{NH}_n^{r,s}$, it is crucial for categorification purposes that we take as a generating set of objects $e_\lambda G(\lambda)$ rather than just $G(\lambda)$, as we will see in Example 8.24.

(3) One can readily check that the category $\mathcal{NH}_n^{r,s}$ does not contain the unique irreducible $(\mathrm{NH}_n^l, \partial)$-module unless $r = n$.

Our next goal is to explicitly construct the generating indecomposable objects of $\mathcal{NH}_n^{r,s}$. We will mostly ignore the differentials and study the indecomposability as modules over NH_n^l in this subsection unless otherwise specified. The p-DG structure and the compatibility with the $\mathfrak{E}$, $\mathfrak{F}$-actions will be studied in the next subsection.

Proposition 8.10. Let $\lambda = (0^a 1^b 0^c 1^d)$ with $c \geq b$. Then $e_\lambda G(\lambda)$ is an indecomposable submodule of $G(\lambda)$.

Proof. The case that $a = 0$ is trivial so assume $a \geq 1$.

First we consider the case $d = 1$. Let $Y' = e_{(b,1)} y^\lambda \mathrm{NH}_{b+1}^{a+b+c+1}$. A routine calculation shows $e_{(b,1)} y^\lambda = y^\lambda e^*_{(b,1)}$ (recall that the star of an idempotent means to reflect its graphical depiction across a horizontal axis). This shows Y' is a submodule of $G(\lambda)$ and in fact a summand since $(1 - e_{(b,1)}) y^\lambda \mathrm{NH}_{b+1}^{a+b+c+1}$ is a complementary summand.

Consider an endomorphism $\Psi := \Psi_{t,h,\mu}^{\lambda\lambda}$ of $G(\lambda)$. Let $\tilde{\Psi}$ be the restriction of Ψ to Y' followed by projection onto Y'. That is $\tilde{\Psi}(e_{(b,1)} y^\lambda) = e_{(b,1)} \Psi(e_{(b,1)} y^\lambda)$.

Write $t = t' t''$ with t' a shortest coset representative in $S_{b+1}/S_b \times S_1$ and $t'' \in S_b \times S_1$. Write $h = h' h''$ in a similar manner.

One then calculates

$$\tilde{\Psi}(e_{(b,1)} y^\lambda) = e_{(b,1)} \Psi(e_{(b,1)} y^\lambda) = e_{(b,1)} \psi_{t''} \psi_{t'} y^\mu \psi_{h'} \psi_{h''} \bar{e}_{(b,1)}.$$

By analyzing the form of the idempotent $e_{(b,1)}$ it is clear that this map is zero if t'' or h'' is not the identity permutation. Thus assume $t = t'$ and $h = h'$. Therefore

$$t, h \in \{e, s_b, s_{b-1} s_b, \ldots, s_1 \cdots s_b\}.$$

Suppose $s = s_i \cdots s_b$. The corresponding filling of the partition must have boxes labeled $1, 2, \ldots, i-1$ each moved at least one to the left from their original positions as prescribed by the standard filling of λ. Note that if $i = 1$ nothing needs to be moved. Now one calculates

$$\begin{aligned} deg(y^\mu) - deg(y^\lambda) &\geq 2(i-1) + 2(c+b-i+2) \\ &= 2c + 2b + 2 \\ &\geq 4b + 2. \end{aligned}$$

The minimal degree of ψ_t or ψ_h is $-2b$. Thus the degree of $\tilde{\Psi}$ is greater than or equal to two. Thus it is impossible to decompose Y'. It is trivial to check that it is a p-DG summand.

The proof for $d > 1$ is similar. We provide some of the details.

Let $Y' = e_{(b,d)} y^\lambda \mathrm{NH}_{b+d}^{a+b+c+d}$. Just as in the proof of Lemma 8.10, let $\Psi = \Psi_{t,h,\mu}^{\lambda\lambda}$ and $\tilde{\Psi}$ be the restriction of Ψ to Y' composed with projection onto Y'.

Just as in the case $d = 1$ we may assume t, h are shortest length coset representatives in $S_{b+d}/S_b \times S_d$.

Consider t for example. We encode it by the tuple $(r_1, \ldots, r_d)$ with $1 \leq r_1 < \cdots < r_d \leq b + d$. This tuple tells us how to shuffle the d boxes into the entire collection of $b + d$ boxes.

Suppose $r_j = i + 1 < b$ and $r_{j+1} \geq b$. This means for the partition μ that we have to push each of the first i boxes in the first collection over at least one spot each. We have to repeat this j times. In the second collection we have to move each of the first j boxes to the left at least $b + c + j - i$ spots. Thus

$$deg(y^\mu) - deg(y^\lambda) \geq 2ij + 2j(b + c + j - i) \geq 2j(2b + j).$$

The minimal degree of ψ_t or ψ_h is $-2jb$. Thus

$$deg(\tilde{\Psi}) \geq 2j(2b + j) - 4jb = 2jb.$$

Since $j \geq 1$, $\tilde{\Psi}$ is a positive degree endomorphism. Thus Y' is indecomposable. ◻

Proposition 8.11. When $a = 0$, the module $e_{(b,d)}G(1^b0^c1^d)$ is indecomposable. Furthermore, as a graded vector space there is an isomorphism

$$\mathrm{END}\left(e_{(b,d)}G(1^b0^c1^d)\right) \cong \mathrm{H}^*(\mathrm{Gr}(d, c+d)),$$

where $\mathrm{H}^*(\mathrm{Gr}(d,c+d))$ is the cohomology of d-planes in $\mathbb{C}^{c+d}$.

Proof. Let $\lambda = (1^b0^c1^d)$ and $Y' = e_{(b,d)}G(\lambda)$.

Consider $\Psi = \Psi^{\lambda\lambda}_{h,t,\mu} \in \mathrm{END}(G(\lambda))$. Let $\tilde{\Psi}$ be the restriction of Ψ to Y' composed with projection onto Y'. It is easy to see that $\tilde{\Psi}$ is trivial unless $h, t \in S_{b+d}/S_b \times S_d$ are shortest length coset representatives. But then h and t cannot fill any possible μ unless $h = t = e$, the identity coset. Thus $deg(\tilde{\Psi}) = 0$ only if $\tilde{\Psi} = \mathrm{Id}$ and otherwise the degree is positive. Thus Y' is indecomposable.

The possible μ which could arise in $\Psi^{\lambda\lambda}_{e,e,\mu}$ are those $\mu \in \mathcal{P}^{b+c+d}_{b+d}$ obtained from shuffling the c entries labeled 0 and the d entries labeled 1 in $\lambda = (1^b0^c1^d)$. There $\binom{c+d}{d}$ such shuffles and it easy to compute the degree of the resulting $\Psi^{\lambda\lambda}_{e,e,\mu}$ which then immediately gives that the graded vector space $\mathrm{END}(e_{(b,d)}G(1^b0^c1^d))$ is isomorphic to $\mathrm{H}^*(\mathrm{Gr}(d,c+d))$. ☐

Lemma 8.12. Suppose $d = 0$ and let $\lambda = (0^a1^b0^c)$. Then

$$e_\lambda G(\lambda) = e_b y^\lambda \mathrm{NH}^{a+b+c}_b = e_b y_1^{b+c-1} \cdots y_b^c \mathrm{NH}^{a+b+c}_b.$$

is an indecomposable summand of $G(\lambda)$. Furthermore

$$Y(\lambda) \cong e_\lambda G(\lambda)$$

Proof. It is easy to see that $e_b y^\lambda \mathrm{NH}^{a+b+c}_b \subset G(\lambda)$ since $e_b y^\lambda = y^\lambda e_b^*$.

We next show that $e_b y^\lambda \mathrm{NH}^{a+b+c}_b$ is a summand of $G(\lambda)$. Clearly

$$G(\lambda) = e_b y^\lambda \mathrm{NH}^{a+b+c}_b + (1 - e_b) y^\lambda \mathrm{NH}^{a+b+c}_b. \tag{8.5}$$

Note that $e_b y^\lambda \mathrm{NH}^{a+b+c}_b \subset e_b \mathrm{NH}^{a+b+c}_b$ and $(1-e_b)y^\lambda \mathrm{NH}^{a+b+c}_b \subset (1-e_b)\mathrm{NH}^{a+b+c}_b$. Since

$$e_b \mathrm{NH}^{a+b+c}_b \cap (1-e_b)\mathrm{NH}^{a+b+c}_b = 0,$$

the decomposition in (8.5) is direct.

Let $\mu \geq \lambda$ and $h, t \in \mathrm{Tab}(\mu)$. In order for the map $\Psi := \Psi^{\lambda\lambda}_{ht\mu}$ to have a non-positive degree, either h or t cannot be the identity permutation.

Define $\tilde{\Psi} \colon e_b y^\lambda \mathrm{NH}^{a+b+c}_b \longrightarrow e_b y^\lambda \mathrm{NH}^{a+b+c}_b$ by

$$\begin{aligned}\tilde{\Psi}(e_b y^\lambda) &= e_b \Psi(e_b y^\lambda)\\ &= y_1^{b-1} \cdots y_b^0 \psi_{w_0} \Psi(y^\lambda e_b^*)\\ &= y_1^{b-1} \cdots y_b^0 \psi_{w_0} \psi_h^* y^\mu \psi_t \psi_{w_0} y_1^{b-1} \cdots y_b^0.\end{aligned}$$

If h and t are both not the identity permutations then the above quantity is zero so there are no negative degree maps. Clearly if h and t are both the identity permuations then $\tilde{\Psi}$ is simply the identity map on $e_b y^\lambda \mathrm{NH}^{a+b+c}_b$. Thus there are no negative degree endomorphisms of $e_b y^\lambda \mathrm{NH}^{a+b+c}_b$ and the only degree zero map is the identity. Therefore $e_b y^\lambda \mathrm{NH}^{a+b+c}_b$ is indecomposable.

Now we show that $e_b y^\lambda \mathrm{NH}^{a+b+c}_b = Y(\lambda)$. Let us assume that $Y(\lambda) \subset G(\mu)$ for some $\mu \in \mathcal{P}^l_n$. Clearly $e_b y^\lambda \psi_{w_0} \in e_b y^\lambda \mathrm{NH}^{a+b+c}_b$. Now we calculate that

$$e_b y^\lambda \psi_{w_0} = y_1^{b+c-1} \cdots y_b^c \psi_{w_0}.$$

We know from the basis theorem of $G(\mu)$ that this basis element is not in $G(\mu)$ for $\mu > \lambda$. Thus $Y(\lambda)$ is not contained in any $G(\mu)$ for $\mu > \lambda$. ☐

For the next construction, we will need the following results. Recall from Definition 6.3 that we have defined an action of the p-DG functors $\mathfrak{E}$ (restriction) and $\mathfrak{F}$ (induction) on $\oplus_{n=0}^{l}(\mathrm{NH}_n^l, \partial)$-mod.

Lemma 8.13. For any $\lambda = (0^a 1^b 0^{c+d}) \in \mathcal{P}_n^{r,s}$, there is an isomorphism of NH_n^l-modules

$$\mathfrak{E}^{(a)} Y(1^{a+b}0^{c+d}) = e_\lambda G(\lambda).$$

Proof. By Proposition 7.26 and Remark 7.27, the indecomposable module $Y(1^{a+b}0^{c+d}) \cong G(1^{a+b}0^{c+d})$ is simple with a basis

$$\{y_1^{l-1} \cdots y_{a+b}^{l-a-b} \psi_w | w \in S_{a+b}\}.$$

Applying $\mathfrak{E}^{(a)}$ to this module has the effect of multiplying the basis elements on the right by the idempotent $e^\star_{(1^b,a)}$ which annihilates some of the basis elements. In fact, a basis for the restricted module $\mathfrak{E}^{(a)} Y(1^{a+b}0^{c+d})$ consists of

$$\{y_1^{l-1} \cdots y_{a+b}^{l-a-b} \psi_w e^\star_{(1^b,a)} | w \in S_{a+b}/(S_1^{\times b} \times S_a)\},$$

where it is understood that $w \in S_{a+b}/(S_1^{\times b} \times S_a)$ ranges over the shortest coset representatives.

On the other hand, using Proposition 7.26, we easily deduce that a basis of $e_\lambda G(\lambda)$ is

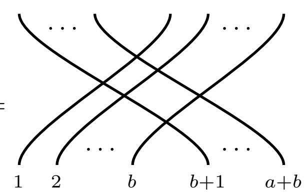
$$\{e_b (y_1 \cdots y_b)^{l-a-b} y_1^{u_1} \cdots y_b^{u_b} \psi_w | w \in S_b, a+b-1 \geq u_1 > \cdots > u_b \geq 0.\}$$

In particular, both spaces $\mathfrak{E}^{(a)} Y(1^{a+b}0^{c+d})$ and $e_\lambda G(\lambda)$ have the same dimension $\binom{a+b}{b} b!$.

