

# Independent arithmetic progressions

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## Abstract

We show that there is a positive constant  $c$  such that any graph on vertex set  $[n]$  with at most  $cn^2/k^2 \log k$  edges contains an independent set of order  $k$  whose vertices form an arithmetic progression. We also present applications of this result to several questions in Ramsey theory.

A classical theorem of Turán [9] shows that any graph on  $n$  vertices with less than  $\frac{n(n-k+1)}{2(k-1)}$  edges contains an independent set of order  $k$ . The celebrated Szemerédi’s theorem [8] states that for  $\delta > 0$ ,  $k \in \mathbb{N}$ , and  $n$  sufficiently large in terms of  $k$  and  $\delta$ , any subset of  $[n] = \{1, \dots, n\}$  of order at least  $\delta n$  contains a  $k$ -term arithmetic progression. Here we marry the themes of these results and deduce as consequences bounds on three other well-studied problems on rainbow arithmetic progressions and set mappings.

Given a graph with vertex set  $[n]$ , a  $k$ -term arithmetic progression is said to be *independent* if it is an independent set in the graph. Our main result is a Turán-type theorem, showing that any sparse graph on vertex set  $[n]$  contains an independent arithmetic progression. Before proving this result, we need a standard estimate from number theory. Note that all logs will be taken to base  $e$ .

**Lemma 1** *There is a positive constant  $\eta$  such that, for all  $n \geq \eta^{-1}k \log k$ , the number of integers from  $[n]$  which are relatively prime to  $1, 2, \dots, k$  is at least  $\eta n / \log k$ .*

**Proof.** Writing  $\Phi(x, y)$  for the number of integers less than or equal to  $x$  all of whose prime factors are greater than  $y$ , a result of Buchstab (see Section 7.2 of [7]) says that

$$\Phi(x, y) = \frac{w(u)x}{\log y} - \frac{y}{\log y} + O\left(\frac{x}{\log^2 x}\right),$$

where  $u$  is defined by  $y = x^{1/u}$  and  $w(u)$  is the Buchstab function, equal to  $1/u$  for  $1 < u \leq 2$  and asymptotic to  $e^{-\gamma}$ , with  $\gamma$  the Euler–Mascheroni constant, as  $u$  tends to infinity. For  $k$  sufficiently large, say  $k \geq k_0$ , and  $n \geq k \log k$ , the required estimate with  $\eta = 1/10$  easily follows by applying this result with  $x = n$  and  $y = k$ . For  $k < k_0$ , the estimate follows by choosing  $\eta$  such that  $\eta^{-1} \geq \max(20 \log k_0, k_0 \log k_0)$ . Then  $n \geq k_0 \log k_0$ , so that  $\Phi(n, k) \geq \Phi(n, k_0) \geq n/10 \log k_0 \geq \eta n / \log k$ .  $\square$

Our main result, which is tight up to the logarithmic factor, is now as follows.

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**Theorem 2** *There is a positive constant  $\varepsilon$  such that any graph  $G$  on  $[n]$  with less than  $\varepsilon \frac{n^2}{k^2 \log k}$  edges contains a  $k$ -term independent arithmetic progression.*

**Proof.** We split into two cases, depending on the size of  $n$ . For  $n \geq 2\eta^{-1}k^2 \log k$ , where  $\eta$  is as in Lemma 1, we consider the set of integers  $X$  which are relatively prime to  $1, 2, \dots, k$  and let  $\mathcal{A}$  be the set of  $k$ -term arithmetic progressions in  $[n]$  whose difference is in  $X$ . We can form an arithmetic progression in  $\mathcal{A}$  by choosing the first term from  $[n/2]$  and the common difference from  $X \cap [n/2k]$ . Therefore, since  $n/2k \geq \eta^{-1}k \log k$ , Lemma 1 applies to show that  $|\mathcal{A}| \geq \eta n^2 / 4k \log k$ . Each pair of integers are in arithmetic progressions with at most one common difference in  $X$  and, hence, are in at most  $k - 1$  arithmetic progressions in  $\mathcal{A}$ . Thus, the number of arithmetic progressions in  $\mathcal{A}$  which contain an edge of  $G$  is at most  $e(G)k$ . Taking  $\varepsilon < \eta/8$ , we have that  $e(G)k < \varepsilon \frac{n^2}{k \log k} < |\mathcal{A}|$ , so there is an arithmetic progression in  $\mathcal{A}$  which forms an independent set.

