

A Coupled Mode Description of the Backward-Wave Oscillator and the Kompfner Dip Condition*

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Summary—The start oscillation condition for the backward-wave oscillator and the operation of the traveling-wave tube amplifier at the Kompfner dip point are described from the point of view of the coupling of two modes of propagation. Growing waves are not involved. Two waves are sufficient when the tube is more than a half plasma wavelength long. Operation in this "large space charge" domain is inherently simpler than in the "low space charge" domain. The start oscillation condition and the Kompfner dip condition are simply expressed in terms of the coupling constant between modes, $\mathcal{K}L = (2n + 1)\pi/2$, where n is an integer. In addition, the uncoupled modes must have the same velocity. The result is also expressed in terms of the more familiar parameters CN and hL . The effect of loss in the circuit mode is calculated.

When the two waves carry energy in opposite directions, growing waves result. This case is discussed briefly.

I. INTRODUCTION

IN TWO RECENT papers^{1,2} Pierce has suggested that the operation of beam-type amplifiers and oscillators can be understood in terms of a coupling of the modes of propagation, or waves, which exist on the electron beam in the absence of the circuit with a mode which exists on the circuit in the absence of the electron beam. It is well known that a cylindrical electron beam supports two space charge waves, one with a phase velocity less than the electron velocity and one with a phase velocity greater than the electron velocity. When the electron beam is surrounded by an electromagnetic circuit, such as a helix, which supports a wave whose velocity is nearly equal to the phase velocity of the space charge waves, the circuit wave and the space charge waves interact, strongly modifying the original waves so as to produce three waves which are, in a sense, combinations of the original waves. Each of the modified waves has some of the attributes of the original waves if the coupling is appreciable. The modified circuit wave has associated with it a modulation of the electron beam while the modified space charge waves have associated with them circuit fields. In many microwave tubes the space charge fields are stronger than the circuit fields so that the velocities of the two space charge waves are sufficiently different that the circuit wave couples strongly with one of the space charge waves and only weakly with the other. The analysis of the operation of the tube is simpler in this large space charge case because one of the waves can be neglected if it is not excited at the beginning of the tube.

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¹ J. R. Pierce, "Coupling of modes of propagation," *Jour. Appl. Phys.*, vol. 25, pp. 179-183; February, 1954.

² J. R. Pierce, "The wave picture of microwave tubes," *Bell Sys. Tech. Jour.*, vol. 33, pp. 1343-1372; November, 1954.

This paper describes (a) the operation of the backward-wave oscillator in terms of the coupling of the slow space charge wave of the beam with a backward circuit wave, (b) the operation of a traveling-wave tube amplifier at the Kompfner dip condition in terms of the coupling of the fast space charge wave with the ordinary circuit wave, and (c) several other interesting mode coupling effects in beam-type tubes. It may be seen from more complete analyses of the backward-wave oscillator^{3,4} that waves which increase with distance are not an essential feature of its operation, and that at start oscillation, the slow space charge wave is approximately in synchronism with the circuit wave.

II. COUPLING OF MODES OF PROPAGATION

Pierce's theory of the coupling of modes of propagation¹ is used as a starting point for this analysis. Consider the coupling of two modes of propagation which are characterized by amplitudes P and Q , defined in such a way that the power flow in the uncoupled modes is PP^* and $\pm QQ^*$ respectively. The upper sign is to be taken if the modes transmit energy in the same direction while the lower sign is to be taken if the modes transmit energy in opposite directions. These two modes are assumed to be coupled together periodically along the transmission systems by linear transducers as indicated in Fig. 1.

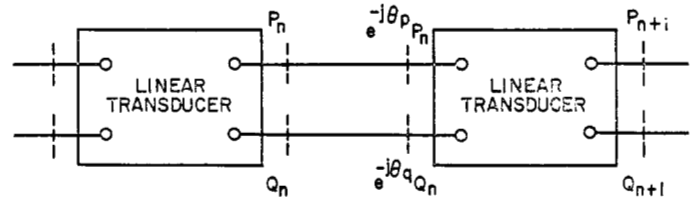


Fig. 1—Equivalent circuit for the coupled mode theory.