Define a map $\Phi : e_\lambda G(\lambda) \longrightarrow \mathfrak{E}^{(a)} Y(1^{a+b}0^{c+d})$ by, for any $e_b y^\lambda x \in e_b y^\lambda \mathrm{NH}_n^l$,

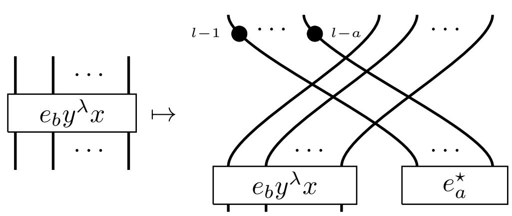
$$\Phi(e_b y^\lambda x) := y_1^{l-1} \cdots y_a^{l-a} \psi_{b,a} e^\star_{(1^b,a)} e_b y^\lambda x,$$

where we recall that $\psi_{b,a}$ is the composition of nilHecke generators associated to the longest minimal coset representatives in $S_{a+b}/S_b \times S_a$:

$\psi_{b,a} =$ 

.

The map is evidently a map of right NH_b^l-modules. It is diagrammatically described by



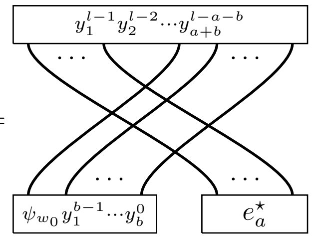
Let us use this description to explain why the map has its image inside

$$\mathfrak{E}^{(a)} G(1^{a+b}0^{c+d}) = y_1^{l-1} \cdots y_{a+b}^{l-a-b} \mathrm{NH}_{a+b}^l e^\star_{(1^b,a)}.$$

It suffices to take $x = 1 \in \mathrm{NH}_b^l$. Notice that we have

$$\begin{aligned} e_b y^\lambda &= e_b y_1^{l-a-1} y_2^{l-a-2} \cdots y_b^{l-a-b} = (y_1 y_2 \cdots y_b)^{l-a-b} e_b y_1^{b-1} y_2^{b-2} \cdots y_b^0 \\ &= y_1^{l-a-1} y_2^{l-a-2} \cdots y_b^{l-a-b} \psi_{w_0} y_1^{b-1} y_2^{b-2} \cdots y_b^0. \end{aligned}$$

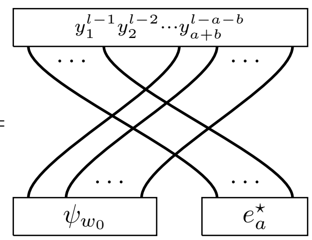
Sliding the monomial $y_1^{l-a-1} y_2^{l-a-2} \cdots y_b^{l-a-b}$ along $\psi_{b,a}$ to the northeastern corner would create lower terms which resolve crossings in $\psi_{b,a}$. Thus, the resulting lower terms involve crossings among the first b or last a strands, and they would in turn kill either ψ_{w_0} of the first b strands, or $e_a^\star$ on the last a strands. Hence we can ignore the lower terms safely and slide up the monomial to obtain

$\Phi(e_b y^\lambda) =$ 

.

Now we see that the image of Φ is contained in the ideal generated by $y_1^{l-1} \cdots y_{a+b}^{l-a-b} \mathrm{NH}_{a+b}^l e^\star_{(1^b,a)}$.

It also follows from this computation that

$\Phi(e_b y^\lambda \psi_{w_0}) =$ 

.

This is the lowest degree element of $\mathfrak{E}^{(a)} G(1^{a+b}0^{c+d})$, and is readily seen to generate the entire module. Hence the map Φ is surjective, and it follows that Φ is an isomorphism since both modules have the same dimension. □

Lemma 8.14. For any $\lambda = (0^a 1^b 0^c 1^d) \in \mathcal{P}_n^{r,s}$, there is an isomorphism of NH_n^l-modules

$$e_\lambda G(\lambda) \cong \mathfrak{F}^{(d)} Y(0^a 1^b 0^{c+d}).$$

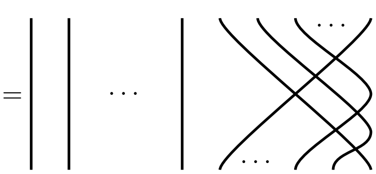
Proof. By Lemma 8.12, $Y(0^a 1^b 0^{c+d}) \cong e_b G(0^a 1^b 0^{c+d})$. We have a sequence of isomorphisms

$$\begin{aligned}
\mathfrak{F}^{(d)} Y(0^a 1^b 0^{c+d}) &\cong \mathfrak{F}^{(d)} (e_b y_1^{b+c+d-1} \cdots y_b^{c+d} \mathrm{NH}_b^l) \\
&\cong e_{(b,d)} y_1^{b+c+d-1} \cdots y_b^{c+d} \mathrm{NH}_{b+d}^l \\
&= e_\lambda (y_1^{b+c+d-1} \cdots y_b^{c+d})(y_{b+1}^{d-1} \cdots y_{b+d}^0) \mathrm{NH}_{b+d}^l \\
&= e_\lambda G(\lambda).
\end{aligned}$$

The third equality above needs some further explanation. It is clear that the right ideal $e_\lambda G(\lambda)$ is contained in the right ideal generated by the element $e_{(b,d)} y_1^{b+c+d-1} \cdots y_b^{c+d}$. The reverse inclusion holds because of the identity

$$e_{(b,d)} y_1^{b+c+d-1} \cdots y_b^{c+d} = e_{(b,d)} (y_1^{b+c+d-1} \cdots y_b^{c+d})(y_{a+b+1}^{d-1} \cdots y_{b+d}^0) \psi_w$$

where ψ_w is the element of NH_{b+d}^l arising from the longest element of the parabolic group $S_1^{\times b} \times S_d \subset S_{b+d}$:

$\psi_w =$ 
.

The lemma follows. □

Proposition 8.15. Suppose $b > c$. Then $\mathfrak{E}^{(a)}(e_{(a+b,d)} G(1^{a+b} 0^c 1^d))$ is indecomposable.

Proof. Let $Y' = Y(1^{a+b}0^{c+d})$ and $Y = \mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y'$. Then, by the previous lemma and Proposition 8.11,

$$\mathfrak{F}^{(d)}Y' \cong e_{(a+b,d)}G(1^{a+b}0^c1^d) \cong Y(1^{a+b}0^c1^d).$$

We will show that $\mathrm{END}(Y)$ is non-negatively graded and that the degree zero subspace is one dimensional. We compute:

$$\begin{aligned}
\mathrm{END}(Y) &\cong \mathrm{HOM}(\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y', \mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y') \\
&\cong \mathrm{HOM}(\mathfrak{F}^{(d)}Y', \mathfrak{F}^{(a)}\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y')q^{a(b+d-c)} \\
&\cong \bigoplus_{j=0}^{a} \mathrm{HOM}(\mathfrak{F}^{(d)}Y', \mathfrak{E}^{(a-j)}\mathfrak{F}^{(a-j)}\mathfrak{F}^{(d)}Y') \begin{bmatrix} a+b+d-c \\ j \end{bmatrix} q^{a(b+d-c)} \\
&\cong \bigoplus_{j=0}^{a} \mathrm{HOM}(\mathfrak{F}^{(d)}Y', \mathfrak{E}^{(a-j)}\mathfrak{F}^{(a+d-j)}Y') \begin{bmatrix} a+b+d-c \\ j \end{bmatrix} \begin{bmatrix} a+d-j \\ d \end{bmatrix} q^{a(b+d-c)} \\
&\cong \bigoplus_{j=0}^{a} \mathrm{HOM}(\mathfrak{F}^{(a-j)}\mathfrak{F}^{(d)}Y', \mathfrak{F}^{(a+d-j)}Y') \begin{bmatrix} a+b+d-c \\ j \end{bmatrix} \begin{bmatrix} a+d-j \\ d \end{bmatrix} q^{a(b+d-c)} q^{(a-j)(2a+b+d-c-j)} \\
&\cong \bigoplus_{j=0}^{a} \mathrm{HOM}(\mathfrak{F}^{(a+d-j)}Y', \mathfrak{F}^{(a+d-j)}Y') \begin{bmatrix} a+b+d-c \\ j \end{bmatrix} \begin{bmatrix} a+d-j \\ d \end{bmatrix}^2 q^{(2a-j)(a+b+d-c-j)} \\
&\cong \bigoplus_{j=0}^{a} \mathrm{END}(Y(1^{a+b}0^{c-a+j}1^{a+d-j})) \begin{bmatrix} a+b+d-c \\ j \end{bmatrix} \begin{bmatrix} a+d-j \\ d \end{bmatrix}^2 q^{(2a-j)(a+b+d-c-j)}.
\end{aligned}$$

The second and fifth isomorphisms follow by taking the biadjoints of $\mathfrak{E}^{(a)}$ and $\mathfrak{E}^{(a-j)}$ using equation (5.13). The third one follows from taking commutation of $\mathfrak{F}^{(a)}$ and $\mathfrak{E}^{(a)}$ (see Proposition 5.7 (ii)). The formula for multiplication of $\mathfrak{F}^{(a-j)}$ and $\mathfrak{F}^{(d)}$ produces the fourth and sixth isomorphism (Proposition 5.7 (i)), while the last one is again a consequence of Lemma 8.14.

It follows from Proposition 8.11 again that $Y(1^{a+b}0^{c-a+j}1^{a+d-j})$ is indecomposable, and

$$\mathrm{END}(Y(1^{a+b}0^{c-a+j}1^{a+d-j})) \cong \mathrm{H}^*(\mathrm{Gr}(a+d-j, c+d)).$$

Since the graded dimension of $\mathrm{H}^*(\mathrm{Gr}(a+d-j, c+d))$ is an element of $1+q\mathbb{N}[q]$ we need only determine the smallest exponent of

$$\begin{bmatrix} a+b+d-c \\ j \end{bmatrix} \begin{bmatrix} a+d-j \\ d \end{bmatrix}^2 q^{(2a-j)(a+b+d-c-j)}.$$

The smallest exponent is

$$-j(a+b+d-c-j) - 2d(a-j) + (2a-j)(a+b+d-c-j) = 2j^2 + (2(c-b)-4a)j + 2a(b-c+a). \quad (8.6)$$

The minimum of this function (with respect to j) occurs when $j = a - \frac{c-b}{2}$. Since $b > c$, it follows that $j = a - \frac{c-b}{2} \geq a$. Since $j \in [0, a]$, the minimum of the function (8.6) must occur at either $j = 0$ or $j = a$.

When $j = 0$, the function (8.6) is equal to $2a(a+b-c) > 0$. When $j = a$, the value of (8.6) is equal to zero. Thus the graded dimension of $\mathrm{END}(Y)$ is an element of $1+q\mathbb{N}[q]$ so Y is indecomposable. □

Remark 8.16. In general, we do not know of a construction of the indecomposable generating p-DG objects in $\mathcal{NH}_n^{r,s}$ directly from NH_n^l except in some very special cases. For instance, for the case $a > 0$, $b > c$, and $d = 1$ it is possible to show that

$$Y(0^a1^b0^c1) \cong (e_{(b,1)}^{(1^{b-c})} + \cdots + e_{(b,1)}^{(1^b)})G(0^a1^b0^c1)$$

and that $e_{(b,1)}^{(1^{b-c})} + \cdots + e_{(b,1)}^{(1^b)}$ is a p-DG idempotent (see the notation (6.9)).

We now state the following result on the classification of indecomposable objects in $\mathcal{NH}_n^{r,s}$. One should compare it with Remark 2.4.

Theorem 8.17. The indecomposable objects $Y(\lambda)$ in $\mathcal{NH}_n^{r,s}$ are parametrized by partitions in $\mathcal{P}_n^{r,s}$. Such a $Y(\lambda)$, when $\lambda = (0^a1^b0^c1^d)$, is isomorphic to either of the following forms:

- $\mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}) \cong e_{(b,d)}G(0^a1^b0^c1^d) \cong \mathfrak{F}^{(d)}Y(0^a1^b0^{c+d})$ if $b \leq c$;
- $\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{c+d}) \cong \mathfrak{E}^{(a)}(e_{(a+b,d)}G(1^{a+b}0^c1^d))$ if $b \geq c$.

If $b = c$ the modules are isomorphic.

Proof. The fact that the modules in the statement of the theorem are indecomposable follows from Propositions 8.10 and 8.15.

It is trivial that the object $e_{(b,d)}G(0^a1^b0^c1^d)$ for $b \leq c$ is in $\mathcal{NH}_n^{r,s}$ since it is a generating object of the category.

Now we will show that $\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{c+d})$ is in $\mathcal{NH}_n^{r,s}$ for $b \geq c$. By Proposition 5.7, the p-DG module $\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{c+d})$ occurs as a subquotient in the fantastic filtration of

$$\mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}) \cong \mathfrak{F}^{(d)}Y(0^a1^b0^{c+d}) \cong e_{(b,d)}G(0^a1^b0^c1^d),$$

where the last isomorphism follows from Lemma 8.14.

Lastly, when $b = c$, the fact that $\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{b+d}) \cong \mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{b+d})$ follows from the special case of Proposition 5.7 that

$$\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}\mathbb{1}_m \cong \mathfrak{F}^{(d)}\mathfrak{E}^{(a)}\mathbb{1}_m$$

whenever $m \leq d - a$. Now m is the weight of the module $Y(1^{a+b}0^{c+d})$, which equals $l - 2(a+b) = d - a$. The result follows. □

Corollary 8.18. The indecomposable objects $Y(\lambda)$ in $\mathcal{NH}_n^{r,s}$ are p-DG modules.