For the second case, when  $n < 2\eta^{-1}k^2 \log k$ , we let  $\mathcal{B}$  be the set of  $k$ -term arithmetic progressions in  $[n]$  whose difference is a prime. By the same argument as in the previous case, the number of arithmetic progressions in  $\mathcal{B}$  which contain an edge of  $G$  is at most  $e(G)k < \varepsilon \frac{n^2}{k \log k}$ . On the other hand, the number of progressions in  $\mathcal{B}$  is at least  $\pi(n/2k)n/2$ , where  $\pi(x)$  is the prime counting function. Since there exist positive constants  $a$  and  $C$  such that  $\pi(x) > a \frac{x}{\log x}$  and  $2\eta^{-1}k^2 \log k < k^C$ , we have that  $\pi(n/2k)n/2 > \frac{a}{2C} \frac{n^2}{k \log k}$ . Therefore, for  $\varepsilon < a/2C$ , there is an independent arithmetic progression.  $\square$

In a coloring of  $[n]$ , an arithmetic progression is *rainbow* if its elements are all different colors. The *sub-Ramsey number*  $sr(m, k)$  is the minimum  $n$  such that every coloring of  $[n]$  in which no color is used more than  $m$  times has a rainbow  $k$ -term arithmetic progression. Alon, Caro, and Tuza [1] proved that there are constants  $c, c' > 0$  such that

$$c' \frac{mk^2}{\log mk} \leq sr(m, k) \leq cmk^2 \log(mk).$$

They also showed that there is an upper bound on  $sr(m, k)$  which is linear in  $m$  but with a worse dependence on  $k$ , namely,  $sr(m, k) \leq cmk^3$ . The lower bound was later improved by Fox, Jungić, and Radoičić [3] to  $sr(m, k) \geq c'mk^2$ . Here we improve on the upper bounds of Alon, Caro, and Tuza [1].

**Corollary 3** *There is a constant  $c$  such that the sub-Ramsey number satisfies*

$$sr(m, k) \leq cmk^2 \log k.$$

**Proof.** Consider a coloring of  $[n]$  with  $n = \varepsilon^{-1}mk^2 \log k$ , with  $\varepsilon$  as in Theorem 2, where no color appears more than  $m$  times. Define a graph on  $[n]$  where two integers are adjacent if they receive the same color. The graph consists of a disjoint union of cliques of order at most  $m$ . Since the maximum of  $\sum_i \binom{x_i}{2}$  under the constraint  $\sum_i x_i$  occurs when each term is as large as possible, the number of edges in this graph is at most  $\frac{n}{m} \binom{m}{2} < \frac{nm}{2}$ . Therefore, by our choice of  $n$ , the number of edges is such that Theorem 2 applies to give an independent  $k$ -term arithmetic progression, which is a rainbow arithmetic progression in our coloring of  $[n]$ .  $\square$

Let  $T_k$  denote the smallest positive integer  $t$  such that for every positive integer  $m$ , every equinumerous  $t$ -coloring of  $[tm]$  contains a rainbow  $k$ -term arithmetic progression. Jungić et al. [5] proved that there are positive constants  $c, c'$  such that

$$c'k^2 \leq T_k \leq ck^3.$$

They conjectured that the lower bound is correct, that is,  $T_k = \Theta(k^2)$ , a problem which was reiterated in the survey [6]. Here we make progress on this conjecture, improving the upper bound to  $ck^2 \log k$ . Note that an equinumerous  $t$ -coloring of  $[tm]$  uses each color exactly  $m$  times, so  $T_k$  is at most the maximum of  $sr(m, k)/m$  over all positive integers  $m$ . Hence, by Corollary 3, we obtain the following corollary.

**Corollary 4** *There is a constant  $c$  such that*

$$T_k \leq ck^2 \log k.$$

Motivated by the set mapping problem of Erdős and Hajnal, Caro [2] proved that for every positive integer  $k$ , there is a minimum integer  $n_0 = n_0(k)$  such that, for all  $n \geq n_0$  and every permutation  $\pi : [n] \rightarrow [n]$ , there is a  $k$ -term arithmetic progression  $A$  such that  $\pi(i) \notin A$  for all  $i \in A$ . Moreover, he showed that there are constants  $c, c' > 0$  such that  $c'k^2/\log k \leq n_0(k) \leq k^2 2^{c \log k / \log \log k}$ . Alon et al. [1] used the same methods they had used to bound  $sr(m, k)$  to improve the earlier upper bound to  $n_0(k) \leq ck^2 \log k$ . Our result gives a simple alternative proof of this.

**Corollary 5** *There is a constant  $c$  such that*

$$n(k) \leq ck^2 \log k.$$

**Proof.** Consider the graph on  $[n]$  with edges  $(i, \pi(i))$  for  $i \in [n]$ . This graph has at most  $n$  edges. By choosing  $c$  large enough, we can make the number of edges such that Theorem 2 applies to give an independent arithmetic progression in this graph. This arithmetic progression has the required property.  $\square$

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