When the transducers are lossless they may be characterized by a single parameter,¹ k , and the coupled system of Fig. 1 is described by the difference equations

$$P_{n+1} = \sqrt{1 \mp k^2} e^{-j(\theta_p - \theta_s)} P_n + k e^{-j(\theta_s - \theta_s)} Q_n$$

$$Q_{n+1} = \mp k e^{-j(\theta_p - \theta_s + \theta_s - \theta_s)} P_n + \sqrt{1 \mp k^2} e^{-j(\theta_s - \theta_s)} Q_n$$

To describe the modes of propagation of the electron beam and helix by giving only their amplitude and propagation constant is certainly an oversimplification, because of their complex nature. Such a simplified picture can be made precise quantitatively by proper definition of the

³ H. R. Johnson, "Backward-wave oscillators," *Proc. IRE*, vol. 43, pp. 684-697; June, 1955.

⁴ H. Heffner, "Analysis of the backward-wave oscillator," *Proc. IRE*, vol. 42, pp. 930-937; June, 1954.

quantities involved. This simplification is qualitatively useful as well.

To go over to the case of continuous coupling, let the transducers be located a short distance Δz apart and make the following substitutions in (1),

$$\begin{aligned} P_n &\rightarrow P(z) & P_{n+1} &\rightarrow P(z) + dP & k &\rightarrow \mathfrak{K}\Delta z \\ Q_n &\rightarrow Q(z) & Q_{n+1} &\rightarrow Q(z) + dQ & \theta_p - \theta_1 &\rightarrow \beta_p \Delta z \\ & & & & \theta_a - \theta_3 &\rightarrow \beta_a \Delta z. \end{aligned}$$

To pass to the limit, $\Delta z \rightarrow 0$, expand the exponentials and square-root factors of (1) in a power series in Δz and retain only the first nonvanishing terms. The following differential equations are obtained:

$$\begin{aligned} \frac{\partial P}{\partial z} + j\beta_p P - \mathfrak{K}Q &= 0 \\ \frac{\partial Q}{\partial z} + j\beta_a Q \pm \mathfrak{K}P &= 0. \end{aligned} \quad (2)$$

These are the fundamental equations of the two-wave theory; \mathfrak{K} will be referred to as the coupling constant. Notice that when \mathfrak{K} is zero the equations have the solutions: $P = C_1 e^{-i\beta_p z}$, and $Q = C_2 e^{-i\beta_a z}$. These are the uncoupled modes in which the amplitudes of P and Q are independent. The following energy equation which is seen to be independent of the coupling, is easily derived from (2)

$$\frac{\partial}{\partial z} (PP^* \pm QQ^*) = 0. \quad (3)$$

This states that the total power flow is independent of z .

To solve these equations when \mathfrak{K} is not zero, assume the solution to be of the form $e^{-i\beta z}$, thus obtaining the algebraic equations,

$$\begin{aligned} j(\beta - \beta_p)P + \mathfrak{K}Q &= 0 \\ \mp \mathfrak{K}P + j(\beta - \beta_a)Q &= 0. \end{aligned} \quad (4)$$

These have a nontrivial solution if the determinant of the coefficients of P and Q vanishes, which means that β must have the values,

$$\begin{aligned} \beta_1 &= \frac{\beta_p + \beta_a}{2} + \sqrt{\left(\frac{\beta_p - \beta_a}{2}\right)^2 \pm \mathfrak{K}^2} \\ \beta_2 &= \frac{\beta_p + \beta_a}{2} - \sqrt{\left(\frac{\beta_p - \beta_a}{2}\right)^2 \pm \mathfrak{K}^2}. \end{aligned} \quad (5)$$

The general solution may be written,

$$\begin{aligned} P &= C_1 e^{-i\beta_1 z} + C_2 e^{-i\beta_2 z} \\ Q &= -jC_1 \frac{\beta_1 - \beta_p}{\mathfrak{K}} e^{-i\beta_1 z} - jC_2 \frac{\beta_2 - \beta_p}{\mathfrak{K}} e^{-i\beta_2 z} \end{aligned} \quad (6)$$

where C_1 and C_2 are arbitrary constants.