Proof. This follows from Theorem 8.17 since $Y(1^{a+b}0^{c+d})$ is a p-DG module and $\mathfrak{E}^{(a)}$, $\mathfrak{F}^{(d)}$ are p-DG functors. □

8.3 Extension of categorical actions III

Except for the last result, we have largely ignored the p-DG structure in the previous subsection, and, in particular, the twists involved in the definition of the functors $\mathfrak{E}^{(a)}$ in Definition 6.3. We will take these matter up in the subsection, and check that the categorical framework of Section 4.4 is suitable for our special situation of two-tensor quiver Schur algebras.

Recall from Lemma 8.13 that there is an isomorphism of right NH_b^l-modules

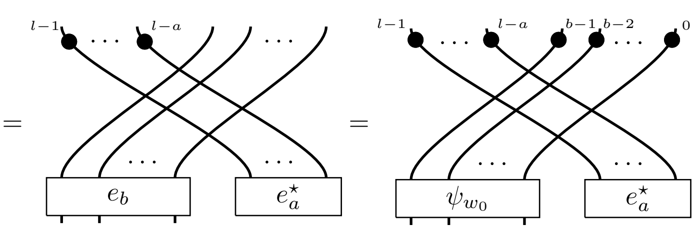
$$\Phi : e_bG(0^a1^b0^{c+d}) \cong \mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}).$$

Our first goal is to upgrade this isomorphism into one of p-DG modules. This will also explain the necessity of twisting the differential for the functor $\mathfrak{E}^{(a)}$ (Definition 6.3).

Lemma 8.19. For the partition $\lambda = (0^a1^b0^{c+d})$, there is a p-DG isomorphism

$$\Phi : e_\lambda G(\lambda) \cong \mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}).$$

Proof. In Lemma 8.13, the map Φ is given by left multiplying any element of $e_\lambda G(\lambda)$ by the element (diagrammatically multiplying on the top)

$A =$ 

.

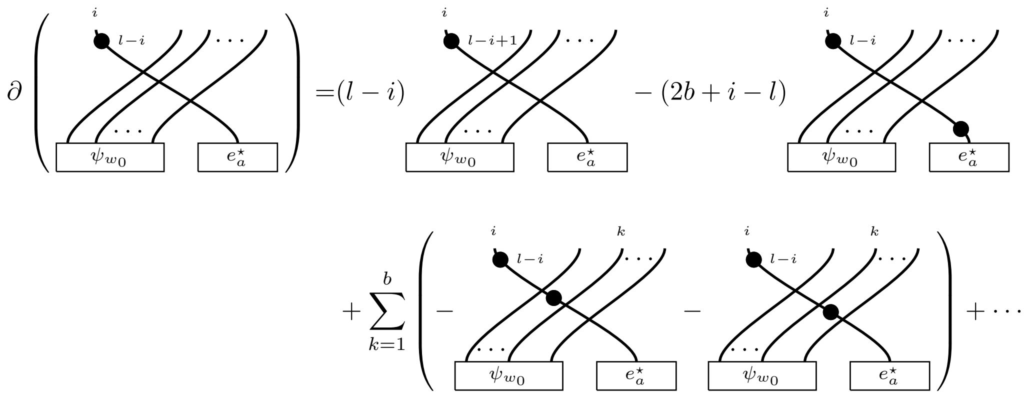
The second equality follows similarly as in the proof of Lemma 8.13.

We need to show that the map Φ is a p-DG map, which is equivalent to checking that $\partial(A)e_b = 0$. To do this, let us count, after applying the differential, the dots appearing on each strand.

By construction (Definition 6.3), we have

$$\partial(e_a^\star) = \sum_{i=1}^{a}(2b+i-l)y_i e_a^\star.$$

Thus, for the ith strand emanating from $e_a^\star$ counting from the left ($1 \leq i \leq a$), we have, upon applying the differential, that it accumulates dots along itself:



The omitted terms involve dots on strands different from the ith one we are focusing on. Now we may slide the dots freely along the ith strand, since the difference between dot-slides would be crossings among the first b or last a strands, thus annihilating either ψ_{w_0} or $e_a^\star$ at the bottom. As the coefficients in front of the diagrams add up to zero, it follows that these terms do not figure in $\partial(A)$.

A similar computation can be made about strands starting at the platform ψ_{w_0}. Summing these two types of terms, we have shown that $\partial(A)e_b = 0$, and the lemma is proven. ☐

Example 8.20. We list a few examples illustrating the necessity of using the twisted restriction functor (Definition 6.3) in Lemma 8.19.

(1) We consider the case when $a = 2$ and $b = 2$. Then there is an isomorphism $\mathfrak{E}^{(2)}Y(1^4 0^{c+d}) \cong e_2 G(0^2 1^2 0^{c+d})$ given by

$$\begin{aligned}
y_1^{l-1}y_2^{l-2}y_3^{l-3}y_4^{l-4}\psi_2\psi_3\psi_1\psi_2\psi_w e^\star_{(1^2,2)} &\mapsto e_2(y_1y_2)^{l-4}y_1\psi_w \\
y_1^{l-1}y_2^{l-2}y_3^{l-3}y_4^{l-4}\psi_3\psi_1\psi_2\psi_w e^\star_{(1^2,2)} &\mapsto e_2(y_1y_2)^{l-4}y_1^2\psi_w \\
y_1^{l-1}y_2^{l-2}y_3^{l-3}y_4^{l-4}\psi_1\psi_2\psi_w e^\star_{(1^2,2)} &\mapsto e_2(y_1y_2)^{l-4}y_1^2y_2\psi_w \\
y_1^{l-1}y_2^{l-2}y_3^{l-3}y_4^{l-4}\psi_3\psi_2\psi_w e^\star_{(1^2,2)} &\mapsto e_2(y_1y_2)^{l-4}y_1^3\psi_w \\
y_1^{l-1}y_2^{l-2}y_3^{l-3}y_4^{l-4}\psi_2\psi_w e^\star_{(1^2,2)} &\mapsto e_2(y_1y_2)^{l-4}y_1^3y_2\psi_w \\
y_1^{l-1}y_2^{l-2}y_3^{l-3}y_4^{l-4}\psi_w e^\star_{(1^2,2)} &\mapsto e_2(y_1y_2)^{l-4}y_1^3y_2^2\psi_w
\end{aligned}$$

where $w \in S_2$.

Note that the element $X = y_1^{l-1}y_2^{l-2}y_3^{l-3}y_4^{l-4}\psi_2\psi_3\psi_1\psi_2 e^\star_{(1^2,2)}$ is a generator for $\mathfrak{E}^{(2)}Y(1^4 0^{c+d})$ and that

$$\partial(X) = (l-3)Xy_1 + (l-5)Xy_2.$$

Similarly, $Z = e_2(y_1y_2)^{l-4}y_1$ generates $e_2G(0^2 1^2 0^{c+d})$ and one computes that

$$\partial(Z) = (l-3)Zy_1 + (l-5)Zy_2.$$

This confirms that the isomorphism in Lemma 8.19 is an isomorphism of p-DG modules.

(2) Another interesting special case of Lemma 8.19 occurs when $c = d = 0$. Then Specht module $e_{a+b}G(1^{a+b}) = Y(1^{a+b})$ is an indecomposable projective module. Then the restriction functor $\mathfrak{E}^{(a)}$ takes the indecomposable projective NH_l^l-module $Y(1^{a+b})$ to the indecomposable NH_b^l-module to $e_bG(0^a1^b)$.

Remark 8.21. In a similar vein as for Lemma 8.19, Lemma 8.14 can also be made into a statement about isomorphism of p-DG modules. The proof is simpler since the functors $\mathfrak{F}^{(d)}$ utilize the natural differential on the idempotent e_d (Definition 6.3).

Our next goal is to show that the indecomposable modules appearing in Theorem 8.17 are naturally contained in the p-DG envelope of the generating modules $e_\lambda G(\lambda)$, $\lambda \in \mathcal{P}_n^{r,s}$. This will be guaranteed by the next result together with Proposition 5.7.

Proposition 8.22. For any $\lambda = (0^a1^b0^c1^d) \in \mathcal{P}_n^{r,s}$, there is a p-DG isomorphism

$$e_\lambda G(\lambda) \cong \mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}).$$

Proof. By Lemma 8.19, we have a p-DG isomorphism

$$\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}) \cong e_bG(0^a1^b0^{c+d}) = e_b(y_1^{l-a-1}\cdots y_b^{l-a-b})\mathrm{NH}_b^l.$$

Now apply the induction functor $\mathfrak{F}^{(d)}$ to the above cyclic module, we have

$$\mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}) \cong e_b(y_1^{l-a-1}\cdots y_b^{l-a-b})\mathrm{NH}_b^l \otimes_{\mathrm{NH}_b^l} e_{(1^b,d)}\mathrm{NH}_{b+d}^l \cong e_{(b,d)}(y_1^{l-a-1}\cdots y_b^{l-a-b})\mathrm{NH}_{b+d}^l,$$

where we have used that the elements in NH_b^l commutes with the idempotent $e_{(1^b,d)}$, and that $e_be_{(1^b,d)} = e_{(b,d)}$. We claim that the last module cyclically generated by $e_{(b,d)}(y_1^{l-a-1}\cdots y_b^{l-a-b})$ is isomorphic to $e_\lambda G(\lambda)$, which is generated by $e_{(b,d)}y_1^{l-a-1}\cdots y_b^{l-a-b}y_{b+1}^{d-1}\cdots y_{b+d}^0$. The inclusion that

$$e_\lambda G(\lambda) \subset e_{(b,d)}(y_1^{l-a-1}\cdots y_b^{l-a-b})\mathrm{NH}_{a+b}^l$$

is clear. The converse follows from the fact that

$$e_{(b,d)}y_1^{l-a-1}\cdots y_b^{l-a-b} = e_{(b,d)}y_1^{l-a-1}\cdots y_b^{l-a-b}y_{b+1}^{d-1}\cdots y_{b+d}^0\psi_w,$$

where $w \in S_1^b \times S_d$ is the longest element of the parabolic subgroup (c.f. the proof of Lemma 8.14). The result now follows. ☐

We are now ready to verify the conditions of Section 4.4 is satisfied in our particular situation.

Theorem 8.23. The p-DG functors $\mathfrak{E}$ and $\mathfrak{F}$ restrict to the p-DG category $\oplus_{n=0}^l \mathcal{NH}_n^{r,s}$.

Proof. This follows from the classification of the indecomposable objects of the category.

Case I: $b < c$. Then $Y(0^a1^b0^c1^d) \cong \mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d})$. In this case

$$\begin{aligned}
\mathfrak{E}Y(0^a1^b0^c1^d) &\cong \mathfrak{E}\mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}) \\
&\cong [a+1]\mathfrak{F}^{(d)}\mathfrak{E}^{(a+1)}Y(1^{a+b}0^{c+d}) \oplus g_1\mathfrak{F}^{(d-1)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}) \\
&\cong [a+1]\mathfrak{F}^{(d)}Y(0^{a+1}1^{b-1}0^{c+d}) \oplus g_1\mathfrak{F}^{(d-1)}Y(0^a1^b0^{c+d}) \\
&\cong [a+1]Y(0^{a+1}1^{b-1}0^c1^d) \oplus g_1Y(0^a1^b0^{c+1}1^{d-1})
\end{aligned}$$

for some multiplicity $g_1 \in \mathbb{N}[q, q^{-1}]$.

Next we have

$$\mathfrak{F}Y(0^a1^b0^c1^d) \cong \mathfrak{F}\mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}) \cong [d+1]\mathfrak{F}^{(d+1)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{c+d}). \tag{8.7}$$

Since $b < c$ then the objects in (8.7) are isomorphic to

$$[d+1]\mathfrak{F}^{(d+1)}Y(0^a1^b0^{c+d}) \cong [d+1]Y(0^a1^b0^{c-1}1^{d+1}).$$

Case II: $b > c$. We compute

$$\begin{aligned}\mathfrak{F}Y(0^a1^b0^c1^d) &\cong \mathfrak{F}\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{c+d}) \\ &\cong [d+1]\mathfrak{E}^{(a)}\mathfrak{F}^{(d+1)}Y(1^{a+b}0^{c+d}) \oplus g_2\mathfrak{E}^{(a-1)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{c+d}) \\ &\cong [d+1]Y(0^a1^b0^{c-1}1^{d+1}) \oplus g_2Y(0^{a-1}1^{b+1}0^c1^d)\end{aligned}$$

for some multiplicity $g_2 \in \mathbb{N}[q, q^{-1}]$.

Next we have

$$\mathfrak{E}Y(0^a1^b0^c1^d) \cong \mathfrak{E}\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{c+d}) \cong [a+1]\mathfrak{E}^{(a+1)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{c+d}). \tag{8.8}$$

Since $b > c$, the objects in (8.8) are isomorphic to

$$[a+1]\mathfrak{E}^{(a+1)}Y(1^{a+b}0^c1^d) \cong [a+1]Y(0^{a+1}1^{b-1}0^c1^d).$$

Case III: $b = c$. In order to show $\mathfrak{F}Y(0^a1^b0^b1^d)$ decomposes as we would like, we proceed as in case II.

$$\begin{aligned}\mathfrak{F}Y(0^a1^b0^b1^d) &\cong \mathfrak{F}\mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{b+d}) \\ &\cong [d+1]\mathfrak{E}^{(a)}\mathfrak{F}^{(d+1)}Y(1^{a+b}0^{b+d}) \oplus g_3\mathfrak{E}^{(a-1)}\mathfrak{F}^{(d)}Y(1^{a+b}0^{b+d}) \\ &\cong [d+1]Y(0^a1^b0^{b-1}1^{d+1}) \oplus g_3Y(0^{a-1}1^{b+1}0^b1^d)\end{aligned}$$

for some multiplicity $g_3 \in \mathbb{N}[q, q^{-1}]$.

In order to show $\mathfrak{E}Y(0^a1^b0^b1^d)$ decomposes as we would like, we proceed as in case I.

$$\begin{aligned}\mathfrak{E}Y(0^a1^b0^b1^d) &\cong \mathfrak{E}\mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{b+d}) \\ &\cong [a+1]\mathfrak{F}^{(d)}\mathfrak{E}^{(a+1)}Y(1^{a+b}0^{b+d}) \oplus g_4\mathfrak{F}^{(d-1)}\mathfrak{E}^{(a)}Y(1^{a+b}0^{b+d}) \\ &\cong [a+1]\mathfrak{F}^{(d)}Y(0^{a+1}1^{b-1}0^{b+d}) \oplus g_4\mathfrak{F}^{(d-1)}Y(0^a1^b0^{b+d}) \\ &\cong [a+1]Y(0^{a+1}1^{b-1}0^b1^d) \oplus g_4Y(0^a1^b0^{b+1}1^{d-1})\end{aligned}$$

for some multiplicity $g_4 \in \mathbb{N}[q, q^{-1}]$.

Finally, the compatibility of the $\mathfrak{E}$-and-$\mathfrak{F}$ actions with the p-differential follows from the higher categorical relations (Proposition 5.7). The direct sum decomposition above will just be replaced by fantastic filtrations, whose assocaited pieces are isomorphic to the occurring summands. ☐

The following example explains the necessity of taking the idempotent truncation for the modules $G(\lambda)$ in the definition of the two-tensor algebra.

Example 8.24. Consider the nilHecke algebra NH_2^3. There are three cyclic modules in this case: $G(110)$, $G(101)$ and $G(011)$. In Example 8.5 we saw that $Y(110) = G(110)$ and $Y(101) = G(101)$. However $G(011) \cong Y(011) \oplus Y(110)$ where $Y(011) = e_2G(011)$.

In order to categorify the second highest weight space of $V_1 \otimes V_2$ we would take the category $\mathcal{NH}_2^{1,2}$. The weight space is two-dimensional and so we would like $\mathcal{NH}_2^{1,2}$ to have exactly two generating non-isomorphic indecomposable objects.

Taking $G(011)$ and $G(101)$ as the generating set for $\mathcal{NH}_2^{1,2}$ is problematic because these two objects actually contain three non-isomorphic indecomposable modules. However, taking the truncated modules $e_{(0,2)}G(011)$ and $e_{(1,1)}G(101) = G(101)$ produces only two non-isomorphic indecomposable objects $Y(011)$ and $Y(101)$.

Finally, we collect two results about the p-DG structure of the module $e_\lambda G(\lambda)$ that will be useful later. The first result shows that a special $Y(\lambda)$ is the projective indecomposable module over NH_n^l, which corresponds to the minimal partition in $\mathcal{P}_n^{r,s}$ (see Definition 8.6).

Proposition 8.25. For the minimal partition $\lambda_0 \in \mathcal{P}_n^{r,s}$, the module $Y(\lambda_0)$ is isomorphic to the indecomposable projective module of NH_n^l.

Proof. The minimal partition $\lambda_0 = (0^a 1^b 0^c 1^d)$ has either $b = 0$ or $c = 0$. Suppose $b = 0$. By Theorem 8.17 we have

$$Y(0^a 0^c 1^d) \cong \mathfrak{F}^{(d)} \mathfrak{E}^{(a)} Y(1^a 0^{c+d}).$$

Note that $\mathfrak{E}^{(a)} Y(1^a 0^{c+d})$ is a module over $\mathrm{NH}_0^l \cong \Bbbk$. By Theorem 8.17, $\mathfrak{F}^{(d)} \mathfrak{E}^{(a)} Y(1^a 0^{c+d})$ is indecomposable and hence so is $\mathfrak{E}^{(a)} Y(1^a 0^{c+d})$. Thus $\mathfrak{E}^{(a)} Y(1^a 0^{c+d}) \cong \Bbbk$. Since $\mathfrak{F}^{(d)}$ is an induction functor, $\mathfrak{F}^{(d)}\Bbbk$ is projective and we already know from Theorem 8.17 that it is indecomposable.

The case $c = 0$ is similar. One uses the fact that NH_l^l is semisimple since it is isomorphic to an $l! \times l!$ matrix algebra over $\Bbbk$. □

Next, recall from Proposition 8.10 that, if $b \leq c$, then the module $e_{(b,d)} G(0^a 1^b 0^c 1^d)$ is indecomposable. When $b \geq c$, the module has a filtered p-DG structure as follows.

Proposition 8.26. Let $b \geq c$. Then $e_{(b,d)} G(0^a 1^b 0^c 1^d) \cong \mathfrak{F}^{(d)} \mathfrak{E}^{(a)} Y(1^{a+b} 0^{c+d})$ has a filtration with subquotients isomorphic to $Y(0^{a-j} 1^{b+j} 0^{c+j} 1^{d-j})$ for $j = 0, \ldots, \min(a, d)$. Furthermore $Y(0^{a-j} 1^{b+j} 0^{c+j} 1^{d-j})$ appears with multiplicity $\begin{bmatrix} b-c \\ j \end{bmatrix}$.

Proof. By Proposition 5.7, the functor $\mathfrak{F}^{(d)} \mathfrak{E}^{(a)}$ has a ∂-stable filtration with graded subquotients $\mathfrak{E}^{(a-j)} \mathfrak{F}^{(d-j)}$ appearing with multiplicities $\begin{bmatrix} b-c \\ j \end{bmatrix}$ since $\mathfrak{E}^{(a-j)} \mathfrak{F}^{(d-j)}$ is acting on a category of weight $\mathbb{1}_{a+b-c-d}$. □

8.4 A basic algebra

Let us consider the basic version of Definition 8.27 in this subsection. We again fix $r, s \in \mathbb{N}$ such that $r+s = l$ and assume $a, b, c, d \in \mathbb{N}$ satisfying conditions of Definition 8.6.

Recall that, by Theorem 8.17, we have a full list of indecomposable p-DG modules $Y(\lambda)$ in $\mathcal{NH}_n^{r,s}$ among the list

$$\left\{ \mathfrak{F}^{(d)} \mathfrak{E}^{(a)} Y(1^{a+b} 0^{c+d}) \quad (b \leq c), \qquad \mathfrak{E}^{(a)} \mathfrak{F}^{(d)} Y(1^{a+b} 0^{c+d}) \quad (b \geq c) \right\}$$

with the identification $\mathfrak{F}^{(d)} \mathfrak{E}^{(a)} Y(1^r 0^s) \cong \mathfrak{E}^{(a)} \mathfrak{F}^{(d)} Y(1^r 0^s)$ if $b = c$. Let us fix, once an for all, an arbitrary grading on the module $Y(1^r 0^s)$, which in turn determines a grading on each $Y(\lambda)$ via the p-DG functors $\mathcal{F}^{(d)}$ and $\mathcal{E}^{(a)}$.

Definition 8.27. Let n, l be two natural numbers, and $(r, s) \in \mathbb{N}^2$ be a decomposition of l. We define the *basic two-tensor Schur algebra* to be the graded endomorphism algebra

$$S_n^b(r, s) := \mathrm{END}_{\mathrm{NH}_n^l} \left(\bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} Y(\lambda) \right).$$

The algebra is equipped with the induced p-differentials on $Y(\lambda)$'s, and therefore is a p-DG algebra.

Theorem 8.28. Let Y_1 and Y_2 be non-isomorphic indecomposable objects of $\mathcal{NH}_n^{r,s}$ with the gradings chosen as above. Then $\mathrm{HOM}_{\mathrm{NH}_n^l}(Y_1, Y_2)$ is positively graded. Consequently, the p-DG algebra $S_n^b(r, s)$ is a positive p-DG algebra.

Proof. Let $u \in \mathbb{Z}$ be a nonzero integer and set

$$Y_1 = Y(0^a 1^b 0^c 1^d), \qquad Y_2 = Y(0^{a+u} 1^{b-u} 0^{c-u} 1^{d+u}), \qquad Y' = Y(1^r 0^s).$$

Assume $b \geq c$. The case $b \leq c$ is similar.

As in the proof of Proposition 8.15, we calculate

$$\mathrm{HOM}(Y_1, Y_2) \cong \mathrm{HOM}(\mathfrak{E}^{(a)} \mathfrak{F}^{(d)} Y', \mathfrak{E}^{(a+u)} \mathfrak{F}^{(d+u)} Y')$$

which is isomorphic to

$$\bigoplus_{j=0}^{\min(a,a+u)} \mathrm{END}(Y(1^{a+b} 0^{c-a+j-u} 1^{a+d-j+u}) M(j) \tag{8.9}$$

where

$$M(j)=\begin{bmatrix}a+b+d-c+u\\ j\end{bmatrix}\begin{bmatrix}a+d-j+u\\ d\end{bmatrix}\begin{bmatrix}a+d-j+u\\ d+u\end{bmatrix}q^{(2a+u-j)(a+b+d-c-j+u)}.$$

The endomorphism algebra in (8.9) is isomorphic to the cohomology of a Grassmannian and in particular non-negatively graded. The smallest power of q occuring in $M(j)$ is

$$\rho(j)=2j^2+(2(c-b)-4a-2u))j+((2a+u)(a+b-c+u)-au).$$

An analysis of $\rho(j)$ similar to the smallest exponent studied in the proof of Proposition 8.15 shows that $\rho(j)>0$. Note that there are two cases here: $u>0$ and $u<0$. They are handled similarly. □

Remark 8.29. We make two comments about the proof above.

(1) Alternatively, the positivity of the p-DG algebra $S_n^b(r,s)$ also follows the work of Hu-Mathas. Indeed, by [HM15, Theorem C], it is proven that the bigger quiver Schur algebra $S_n(l)$ is graded Morita equivalent to a basic algebra which is Koszul, and therefore is positively graded. On the other hand, the modules $Y(\lambda)$, $\lambda\in\mathcal{P}_n^{r,s}$ constitute a subset of the indecomposable modules for defining the basic algebra for $S_n(l)$. It follows that, with respect to an implicit grading choice, the smaller endomorphism algebra $S_n^b(r,s)$ is also positively graded.

(2) The proof given above is more explicit and exhibits a natural choice of gradings for $Y(\lambda)$'s arising from the categorical quantum $\mathfrak{sl}_2$ action. The calculations above are motivated from similar considerations appearing in [Lau10, Section 9.2].

8.5 A basis for truncated modules

We now describe a basis of $e_\lambda G(\lambda)$ inherited from the basis for $G(\lambda)$ constructed by Hu-Mathas [HM15].

Lemma 8.30. If $\lambda=(0^a1^b0^c1^d)\in\mathcal{P}_n^{r,s}$, then the module $e_\lambda G(\lambda)$ is spanned by $B_\lambda:=\bigcup_{j=0}^{\mathrm{min(a,d)}}B_j$ where

$$B_j=\left\{e_{(b,d)}\psi^*_{w_1}y_1^{u_1}\cdots y_{b+j}^{u_{b+j}}y_{b+j+1}^{v_1}\cdots y_{b+d}^{v_{d-j}}\psi_{w_2}\;\middle|\;\begin{array}{l}w_1\in S_{b+j}/S_b\times S_j,\\ r+s>u_1>\cdots>u_{b+j}>s-1,\\ s>v_1>\cdots>v_{d-j}>-1\\ w_2\in S_{b+d}\end{array}\right\}.$$

Proof. Let $\lambda=(0^a1^b0^c1^d)$. Proposition 7.26 provides a basis of $G(0^a1^b0^c1^d)$ of the form

$$\{\psi_{ht}|h\in\mathrm{Tab}^\lambda(\mu),t\in\mathrm{Tab}(\mu),\mu\in\mathcal{P}_n^l\}.$$

Recall that $h\in\mathrm{Tab}^\lambda(\mu)$ means that h is a filling of a partition μ greater than or equal to the standard filling t^λ of λ.

Let j be the number of the last d entries that we move into the first $a+b$ spots. Clearly $0\le j\le\mathrm{min(a,b)}$. This means that there will be $b+j$ entries in the first $r=a+b$ spots and $d-j$ entries in the last $s=c+d$ spots.

Assume for a moment that all of the $b+j$ entries in the first r spots are as far to the right as possible and all of the remaining $d-j$ entries in the last s spots are as far to the right as possible.

Since we are look for a spanning set of $e_\lambda G(\lambda)$, there are restrictions on a filling h of such a partition so that $e_{(b,d)}\psi_h^*$ is non-zero. Namely, w_1 must be a minimal length representative in $S_{b+j}/S_b\times S_j$.