III. THE BACKWARD-WAVE OSCILLATOR

The operation of the backward-wave oscillator may be understood in terms of the coupling of the slow space

charge wave and the backward circuit wave. A backward circuit wave is one whose group velocity or direction of energy flow is in the opposite direction from the phase velocity. This situation may be obtained with a periodic circuit where space harmonic components of a wave may have a phase velocity which is opposite to the direction of energy flow. The electron beam transmits less total energy in the direction of electron flow when it is modulated with a slow space charge wave than it does when it is unmodulated. If the ac power flow is taken to be the total power flow less the dc power flow, the ac power flow associated with the slow space charge wave can be said to be in the opposite direction from the direction of electron flow. The relationship:

$$\begin{aligned} \text{ac power flow} &= (\text{ac energy stored per unit length}) \\ &\quad \times (\text{group velocity}) \end{aligned}$$

can be shown to hold in the case of one-dimensional space charge waves. Because the ac energy per unit length of the slow space charge wave is negative, the power flow is said to be in the opposite direction from the group velocity.

Let P represent the amplitude of the slow space charge wave with phase velocity in the $+z$ direction and power flow in the $-z$ direction. Let Q be the amplitude of a circuit wave or space harmonic component with phase velocity also in the $+z$ direction and power flow in the $-z$ direction. Since both waves transmit energy in the $-z$ direction, the upper signs should be used when applying the results of the preceding section. The electron beam enters the interaction region unmodulated at $z = 0$, so one boundary condition to be applied to (6) is

$$P = 0 \quad \text{at} \quad z = 0.$$

When the circuit is perfectly matched at $z = L$ the power flow in the negative direction at this point is zero, hence

$$Q = 0 \quad \text{at} \quad z = L.$$

This boundary condition is also to be applied to (6). These two boundary conditions yield the equations:

$$\begin{aligned} C_1 + C_2 &= 0 \\ C_1 \frac{\beta_1 - \beta_p}{\mathfrak{K}} e^{-i\beta_1 L} + C_2 \frac{\beta_2 - \beta_p}{\mathfrak{K}} e^{-i\beta_2 L} &= 0 \end{aligned} \quad (7)$$

which have a nontrivial solution only if

$$(\beta_1 - \beta_p)e^{-i\beta_1 L} - (\beta_2 - \beta_p)e^{-i\beta_2 L} = 0. \quad (8)$$

Upon substituting for β_1 and β_2 the values given by (5) and performing some minor manipulations this condition can be written in the following form:

$$\begin{aligned} \sqrt{\mathfrak{K}^2 L^2 + \left(\frac{\beta_a - \beta_p}{2}\right)^2 L^2} \cot \sqrt{\mathfrak{K}^2 L^2 + \left(\frac{\beta_a - \beta_p}{2}\right)^2 L^2} \\ = j \frac{\beta_a - \beta_p}{2} L. \end{aligned} \quad (9)$$

The solutions of this equation in the absence of loss (β_a and β_p real) are:

$$\beta_a = \beta_p \text{ and } L = (2n + 1) \frac{\pi}{2}, \quad m = 0, 1, 2, \dots \quad (10)$$

Thus from (5),

$$\beta_1 = \beta_p + \mathfrak{C} \quad \beta_2 = \beta_p - \mathfrak{C}. \quad (11)$$

The uncoupled mode amplitudes are easily shown to be

$$\begin{aligned} P &= -2jC_1 e^{-i\beta_p z} \sin \mathfrak{C}z \\ Q &= -2jC_1 e^{-i\beta_p z} \cos \mathfrak{C}z \end{aligned} \quad (12)$$