Next, we have freedom to choose which of the first r spots contains the first $b+j$ entries. This determines the exponents $u_1,\ldots,u_{b+j}$. Similarly the remaining $d-j$ entries may occupy any of the last s spots. This determines the exponents $v_1,\ldots,v_{d-j}$.

Finally, $t\in\mathrm{Tab}(\mu)$ may be anything which corresponds to $w_2\in S_{b+d}$. □

Proposition 8.31. The spanning set B_λ in Lemma 8.30 is a basis of $e_\lambda G(\lambda)$.

Proof. Via Lemma 8.14 we know that $e_{(b,d)}G(0^a1^b0^c1^d) \cong \mathfrak{F}^{(d)}Y(0^a1^b0^{c+d})$. From Hu and Mathas we know the number of composition factors in a Jordan-Holder series for this module. We thus easily compute that it has dimension

$$\binom{a+b}{a}\binom{a+c+d}{d}(b+d)!.$$

The number of elements in the spanning set in Lemma 8.30 is

$$\sum_{j=0}^{\min(a,d)} \binom{c+d}{d-j}\binom{b+j}{j}\binom{a+b}{b+j}(b+d)!.$$

The proposition follows from

$$\binom{a+b}{a}\binom{a+c+d}{d} = \sum_{j=0}^{\min(a,d)} \binom{c+d}{c+j}\binom{b+j}{j}\binom{a+b}{j+b}. \tag{8.10}$$

Equation (8.10) is easily proved by induction on c where the base case $c = 0$

$$\binom{a+d}{a} = \sum_{j=0}^{a} \binom{d}{j}\binom{a}{j}$$

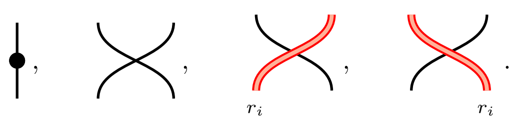
is Vandermonde's identity. □

9 p-DG Webster algebras

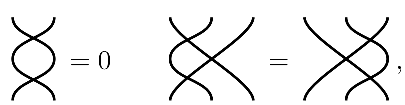
9.1 Definitions

We begin by recalling the definition of a particular Webster algebra. More general versions of these algebras, which are associated with arbitrary finite Lie types, can be found in [Web17].

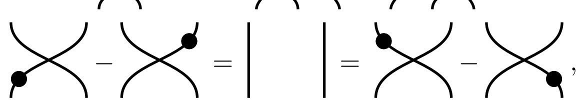
Definition 9.1. Set $\mathbf{r} = (r_1, \ldots, r_m)$ with each $r_i \in \mathbb{N}$, and let $n \in \mathbb{N}$ be in the range $0 \le n \le \sum_{i=1}^m r_i$. The Webster algebra $W_n(\mathbf{r})$ in this case is an algebra with m red strands of widths (labels) $r_1, \ldots, r_m$ (read from left to right) and n black strands. Far away generators commute with each other. The black strands are allowed to carry dots, and red strands are not allowed to cross each other. We depict the local generators of this algebra by



Multiplication in this algebra is vertical concatenation of diagrams. When the colors of the boundary points of one diagram do not match the colors of the boundary points of a second diagram, their product is taken to be zero. The relations between the local generators are given by the usual nilHecke algebra relations among black strands

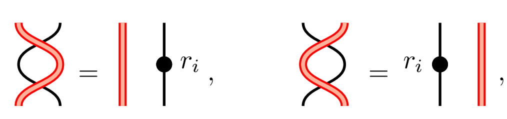


(9.1a)

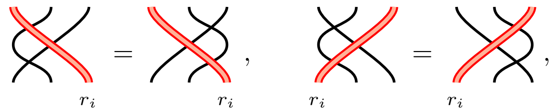


(9.1b)

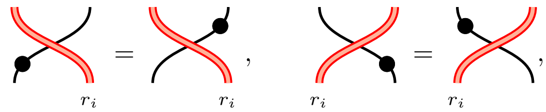
and local relations among red-black strands



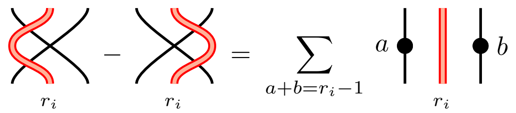
(9.1c)



(9.1d)

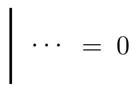


(9.1e)



(9.1f)

together with the *cyclotomic relation* that a black strand, appearing on the far left of any diagram, annihilates the entire picture:

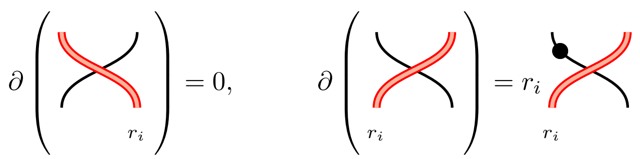


(9.1g)

When $m = 2$ and $\mathbf{r} = (r, s)$, we will write $W_n(\mathbf{r})$ as $W_n(r, s)$.

A family of differentials was introduced on $W_n(\mathbf{r})$ in [KQ15]. Among the family, a unique differential, up to conjugation by (anti)-automorphisms of $W_n(\mathbf{r})$, is determined in [QS16]. This is the differential which is compatible with the natural categorical half-quantum $\mathfrak{sl}_2$ action. We give this choice in the lemma below. The fact that ∂ is indeed a p-differential is a straightforward calculation.

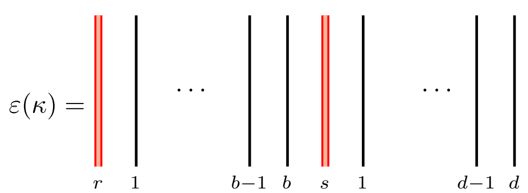
Lemma 9.2. The Webster algebra has a p-DG structure given by



(9.2)

Proof. See [QS16, Section 4.2]. □

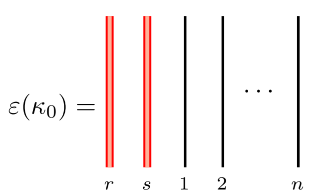
To a sequence κ of vertical lines read from left to right: red strand labeled by r, followed by b black strands, followed by red strand labeled by s, followed by d black strands, we associate an idempotent $\varepsilon(\kappa) \in W_n(r, s)$:



(9.3)

Remark 9.3. By the cyclotomic condition, $\varepsilon(\kappa) = 0$ if $b > r$. More generally, one may show that $\varepsilon(\kappa) = 0$ if $b + d > r + s$.

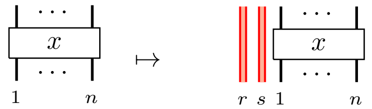
A special sequence κ_0, having all the black strands to the right of the red strands, will play an important role later:



(9.4)

Lemma 9.4. There is a p-DG isomorphism

$$\mathrm{NH}_n^l \longrightarrow \varepsilon(\kappa_0)W_n(r,s)\varepsilon(\kappa_0),$$



Proof. See [Web17, Proposition 5.31]. The compatibility with p-differentials is clear from the definitions of the differentials on both sides. □

Definition 9.5. To any sequence κ there is a projective right module $W_n(r,s)$-module

$$Q(\kappa) := \varepsilon(\kappa)W_n(r,s).$$

The projective module $Q(\kappa)$ carries a p-DG structure inherited from that of $W_n(r,s)$.

9.2 Connection to quiver Schur algebras

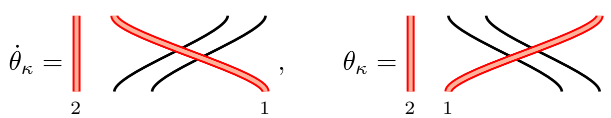
Our next goal is to give a diagrammatic description of the two-tensor quiver Schur algebras as certain blocks of Webster algebras.

Definition 9.6. Let κ be a sequence of vertical lines read from left to right: red strand labeled by r, followed by b black strands, followed by red strand labeled by s, followed by d black strands.

(1) The element $\dot{\theta}_\kappa$ is obtained from the diagram of minimal degree with no black strands intersecting which takes the sequence of black and red boundary points at the bottom of the diagram governed by κ to the sequence at the top where all red boundary points are to the left of the black boundary points.

(2) The element θ_κ is obtained by reflecting the diagram for $\dot{\theta}_\kappa$ in the horizontal axis at the bottom of the diagram.

Example 9.7. Suppose $r = 2$, $s = 1$ and κ is the sequence: red strand labeled by 2, followed by two black strands, followed by a red strand labeled by 1. Then



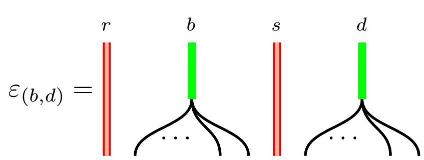
We start by constructing a collection of submodules of the p-DG Webster algebra that restricts to the direct sum of modules

$$G := \bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} e_\lambda G(\lambda)$$

over NH_n^l as p-DG modules.

Let $\lambda = (0^a 1^b 0^c 1^d) \in \mathcal{P}_n^{r,s}$ as in Definition 8.6. To such a λ we associate a sequence κ_λ of vertical lines read from left to right: red strand labeled by r, followed by b black strands, followed by red strand labeled by s, followed by d black strands.

Recall from the previous subsection that, for a multi-partition $\lambda = (0^a 1^b 0^c 1^d) \in \mathcal{P}_n^{r,s}$ we have attached to it an idempotent $\varepsilon(\kappa_\lambda)$ (equation (9.3)) and the corresponding projective module $Q(\kappa_\lambda)$. Now, for any two natural numbers b, d, we construct a thick version of the idemptent of (9.3) as follows. We first place a diagram for the thick idempotent e_b (equation (6.6b)) to the right of a red strand labeled r, then, to the right, we place a second red strand labeled by s and another idempotent e_d:



(9.5)

which then generates a (p-DG) right projective module over $W_n(r,s)$.

Definition 9.8. Given $\lambda = (0^a1^b0^c1^d) \in \mathcal{P}_n^{r,s}$, we will associate with it $\varepsilon_\lambda := \varepsilon_{(b,d)}$ and define the corresponding projective module

$$Q(\lambda) = \varepsilon_\lambda Q(\kappa_\lambda) \subset Q(\kappa_\lambda).$$

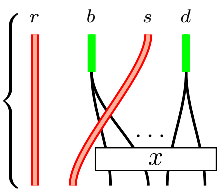
The module $Q(\lambda)$ inherits a p-DG submodule structure from $Q(\kappa_\lambda)$.

Now, for any $b, d \in \mathbb{N}$, we consider

$$\mathrm{HOM}_{W_n(r,s)}(Q(\kappa_0), Q(\lambda)) \cong \varepsilon_{(b,d)} W_n(r,s)\varepsilon(\kappa_0).$$

The right hand side is naturally a module over $\varepsilon(\kappa_0)W_n(r,s)\varepsilon(\kappa_0) \cong \mathrm{NH}_n^l$ (Lemma 9.4). Diagrammatically, the module consists of diagrams of the form

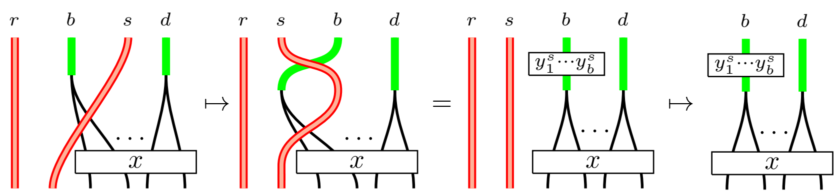
$$\varepsilon_\lambda W_n(r,s)\varepsilon(\kappa_0) \cong \left\{ \quad \middle| x \in \mathrm{NH}_n^l \right\}. \tag{9.6}$$



In particular, for $\lambda = (0^a1^b0^c1^d) \in \mathcal{P}_n^{r,s}$, we have an isomorphism of right NH_n^l-modules

$$\varepsilon_\lambda W_n(r,s)\varepsilon(\kappa_0) \cong e_\lambda G(\lambda).$$

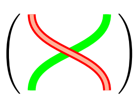
This is given as follows. We "sweep" the thickness-b strand to the right of the red strand labeled s, i.e., multiplying on top of the diagram the element $\dot{\theta}_\kappa$. Then we simplify the diagrams obtained using relation (9.1c). Finally we utilize the isomorphism of Lemma 9.4:



(9.7)

The sweeping map is always an injection [Web17, Lemma 5.25]. Furthermore, it is a p-DG homomorphism since, by Lemma 9.2, we have

$$\partial \left(\quad \right) = 0.$$



We are now ready to establish the following.

Proposition 9.9. For each $\lambda = (0^a1^b0^c1^d) \in \mathcal{P}_n^{r,s}$, there is an isomorphism of right p-DG modules over NH_n^l

$$\mathrm{HOM}_{W_n(r,s)}(Q(\kappa_0), Q(\lambda)) \cong \varepsilon_\lambda W_n(r,s)\varepsilon(\kappa_0) \cong e_\lambda G(\lambda).$$

Proof. By the discussion above, it suffices to show that the element $e_{(b,d)}(y_1 \cdots y_b)^s$ generates the same p-DG right ideal of NH_n^l as $e_\lambda y^\lambda$. It is clear that $e_\lambda G(\lambda) \subset e_{(b,d)}(y_1 \cdots y_b)^s \mathrm{NH}_n^l$. The reverse inclusion holds because

$$e_{(b,d)}(y_1 \cdots y_b)^s = e_{(b,d)} \cdot e_{(b,d)}(y_1 \cdots y_b)^s = e_{(b,d)}(y_1 \ldots y_b)^s e_{(b,d)} = e_\lambda y^\lambda \psi_{w_{(b,d)}},$$

where $w_{(b,d)}$ is the longest element in the parabolic subgroup $S_b \times S_d \subset S_n$. □

Summing over $\lambda \in \mathcal{P}_n^{r,s}$, we set

$$Q := \bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} Q(\lambda). \tag{9.8}$$

We have shown that

$$\mathrm{HOM}_{W_n(r,s)}(Q(\kappa_0), Q) \cong \bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} e_\lambda G(\lambda) = G. \tag{9.9}$$

Theorem 9.10. There is an isomorphism of p-DG algebras

$$S_n(r,s) \cong \mathrm{END}_{W_n(r,s)}(Q).$$

Proof. The result follows by applying the general results of Section 4. More specifically, observe that the NH_n^l-module $G = \oplus_{\lambda \in \mathcal{P}_n^{r,s}} e_\lambda G(\lambda)$ is a faithful representation. Indeed, by Proposition 8.25, the module $e_{(0^{l-n}1^n)} G(0^{l-n}1^n)$ contains $Y(0^{l-n}1^n)$. By Theorem 4.12, the functor

$$\mathrm{HOM}_{S_n(r,s)}(G,-) : (S_n(r,s),\partial)\text{-mod} \longrightarrow (\mathrm{NH}_n^l, \partial)\text{-mod}$$

is fully faithful on cofibrant summands of $S_n(r,s)$. By Lemma 4.11, we have

$$\mathrm{HOM}_{S_n(r,s)}(G, S_n(r,s)) \cong \mathrm{HOM}_{\mathrm{NH}_n^l}(G, \mathrm{NH}_n^l) \cong G^* \cong G$$

since G is graded self-dual (Proposition 7.30). Now the result follows from the previous Proposition identifying the space with $\mathrm{HOM}_{W_n(r,s)}(Q(\kappa_0), Q)$. □

9.3 Examples

In this subsection, we give some examples of the two-tensor quiver Schur algebras and point out a subtle difference with the p-DG (thin) Webster algebra.

Example: $s = 0$**.** We first consider this very special case of the two-tensor quiver Schur algebra, and compare the results with those in Section 6.4.

In this situation, the set $\mathcal{P}_n^{r,0} = \mathcal{P}_n^{l,0}$ consists of a unique partition $\lambda_n = (0^{l-n}1^n)$ for each $n \in \{0, \ldots, l\}$, associated with which are

$$y^{\lambda_n} = y_1^{n-1} y_2^{n-2} \cdots y_n^0, \qquad e_{\lambda_n} = y_1^{n-1} y_2^{n-2} \cdots y_n^0 \psi_{w_0} = e_n.$$

We then have, by Lemma 8.12, that

$$Y(\lambda_n) \cong e_{\lambda_n} G(\lambda_n) = e_n y_1^{n-1} y_2^{n-2} \cdots y_n^0 \mathrm{NH}_n^l.$$

As in the proof of Lemma 8.14, we see that

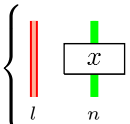
$$Y(\lambda_n) \cong e_n \mathrm{NH}_n^l \tag{9.10}$$

is in fact, up to grading shifts, the unique indecomposable projective module over NH_n^l equipped with the right ideal p-DG module structure. It follows that

$$S_n(l,0) = \mathrm{END}_{\mathrm{NH}_n^l}(Y(\lambda_n)) \cong \mathrm{H}^*(\mathrm{Gr}(n,l)), \tag{9.11}$$

with the latter ring identified with the center of NH_n^l. The diagrammatics of the two-tensor quiver Schur algebra specialize into

$$S_n(l,0) \cong \left\{ \text{(diagram)} \;\middle|\; x \in \mathrm{H}^*(\mathrm{Gr}(n,l)) \right\}. \tag{9.12}$$



Remark 9.11. Since the cohomology ring of $\mathrm{Gr}(n,l)$ is a positively graded local p-DG algebra, Theorem 3.9 implies immediately that

$$K_0(\mathcal{D}^c(S_n(l,0))) \cong \mathbb{O}_p[Y(\lambda_n)],$$

where $[Y(\lambda_n)]$ stands for the symbol of the p-DG module $Y(\lambda)$ in the Grothendieck group. One should compare this with Remark 6.6 for cyclotomic nilHecke algebras.

As we will see as in the next section (Theorem 10.3), the direct sum of the derived categories

$$\mathcal{D}^c(S(l,0)) := \bigoplus_{n=0}^{l} \mathcal{D}^c(S_n(l,0))$$

categorifies the Weyl module for $\dot{U}_{\mathbb{O}_p}$ at a prime root of unity, while $\mathcal{D}^c(\mathrm{NH}^l)$ only categorifies the submodule generated by the highest weight vector.

Example: $r = 2$ **and** $s = 1$. Consider the data $r = 2, s = 1$ and $n = 2$, which is a continuation of Example 8.24. In this case $\mathcal{P}_2^{2,1}$ contains only two elements

$$\mu = (\square, \square, \emptyset) \qquad \lambda = (\emptyset, \square, \square).$$

There is a third element (Example 7.2)

$$\gamma = (\square, \emptyset, \square) \in \mathcal{P}_2^3$$

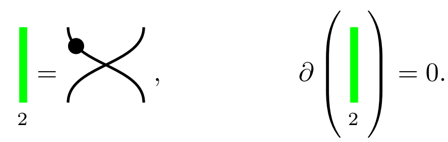
which is not in $\mathcal{P}_2^{2,1}$. The corresponding generating objects of the category $\mathcal{NH}_2^{2,1}$ are:

$$e_2 G(\mu) \cong G(\mu) \qquad e_{(1,1)} G(\lambda) = G(\lambda).$$

A basis of the endomorphism algebra of $G(\mu) \oplus G(\lambda)$ is given by

$$\begin{aligned}
\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\mu), G(\mu)) &= \{\Psi^{\mu\mu}_{ee\mu}\} \\
\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\mu), G(\lambda)) &= \{\Psi^{\lambda\mu}_{ee\mu}, \Psi^{\lambda\mu}_{s_1e\mu}\} \\
\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\lambda), G(\mu)) &= \{\Psi^{\mu\lambda}_{ee\mu}, \Psi^{\mu\lambda}_{es_1\mu}\} \\
\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\lambda), G(\lambda)) &= \{\Psi^{\lambda\lambda}_{ee\lambda}, \Psi^{\lambda\lambda}_{ee\gamma}, \Psi^{\lambda\lambda}_{ee\mu}, \Psi^{\lambda\lambda}_{s_1e\mu}, \Psi^{\lambda\lambda}_{es_1\mu}, \Psi^{\lambda\lambda}_{s_1s_1\mu}\}.
\end{aligned}$$

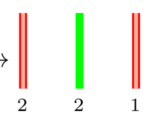
The subalgebra $\mathrm{END}_{\mathrm{NH}_2^3}(G(\mu) \oplus G(\lambda))$ is isomorphic to the subalgebra $\mathrm{END}_{W_2(2,1)}(Q(\mu) \oplus Q(\lambda))$ inside the Webster algebra via the isomorphism of Theorem 9.10. It is explicitly given as follows. Let us recall from equation (6.6b) that a thickness-2 strand represents the idempotent e_2, and with the induced endomorphism differential acting by zero:



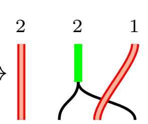
Thickness-1 strands are represented by thin ones.

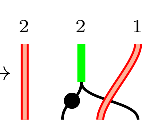
Then we have the following identifications defined on bases:

- $\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\mu), G(\mu)) \cong \mathrm{HOM}_{W_2(2,1)}(Q(\mu), Q(\mu))$:

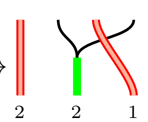
$\Psi^{\mu\mu}_{ee\mu} \mapsto$ 

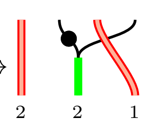
- $\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\lambda), G(\mu)) \cong \mathrm{HOM}_{W_2(2,1)}(Q(\lambda), Q(\mu))$:

$\Psi^{\mu\lambda}_{es_1\mu} \mapsto$ 

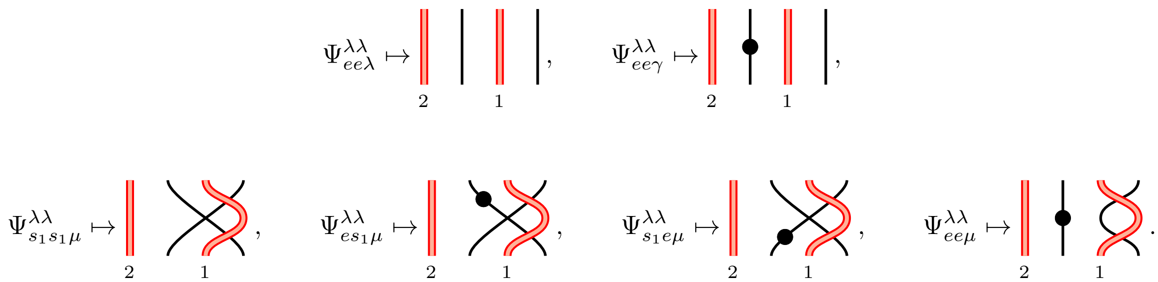
, $\Psi^{\mu\lambda}_{ee\mu} \mapsto$ 

- $\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\mu), G(\lambda)) \cong \mathrm{HOM}_{W_2(2,1)}(Q(\mu), Q(\lambda))$:

$\Psi^{\lambda\mu}_{s_1e\mu} \mapsto$ 

, $\Psi^{\lambda\mu}_{ee\mu} \mapsto$ 

- $\mathrm{HOM}_{\mathrm{NH}_2^3}(G(\lambda), G(\lambda)) \cong \mathrm{HOM}_{W_2(2,1)}(Q(\lambda), Q(\lambda))$:



It is straightforward to check that this vector space isomorphism is in fact an algebra isomorphism as well. We leave the details as an exercise to the reader.

10 A categorification of a tensor product

10.1 The main theorem

We start by collecting some easily deduced consequences of the previous sections. For convenience, we define

$$(S(r,s),\partial)\text{-mod} := \bigoplus_{n=0}^{l} (S_n(r,s),\partial)\text{-mod}, \qquad (S^b(r,s),\partial)\text{-mod} := \bigoplus_{n=0}^{l} (S^b_n(r,s),\partial)\text{-mod}. \tag{10.1}$$

Likewise, we define their p-DG derived categories as

$$\mathcal{D}(S(r,s)) := \bigoplus_{n=0}^{l} \mathcal{D}(S_n(r,s)), \qquad \mathcal{D}(S^b(r,s)) := \bigoplus_{n=0}^{l} \mathcal{D}(S^b_n(r,s)). \tag{10.2}$$

We consider a collection of cofibrant p-DG modules over $S^b_n(r,s)$ as follows. Let $\mu \in \mathcal{P}^{r,s}_n$. Then the composition of the natural projection and inclusion defines an idempotent

$$\xi_\mu : \bigoplus_{\lambda \in \mathcal{P}^{r,s}_n} Y(\lambda) \to Y(\mu) \hookrightarrow \bigoplus_{\lambda \in \mathcal{P}^{r,s}_n} Y(\lambda) \tag{10.3}$$

in the p-DG algebra $S^b_n(r,s)$. Thus we define the left $(S^b_n(r,s),\partial)$-module

$$P^b(\mu) := S^b_n(r,s)\xi_\mu \cong \mathrm{HOM}_{\mathrm{NH}^l_n}\left(Y(\mu), \bigoplus_{\lambda \in \mathcal{P}^{r,s}_n} Y(\lambda)\right). \tag{10.4}$$

It is clearly a direct summand in $S^b_n(r,s)$, and hence is a cofibrant p-DG module.

Proposition 10.1. (i) There is an action of the p-DG 2-category $(\dot{\mathcal{U}},\partial)$-mod on $(S^b(r,s),\partial)$-mod. Under localization, the action induces an action of the derived p-DG 2-category $\mathcal{D}(\dot{\mathcal{U}})$ on $\mathcal{D}(S^b(r,s))$.

(ii) On the level of Grothendieck groups, there is an isomorphism of modules over $K_0(\mathcal{D}^c(\dot{\mathcal{U}})) \cong \dot{U}_{\mathbb{O}_p}$:

$$K_0(\mathcal{D}^c(\oplus_{n=0}^{l} S^b_n(r,s))) \cong V_r \otimes_{\mathbb{O}_p} V_s$$

where V_r and V_s are the Weyl modules of rank $r+1$ and $s+1$ over $\mathbb{O}_p$ respectively.

(iii) For each $\lambda = (0^a1^b0^c1^d) \in \mathcal{P}^{r,s}_n$, the indecomposable module $P^b(\lambda)$ descends to the canonical basis element $v_b \diamond v_d$ in the Grothendieck group.