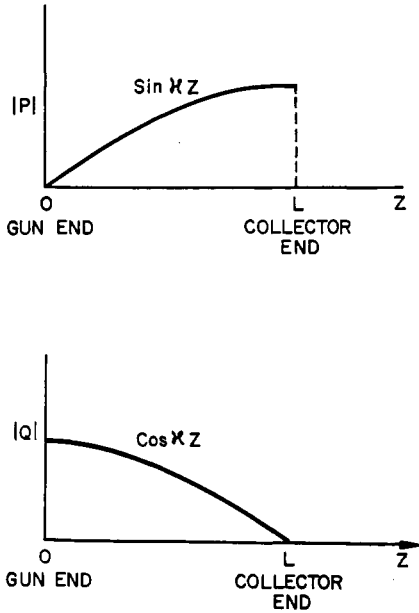


Fig. 2—Mode amplitudes at start oscillation of the backward-wave oscillator.

which are shown in Fig. 2 for the case $n = 0$. This is the condition that prevails at start oscillation for the lowest mode of operation. The slow space charge wave is in synchronism with the circuit wave and there is a beating or interference between the two modified waves of wavelength $2\pi/\mathfrak{C}$. The tube is a quarter beat wavelength long at start oscillation. The power flow along the tube is given by

$$\begin{aligned} PP^* + QQ^* &= |-2jC_1|^2 \sin^2 \mathfrak{C}z \\ &+ |-2jC_1|^2 \cos^2 \mathfrak{C}z = 4C_1^2 \end{aligned} \quad (13)$$

and is constant along the tube. The power may be said to flow in on the electron beam at the collector end of the tube, as a result of the beam leaving the interaction region in a modulated condition, and is completely transferred to the circuit upon reaching the gun or output end of the tube. When operating in the next lowest mode [$n = 1$ in (10)], the tube is long enough to transfer the energy from the beam to the circuit, back to the beam, and then back to the circuit again.

So far, one of the start oscillation conditions is not in the form in which it is usually expressed. It is, in fact,

much simpler in this special case. Although one could calculate \mathfrak{C} directly from the fields and currents of the uncoupled modes, \mathfrak{C} can also be expressed in terms of the usual traveling-wave tube parameters CN and hL by comparing the equation for the propagation constants obtained here with that obtained when all four waves are present:

$$[(\beta - \beta_e)^2 - h^2][\beta^2 - \beta_1^2] = 2\beta_e \beta^2 \beta_1 C^3 \quad (14)$$

where C is the interaction parameter ($E^2/\beta^2 P$)($I_0/4V_0$), N is the electronic wavelength of the tube, β_1 is the propagation constant of the circuit, $\beta_e = \omega/u_0$ is the electron propagation constant, and $h = \omega_0/u_0$ is the reduced plasma wave number.

To specialize (14) to the case where only two waves are important, assume that $\beta \approx \beta_1 \approx \beta_e + h$ and replace β and β_1 by β_e except where differences between nearly equal quantities are involved. This gives

$$(\beta - \beta_e - h)(\beta - \beta_1) - \frac{\beta_e^3 C^3}{2h} = 0 \quad (15)$$

which is to be compared with the coupled mode result,

$$(\beta - \beta_p)(\beta - \beta_q) - \mathfrak{C}^2 = 0. \quad (16)$$

Thus we may identify our \mathfrak{C}^2 with $\beta^3 C^3/2h$ and the start-oscillation condition may be written,

$$\mathfrak{C}^2 L^2 = \frac{\beta_e^3 C^3 L^3}{2hL} = \frac{4\pi^3 (CN)^3}{hL} = \frac{\pi^2}{4}, \quad (17)$$

or

$$CN = \left(\frac{hL}{16\pi}\right)^{1/3}.$$

This result is shown in Fig. 3. Condition $\beta_a = \beta_p$ is shown in Fig. 4 (next page). Results of the three-wave theory⁴ are also shown for purposes of comparison. It is seen that the

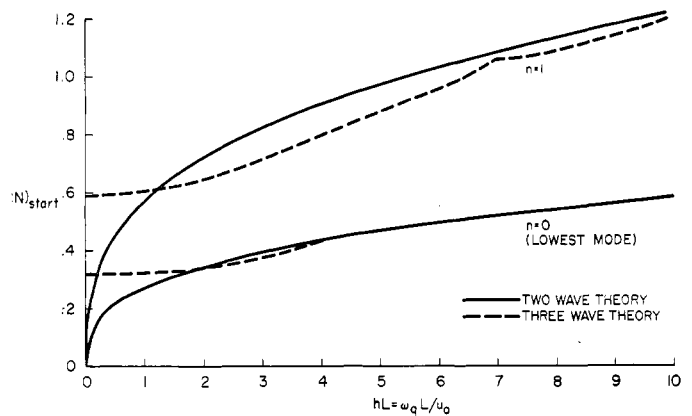


Fig. 3—A comparison of the start oscillation current condition obtained from the coupled mode theory with that obtained from the three-wave theory.

CN_{start} condition for the lowest mode of operation is in good agreement if hL is greater than about 1.5 and the synchronism condition is in good agreement if hL is greater than about 3.0.

Thus this theory is applicable when the tube length is greater than a quarter or half plasma wavelength.

IV. THE KOMPFFNER DIP CONDITION⁵

By proper choice of beam voltage and current, a traveling-wave tube amplifier may be adjusted so as to produce no output. This is known as the Kompfner dip condition and it is useful because it permits direct measurement of the traveling-wave tube parameters. In the large space charge case the operation of a traveling-wave tube in this manner can be understood in terms of the coupling of the fast space charge wave with the circuit wave. The ac power flow of the fast space charge wave is in the same direction as the electron flow and the power flow on the circuit is also in this direction so the upper signs in Section II are again to be used. Letting P stand for the amplitude of the fast space charge wave whose phase velocity and energy flow are in the $+z$ direction,

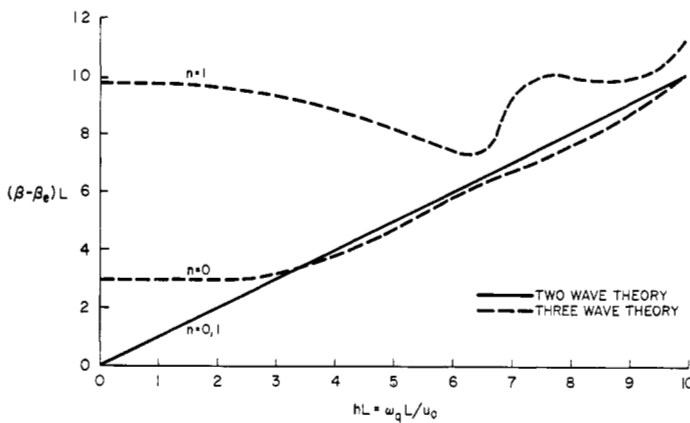


Fig. 4—A comparison of the start oscillation synchronism condition obtained from the coupled mode theory with that obtained from the three-wave theory.

and Q stand for the circuit wave amplitude whose phase velocity and energy flow are also in the $+z$ direction, the boundary conditions are

$$P = 0 \text{ at } z = 0 \quad \text{and} \quad Q = Q_0 \text{ at } z = 0;$$

i.e., the beam enters the interaction region unmodulated but a signal, Q_0 , is impressed on the circuit. We inquire when the output, the circuit amplitude at $z = L$, may be zero. This condition will be independent of Q_0 , hence the condition $Q = Q_0$ at $z = 0$ may be disregarded, only

$$P = 0 \text{ at } z = 0 \quad \text{and} \quad Q = 0 \text{ at } z = L$$

being required. These are formally identical with the boundary conditions of the backward-wave oscillator.

Because the differential equations and boundary conditions are the same as for the backward-wave oscillator, (7) through (13) apply also to the Kompfner dip condition. Their interpretation is different, however. In the latter case a signal is applied to the input of the tube in the form of a pure circuit wave and the fast space charge wave is not excited; at $z = L$ the waves have interacted just long

enough to convert the circuit wave to a pure space charge wave, so that no output is observed. The power flow along the tube is again constant, the energy being carried by the circuit near the gun end of the tube and by the electron beam near the collector end.