Proof. By Proposition 4.21 and Theorem 8.23, the action of $(\dot{\mathcal{U}}, \partial)$ on $\oplus_{n=0}^{l}(\mathrm{NH}_n^l, \partial)\text{-mod}$ (Theorem 6.5) extends to the category $\oplus_{n=0}^{l}(S_n^b(r,s), \partial)\text{-mod}$.

The second statement follows is a consequence of Theorem 8.28. The positivity of the p-DG algebra $S_n^b(r,s)$ allows one to compute the Grothendieck group directly from Theorem 3.9.

The third statement is a consequence of the construction of $Y(0^a1^b0^c1^d)$ as a step-by-step categorification of the canonical basis elements. Recall from Remark 2.4 that

$$v_b \diamond v_d = \begin{cases} F^{(d)}E^{(a)}(v_r \otimes v_0) & \text{if } b \leq c, \\ E^{(a)}F^{(d)}(v_r \otimes v_0) & \text{if } b \geq c. \end{cases}$$

Also recall from Theorem 8.17 that

$$Y(\lambda) = \begin{cases} \mathfrak{F}^{(d)}\mathfrak{E}^{(a)}Y(1^r0^s) & \text{if } b \leq c, \\ \mathfrak{E}^{(a)}\mathfrak{F}^{(d)}Y(1^r0^s) & \text{if } b \geq c. \end{cases}$$

Thus the map

$$K_0(\mathcal{D}^c(\oplus_{n=0}^{l}S_n^b(r,s))) \to V_r \otimes_{\mathbb{O}_p} V_s$$

sending $[P^b(\lambda)]$ to $v_b \otimes v_d$ intertwines the action of $[\mathfrak{E}]$ and $[\mathfrak{F}]$ on $K_0(\mathcal{D}^c(\oplus_{n=0}^{l}S_n^b(r,s)))$ with the action of E and F on $V_r \otimes_{\mathbb{O}_p} V_s$. □

We then extend the theorem to the full two-tensor quiver Schur algebra. This requires us to set up a chain of intermediate p-DG algebras between $S_n^b(r,s)$ and $S_n(r,s)$, and show that they are all p-DG Morita equivalent. First we make a definition.

Definition 10.2. For each $\mu = (0^a1^b0^c1^d) \in \mathcal{P}_n^{r,s}$, the p-DG module

$$P(\mu) := \mathrm{HOM}_{\mathrm{NH}_n^l}\left(Y(\mu), \bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} e_\lambda G(\lambda)\right)$$

will be referred to as the *canonical module* associated to the partition μ.

Choose an arbitrary total order "$>$" which refines the partial order of Definition 7.3. For each fixed $\mu \in \mathcal{P}_n^{r,s}$, we take the module

$$G^{\geq\mu} := \left(\bigoplus_{\lambda<\mu} Y(\lambda)\right) \bigoplus \left(\bigoplus_{\lambda\geq\mu} e_\lambda G(\lambda)\right).$$

In other words, if $\mu > \nu$ are two neighboring terms in the total order, then $G^{\geq\nu}$ is obtained from $G^{\geq\mu}$ by replacing the summand $Y(\nu) \subset G^{\geq\mu}$ by $e_\nu G(\nu)$. By Proposition 8.26, $Y(\nu)$ appears in $e_\nu G(\nu)$ as a p-DG quotient module with multiplicity one. Furthermore, each time when making this replacement, we only introduce an extra filtered p-DG submodule G' whose subquotients are grading shifts of $Y(\lambda)$ with $\lambda > \nu$. Let us write

$$e_\nu G(\nu) = Y(\nu) \bigoplus \left(\bigoplus_{\lambda>\nu} g_\lambda Y(\lambda)\right) = Y(\nu) \oplus G',$$

so that there is a filtered direct sum

$$G^{\geq\nu} \cong G' \oplus G^{\geq\mu}.$$

Here it is understood the summands are associated graded pieces of the p-DG filtration.

Now let us compare the endomorphism p-DG algebras of $G^{\geq\mu}$ and $G^{\geq\nu}$ over NH_n^l. Call the former $S^{\geq\mu}$ and the latter $S^{\geq\nu}$. We may identify $S^{\geq\nu}$ as a "block matrix" involving the $S^{\geq\mu}$ as follows (the possible external differentials between blocks are indicated on the arrows):

$$S^{\geq\nu} = \mathrm{END}(G^{\geq\nu}) \cong \begin{pmatrix} \mathrm{END}(G') & \xrightarrow{\partial} & \mathrm{HOM}(G', G^{\geq\mu}) \\ \partial\big\uparrow & & \big\uparrow\partial \\ \mathrm{HOM}(G^{\geq\mu}, G') & \xrightarrow{\partial} & S^{\geq\mu} \end{pmatrix}$$

The last column of the formal matrix is isomorphic to $\mathrm{HOM}(G^{\geq \nu}, G^{\geq \mu})$, and the last row may be identified with $\mathrm{HOM}(G^{\geq \mu}, G^{\geq \nu})$, which are respectively p-DG bimodules over $(S^{\geq \mu}, S^{\geq \nu})$ and $(S^{\geq \mu}, S^{\geq \nu})$. We are then reduced to the situation of Proposition 4.22, which inductively allows us to conclude:

Theorem 10.3. Fix two numbers $r, s \in \mathbb{N}$.

(i) There is p-DG Morita equivalence between $(S(r,s), \partial)\text{-mod}$ and $(S^b(r,s), \partial)\text{-mod}$. For each fixed $n \in \{0, 1, \ldots, n\}$, the functor is given by tensor product with the p-DG bimodule over $(S_n(r,s), S_n^b(r,s))$:

$$\mathrm{HOM}_{\mathrm{NH}_l}\left(\bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} Y(\lambda), \bigoplus_{\mu \in \mathcal{P}_n^{r,s}} G(\mu)\right) \otimes_{S_n^b(r,s)} (-) : (S_n^b(r,s), \partial)\text{-mod} \longrightarrow (S_n(r,s), \partial)\text{-mod}.$$

Summing over n, the equivalences induce derived equivalence between $\mathcal{D}(S(r,s))$ and $\mathcal{D}(S^b(r,s))$.

(ii) The p-DG 2-category $(\dot{\mathcal{U}}, \partial)$-mod acts on $(S(r,s), \partial)\text{-mod}$, inducing an action of the derived p-DG 2-category $\mathcal{D}(\dot{\mathcal{U}})$ on $\mathcal{D}(S(r,s))$. The derived action categorifies the action of $\dot{U}_{\mathbb{O}_p}$ on the tensor product representation $V_r \otimes_{\mathbb{O}_p} V_s$.

(iii) For each $\mu = (0^a 1^b 0^c 1^d) \in \mathcal{P}_n^{r,s}$, the canonical module $P(\mu)$ descends in the Grothendieck group $K_0(S_n(r,s))$ to the canonical basis element $v_b \diamond v_d$.

Proof. The first statement now follows easily by an induction on the $\lambda \in \mathcal{P}_n^{r,s}$ via Proposition 4.22 with respect to the total ordering chosen above. The induction also shows that the $(S_n(r,s), S_n^b(r,s))$-bimodule

$$\mathrm{HOM}_{\mathrm{NH}_l}\left(\bigoplus_{\lambda \in \mathcal{P}_n^{r,s}} Y(\lambda), \bigoplus_{\mu \in \mathcal{P}_n^{r,s}} G(\mu)\right)$$

is cofibrant as both a left and right p-DG module. Thus the derived tensor functor is equal to the underived tensor product.

The categorical $\dot{U}_{\mathbb{O}_p}$-action extension result follows, similar as in the proof of the previous proposition, by applying Proposition 4.21 to the current situation.

Finally, the above functor, when applied to the cofibrant module $P^b(\mu)$ for a fixed $\mu \in \mathcal{P}_n^{r,s}$, truncates the factor $\oplus_\lambda Y(\lambda)$ in the definition of the bimodule through the idempotent ξ_μ, i.e., it replaces $\oplus_\lambda Y(\lambda)$ just by $Y(\mu)$. The claim follows. ☐

10.2 Stratified structure

The two-tensor quiver Schur algebra is naturally equipped with a stratified structure (see, for instance, [Kle15]) that is compatible with the differential. We sketch the construction here, starting with the basic algebra case. Recall that, for $\lambda \in \mathcal{P}_n^{r,s}$, we have associated with the cofibrant p-DG module over $S_n^b(r,s)$:

$$P^b(\lambda) := \mathrm{HOM}_{\mathrm{NH}_n^l}\left(Y(\lambda), \bigoplus_{\mu \in \mathcal{P}_n^{r,s}} Y(\mu)\right).$$

Definition 10.4. Let $\lambda \in \mathcal{P}_n^{r,s}$. Define the standard left $S_n^b(r,s)$-module $\Delta^b(\lambda) = P^b(\lambda)/P^{>\lambda}$ where $P^{>\lambda}$ is the left submodule generated by

$$\mathrm{HOM}_{\mathrm{NH}_n^l}\left(Y(\lambda), \bigoplus_{\substack{\mu \in \mathcal{P}_n^{r,s} \\ \mu > \lambda}} Y(\mu)\right).$$

We would like to construct a filtration on each projective module $P^b(\lambda)$ which has subquotients of the form $\Delta^b(\gamma)$ for $\gamma \geq \lambda$.

Definition 10.5. (1) Let $\mathrm{HOM}^{\geq\gamma}_{\mathrm{NH}^l_n}(Y(\lambda), Y(\mu))$ be the left submodule of $P^b(\lambda)$ generated by all maps in $\mathrm{HOM}_{\mathrm{NH}^l_n}(Y(\lambda), Y(\mu))$ which factor through $Y(\gamma')$ for some $\gamma' \geq \gamma$. Then define the submodule $P^{\geq\gamma}(\lambda) \subset P^b(\lambda)$ to be the left submodule generated by all maps factoring through $Y(\gamma')$ for some $\gamma' \geq \gamma$. That is,

$$P^{\geq\gamma}(\lambda) = \mathrm{HOM}^{\geq\gamma}_{\mathrm{NH}^l_n}\left(Y(\lambda), \bigoplus_{\substack{\mu\in\mathcal{P}^{r,s}_n \\ \mu>\lambda}} Y(\mu) \right).$$

(2) Let $\mathrm{HOM}^{>\gamma}_{\mathrm{NH}^l_n}(Y(\lambda), Y(\mu))$ be the left submodule of $P(\lambda)$ generated by all maps in $\mathrm{HOM}_{\mathrm{NH}^l_n}(Y(\lambda), Y(\mu))$ which factor through $Y(\gamma')$ for some $\gamma' > \gamma$. Then define the submodule $P^{\geq\gamma}(\lambda) \subset P^b(\lambda)$ to be the left submodule generated by all maps factoring through $Y(\gamma')$ for some $\gamma' > \gamma$. That is,

$$P^{>\gamma}(\lambda) = \mathrm{HOM}^{>\gamma}_{\mathrm{NH}^l_n}\left(Y(\lambda), \bigoplus_{\substack{\mu\in\mathcal{P}^{r,s}_n \\ \mu>\lambda}} Y(\mu) \right).$$

Lemma 10.6. The submodules $P^{\geq\gamma}(\lambda)$ and $P^{>\gamma}(\lambda)$ of $P^b(\lambda)$ are stable under ∂.

Proof. If $\phi \in P^{\geq\gamma}(\lambda)$ then $\phi = \phi_2 \circ \phi_1$ with $\phi_1 \colon Y(\lambda) \to Y(\gamma')$ and $\phi_2 \colon Y(\gamma') \to Y(\mu)$ for some μ and some $\gamma' \geq \gamma$. By the definition of ∂ on ϕ_1 and ϕ_2, $\partial(\phi_1) \colon Y(\lambda) \to Y(\gamma')$ and $\partial(\phi_2) \colon Y(\gamma') \to Y(\mu)$. Thus $\partial(\phi) = \partial(\phi_2)\phi_1 + \phi_2\partial(\phi_1)$ factors through $Y(\gamma')$.

The proof that $P^{>\gamma}(\lambda)$ is a p-DG submodule is the same. □

Corollary 10.7. The standard module $\Delta(\lambda)$ is a p-DG module over $S^b_n(r,s)$.

Proof. The p-DG structure is determined by the p-DG structure on the $Y(\mu)$ and $Y(\lambda)$ using the formula $\partial f(x) = \partial(f(x)) - f(\partial(x))$. The fact that $P(>\lambda)$ is stable under ∂ follows from the proof of Lemma 10.6. □

The cofibrant module $P^b(\lambda)$ should have a filtration $P^{\geq\lambda}$ with subquotients $P^{\geq\gamma}(\lambda)/P^{>\gamma}(\lambda) \cong \Delta(\gamma)^{f(\gamma)}$ for some graded multiplicities $f(\gamma)$.

The formula for $f(\gamma)$ should be determined by the coefficient of a standard basis in a canonical basis element $v_b \diamond v_d$ from Proposition 2.3 where $\lambda = (0^a 1^b 0^c 1^d)$.

We provide a couple of examples illustrating this.