V. THE EFFECT OF CIRCUIT LOSS

In the discussions of Sections III and IV, circuit loss was neglected. To include circuit loss, replace $j\beta_c$ by $j\beta_c - \alpha$. A positive value of α represents loss in the backward-wave oscillator circuit, whereas a negative value of α represents loss in an ordinary traveling-wave tube, to which the Kompfner dip condition applies.⁶ Eq. (9) may be rewritten:

$$\begin{aligned} & \sqrt{3\mathcal{C}^2 L^2 + \frac{1}{4}(\beta_c - \beta_p + j\alpha)^2 L^2} \\ & \cdot \cot \sqrt{3\mathcal{C}^2 L^2 + \frac{1}{4}(\beta_c - \beta_p + j\alpha)^2 L^2} \\ & = \frac{j(\beta_c - \beta_p) \pm \alpha}{2} L \end{aligned} \quad (18)$$

which has a solution when $\beta_p = \beta_c$ and

$$\sqrt{3\mathcal{C}^2 L^2 - \frac{1}{4}\alpha^2 L^2} \cot \sqrt{3\mathcal{C}^2 L^2 - \frac{1}{4}\alpha^2 L^2} = -\frac{1}{2}\alpha L. \quad (19)$$

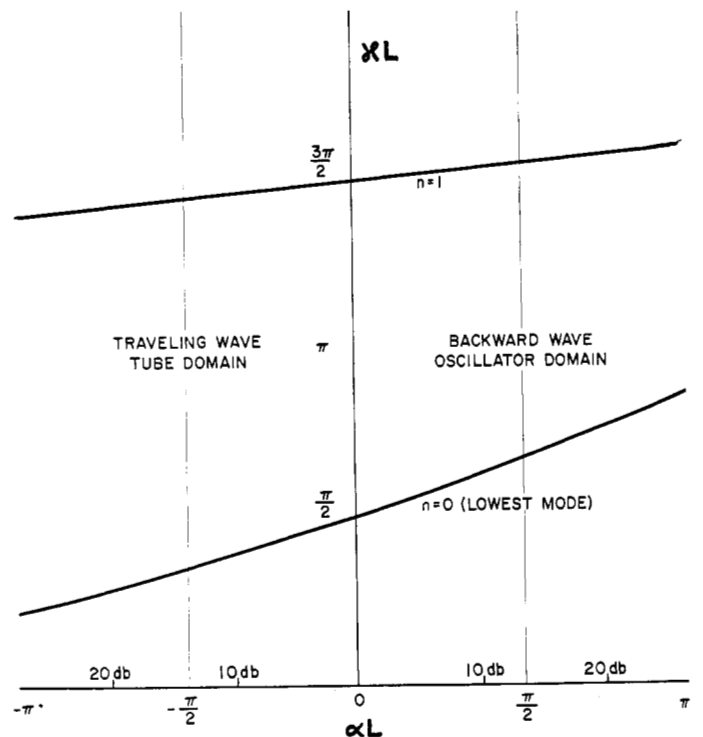


Fig. 5—Effect of loss on the backward-wave oscillator start oscillation condition and the Kompfner dip condition of a TWT amplifier. $3\mathcal{C}L$ is plotted against αL , the total loss in nepers.

The solution of this equation is given in Fig. 5. It may be seen that circuit loss increases the value of $3\mathcal{C}L$ at start oscillation of the backward-wave oscillator, but decreases the value of $3\mathcal{C}L$ for the Kompfner dip condition. When $\alpha > 3\mathcal{C}$, backward-wave oscillation is not possible irrespec-

⁵ R. Kompfner, "On the operation of the traveling wave tube at low level," *British Jour. IRE*, vol. 10, pp. 283-289; August, September, 1950.