Example 10.8. Consider $S^b_1(2,1)$. In this case $\mathrm{NH}^3_1 = \Bbbk[y]/y^3$ and there are two elements of $\mathcal{P}^{2,1}_1$:

$$\mu = (\,\emptyset\,,\,\square\,,\,\emptyset\,) \qquad\qquad \lambda = (\,\emptyset\,,\,\emptyset\,,\,\square\,).$$

Then we have

$$Y(\mu) = y\Bbbk[y]/y^3 \qquad\qquad Y(\lambda) = \Bbbk[y]/y^3.$$

$$\Delta(\mu) = P^b(\mu) = \mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\mu), Y(\lambda)) \oplus \mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\mu), Y(\mu))$$

with

$$\mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\mu), Y(\lambda)) = \Bbbk\langle y \mapsto y, y \mapsto y^2\rangle$$
$$\mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\mu), Y(\mu)) = \Bbbk\langle y \mapsto y, y \mapsto y^2\rangle.$$

$$P^b(\lambda) = \mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\lambda), Y(\lambda)) \oplus \mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\lambda), Y(\mu))$$

with

$$\mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\lambda), Y(\lambda)) = \Bbbk\langle 1 \mapsto 1, 1 \mapsto y, 1 \mapsto y^2\rangle$$
$$\mathrm{HOM}_{\Bbbk[y]/y^3}(Y(\lambda), Y(\mu)) = \Bbbk\langle 1 \mapsto y, 1 \mapsto y^2\rangle.$$

All of the maps in $P^b(\lambda)$ factor through $Y(\mu)$ except for the identity map of $Y(\lambda)$. Thus $P^{\geq\mu}(\lambda) \cong \Delta(\mu)$ and $P^b(\lambda)/P^{\geq\mu}(\lambda) \cong \Delta(\lambda)$ where $\Delta(\lambda)$ is a simple one-dimensional module.

Note that $S_1^b(2,1) \cong ((2)+(3))A_3^!((2)+(3))$.

Example 10.9. Consider $S_2^b(2,1)$. In this case there are two elements of $\mathcal{P}_2^{2,1}$:

$$\mu = (\,\square\,,\,\square\,,\,\emptyset\,) \qquad\qquad \lambda = (\,\emptyset\,,\,\square\,,\,\square\,).$$

One easily checks that

$$Y(\mu) = y_1^2 y_2 \mathrm{NH}_2^3 \qquad\qquad Y(\lambda) = y_1\psi_1\mathrm{NH}_2^3.$$

$$\Delta(\mu) = P^b(\mu) = \mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\mu),Y(\lambda)) \oplus \mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\mu),Y(\mu))$$

with

$$\mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\mu),Y(\lambda)) = \Bbbk\langle y_1^2y_2 \mapsto y_1^2y_2\psi_1y_1\rangle$$
$$\mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\mu),Y(\mu)) = \Bbbk\langle y_1^2y_2 \mapsto y_1^2y_2\rangle.$$

$$P^b(\lambda) = \mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\lambda),Y(\lambda)) \oplus \mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\lambda),Y(\mu))$$

with

$$\mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\lambda),Y(\lambda)) = \Bbbk\langle y_1\psi_1 \mapsto y_1^2y_2\psi_1,\, y_1\psi_1 \mapsto y_1\psi_1y_1^2\psi_1,\, y_1\psi_1 \mapsto y_1\psi_1\rangle$$
$$\mathrm{HOM}_{\mathrm{NH}_2^3}(Y(\lambda),Y(\mu)) = \Bbbk\langle y_1\psi_1 \mapsto y_1^2y_2\psi_1\rangle.$$

Notice that only the maps in $P^b(\lambda)$ that factor through $Y(\mu)$ are

$$y_1\psi_1 \mapsto y_1^2y_2\psi_1 \qquad\qquad y_1^2y_2 \mapsto y_1^2y_2\psi_1y_1.$$

This submodule is isomorphic to $\Delta(\mu)$ and the quotient of $P^b(\lambda)$ by this submodule is isomorphic to $\Delta(\lambda)$. If $S_2^b(2,1)$ were cellular then $\Delta(\lambda)$ would be simple (which it is not).

Note that $S_2^b(2,1) \cong ((1)+(3))A_3^!((1)+(3))$.

Remark 10.10. (1) Definition 10.5 is equivalent, in the language of [Kle15], to the following description on the algebra $S_n^b(r,s)$ itself. Pick an arbitrary total order refining the dominance order (Definition 7.3) on $\mathcal{P}_n^{r,s}$, then there is a chain of two-sided ideals in $S_n^b(r,s)$ defined as

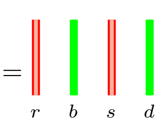
$$I_{\geq\lambda}^b := \sum_{\mu\geq\lambda} S_n^b(r,s)\xi_\mu S_n^b(r,s) \qquad\qquad (\text{resp. } I_{>\lambda}^b := \sum_{\mu>\lambda} S_n^b(r,s)\xi_\mu S_n^b(r,s)).$$

Since $\partial(\xi_\mu)=0$, it is clear that the chain is ∂-stable, which induces a differential on the quotient algebra. Then one can show that $I_{\geq\lambda}^b/I_{>\lambda}^b$ is a p-DG matrix algebra with coefficients in $\mathrm{END}_{S_n^b(r,s)}(\Delta^b(\lambda))$.

(2) One can also make this description on the two-tensor quiver Schur algebra itself by replacing the idempotent ξ_μ above with the idempotent $\varepsilon_\mu \in S_n(r,s)$:

$$\varepsilon_\mu : \bigoplus_{\lambda\in\mathcal{P}_n^{r,s}} e_\lambda G(\lambda) \to e_\mu G(\mu) \hookrightarrow \bigoplus_{\lambda\in\mathcal{P}_n^{r,s}} e_\lambda G(\lambda).$$

Using the Webster diagrammatics of Theorem 9.10, the idempotent translates into ($\mu = (0^a1^b0^c1^d)$),

$\varepsilon_\mu =$  ,

while the corresponding chain of ideals in $S_n(r,s)$ is taken to be

$$I_{\geq\mu} := \sum_{\lambda\geq\mu} S_n(r,s)\varepsilon_\lambda S_n(r,s) \qquad\qquad (\text{resp. } I_{>\lambda} := \sum_{\mu>\lambda} S_n(r,s)\varepsilon_\mu S_n(r,s)).$$

The p-DG stratified structure will play an important role in subsequent works.

10.3 Future directions

As a conclusion, we formulate a conjecture for categorifying a general m-fold tensor product $V_{r_1} \otimes_{\mathbb{O}_p} V_{r_2} \otimes_{\mathbb{O}_p} \cdots \otimes_{\mathbb{O}_p} V_{r_m}$, where each V_{r_i} is the rank-(r_i+1) $\mathbb{O}_p$-integral Weyl module over $\dot{U}_{\mathbb{O}_p}$.

To do this, we first generalize Definition 8.6. Set $\mathbf{r} = (r_1, r_2, \dots, r_m)$ and write $l = r_1 + r_2 + \cdots + r_m$.

Definition 10.11. Let $\mathcal{P}_n^{\mathbf{r}}$ be the subset of all partitions $\lambda \in \mathcal{P}_n^l$ of the form $\lambda = (0^{a_1}1^{b_1}0^{a_2}1^{b_2}\dots 0^{a_m}1^{b_m})$ satisfying

$$a_i + b_i = r_i, \quad (i = 1, 2, \dots, m) \qquad \text{and} \qquad \sum_{i=1}^{m} b_i = n.$$

We think of such a sequence also as a partition

$$\lambda = (\underbrace{\emptyset, \dots, \emptyset}_{a_1}, \underbrace{\square, \dots, \square}_{b_1} \mid \underbrace{\emptyset, \dots, \emptyset}_{a_2}, \underbrace{\square, \dots, \square}_{b_2} \mid \cdots \mid \underbrace{\emptyset, \dots, \emptyset}_{a_m}, \underbrace{\square, \dots, \square}_{b_m}).$$

Note that the minimal partition $\lambda_0 \in \mathcal{P}_n^l$ (Definition 7.3) always belongs to $\mathcal{P}_n^{\mathbf{r}}$.

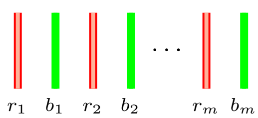
To any partition $\lambda \in \mathcal{P}_n^{\mathbf{r}}$ we have associated a thick idempotent in NH_n^l as in equation (6.8)

$$e_\lambda = e_{b_1} \otimes e_{b_2} \otimes \cdots \otimes e_{b_m}.$$

We can then construct the right NH_n^l-module $e_\lambda G(\lambda)$ as the idemptoent truncation of $G(\lambda)$ in Definition 7.24 by the idemptotent e_λ. It is a right p-DG module by Proposition 7.31.

Likewise, for the Webster algebra $W_n(\mathbf{r})$ (Definition 9.1), we associate the idempotent

$$\varepsilon_\lambda =$$



to each $\lambda \in \mathcal{P}_n^{\mathbf{r}}$, and the corresponding projective module $Q(\lambda) := \varepsilon_\lambda \cdot W_n(\mathbf{r})$. It is clear that $Q(\lambda)$ is a right p-DG module over $W_n(\mathbf{r})$.

As in Proposition 8.25, the minimal partition λ_0 gives rise to $e_{\lambda_0}G(\lambda_0)$, which is a projective module of NH_n^l. This implies that the action of NH_n^l on $\oplus_{\lambda \in \mathcal{P}_n^{\mathbf{r}}} e_\lambda G(\lambda)$ is faithful.

Definition 10.12. For any $n \in \{0, 1, \dots, l\}$, the p-DG *m-tensor quiver Schur algebra* is

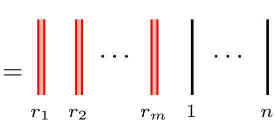
$$S_n(\mathbf{r}) := \mathrm{END}_{\mathrm{NH}_n^l}\left(\bigoplus_{\lambda \in \mathcal{P}_n^{\mathbf{r}}} e_\lambda G(\lambda) \right),$$

equipped with the natural p-differential as endomorphism algebra of a p-DG module.

The faithfulness of the NH_n^l representation exhibits the commuting $S_n(\mathbf{r})$ and NH_n^l actions on $\oplus_{\lambda \in \mathcal{P}_n^{\mathbf{r}}} e_\lambda G(\lambda)$ as double commutants, so that the framework of Section 4 applies.

Similar to what we have seen in Section 9.2, the collection of modules $\oplus_{\lambda \in \mathcal{P}_n^{\mathbf{r}}} e_\lambda G(\lambda)$ may be reconstructed using the Webster idempotent

$$\varepsilon(\kappa_0) =$$



as

$$\bigoplus_{\lambda \in \mathcal{P}_n^{\mathbf{r}}} e_\lambda G(\lambda) \cong \mathrm{HOM}_{W_n(\mathbf{r})}\left(Q(\kappa_0), \bigoplus_{\lambda \in \mathcal{P}_n^{\mathbf{r}}} Q(\lambda) \right), \tag{10.5}$$

where $Q(\kappa_0) := \varepsilon(\kappa_0) \cdot W_n(\mathbf{r})$. Parallel to Theorem 9.10, we have the following.

Theorem 10.13. There is an isomorphism of p-DG algebras

$$\mathrm{END}_{\mathrm{NH}_n^l}\left(\bigoplus_{\lambda\in\mathcal{P}_n^{\mathbf{r}}} e_\lambda G(\lambda)\right) \cong \mathrm{END}_{W_{\mathbf{r}}}\left(\bigoplus_{\lambda\in\mathcal{P}_n^{\mathbf{r}}} Q(\lambda)\right).$$

Proof. The proof is almost identical to that of Theorem 9.10. We leave the details to the reader. □

The theorem gives rise to a diagrammatic description of the m-tensor quiver Schur algebra in terms of Webster diagrammatics.

By the framework in Section 4, there exists a categorical action of $(\dot{\mathcal{U}},\partial)$ on the p-DG category

$$(S(\mathbf{r}),\partial)\text{-mod} := \bigoplus_{n=0}^{l}(S_n(\mathbf{r}),\partial)\text{-mod}$$

induced from that on $\oplus_{n=0}^{l}(\mathrm{NH}_n^l,\partial)\text{-mod}$. We propose the following.

Conjecture 10.14. The compact derived category $\mathcal{D}^c(S_n(\mathbf{r}))$ categorifies the weight $l-2n$ subspace in the m-fold tensor product representation $V_{r_1}\otimes_{\mathbb{O}_p} V_{r_2}\otimes_{\mathbb{O}_p}\cdots\otimes_{\mathbb{O}_p} V_{r_m}$ of $\dot{U}_{\mathbb{O}_p}(\mathfrak{sl}_2)$.

So far, we have seen that the conjecture holds in the case when $m=1$ (see the first example of Section 9.3) and $m=2$ (Theorem 10.3).

In a sequel to this work, we will prove the conjecture for the extreme case when $m=l$, which corresponds to categorifying $V_1^{\otimes m}$. We expect that the stratified (p-DG) structure of the previous subsection generalizes to the m-tensor case, and will play an important role towards proving the conjecture. In particular, for the case of categorifying $V_1^{\otimes m}$, the stratified p-DG structure upgrades into a quasi-hereditary p-DG structure. We will develop further the related machinery in a sequel to this work, which will enable us to identify the Grothendieck groups of $\mathcal{D}^c(S(\mathbf{r}))$ explicitly.

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