⁶ Circuit power flow is in the $-z$ direction in the first case, and in the $+z$ direction in the second case.

tive of the length of the tube. The value of CN of (17) should be multiplied by the two-thirds power of the ordinate of Fig. 5 to account for the effect of loss. A useful analytic approximation, valid for less than about 20 db loss, is

$$CN = \left(\frac{hL}{16\pi}\right)^{2/3} \left(1 + \frac{2\alpha L}{\pi^2}\right)^{2/3}. \quad (20)$$

VI. REMARKS ON THE APPLICATION OF BOUNDARY CONDITIONS

In the absence of coupling, it is clear that the point at which the boundary condition on a particular wave is to be applied depends on direction of the group velocity of that wave. If, for example, the group velocity of the wave is to the right, the boundary condition on this wave should be applied on the left end of the region where the solution is desired. When two or more waves interact and boundary conditions are applied to the uncoupled mode amplitudes, the group velocity of each of the uncoupled modes is assumed to determine the point where the boundary condition on that wave is applied. This is certainly reasonable, since the group velocity is the direction in which one can signal or propagate a disturbance.

When the power flows of the uncoupled modes are in the same direction, we have seen that there is an interference or beating phenomena between the two waves. Two situations arise: the group velocities of the two waves may be in opposite directions or the same direction. The backward-wave oscillator is an example of the former situation, since the slow space charge wave has a group velocity in the $+z$ direction, and the circuit wave has a group velocity in the $-z$ direction. In applying the boundary conditions in Section III, we have followed the above convention. The Kompfner dip condition is an example of the second situation since the circuit wave and the fast space charge wave both have group velocities in the $-z$ direction. Both boundary conditions are applied at $z = 0$.

If the power flows of the uncoupled waves are in opposite directions, one of the modified waves increases with z while the other decreases with z , provided the phase velocities of the uncoupled waves are approximately equal. The application of the coupled mode theory to this very important situation has been discussed by Pierce.² In this section a few remarks on the application of boundary conditions will be made and a few examples given.

As in the case of power flows in the same direction, two situations also arise: the group velocities may be in the same or opposite directions. The former situation arises only when one of the waves is the slow space charge wave of an electron beam, since this is the only wave for which the group velocity and power flow are in opposite directions. Both increasing and decreasing waves can be excited at the same end of the coupling region. This is the situation that exists in traveling-wave tube amplifiers, and it suffices to say that the analysis of Section II can be applied to obtain simple gain expressions, valid for large space charge conditions.

The other situation arises when the group velocities of the two waves are in opposite directions. One boundary condition is applied at each end of the region of coupling. For example, consider the coupling of the fast space charge waves of two electron beams whose velocities are in opposite directions. The ω vs β diagram of this system is shown in Fig. 6. The coupling of the two waves produces an increasing and a decreasing wave in the region of $\beta = 0$.

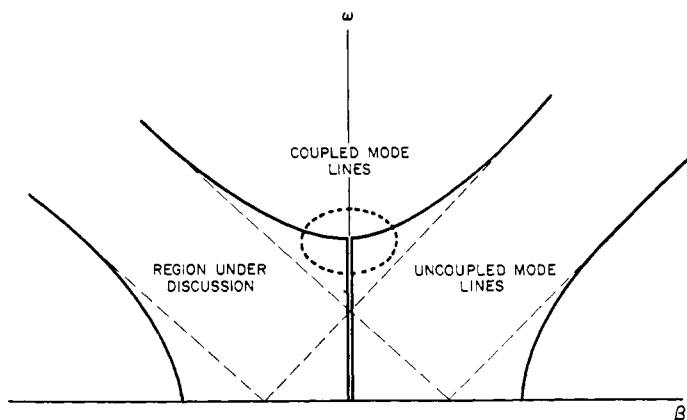


Fig. 6—Diagram of " ω versus β " for two electron beams in opposite directions. When the frequency is less than $\sqrt{2} \omega_p$ a pair of waves result, one of which increases with z , and one of which decreases.

This is indicated by the double line in the figure, where only the real part of β is shown. Application of the boundary conditions shows that the boundary condition at $z = 0$ affects primarily the wave which decreases with z , while the boundary condition at $z = L$ affects primarily the wave which increases with z , or decreases in the $-z$ direction. The situation is similar to that of a waveguide slightly below cutoff. Only waves which decrease away from the source are excited, and this phenomena cannot be used to obtain microwave amplification.

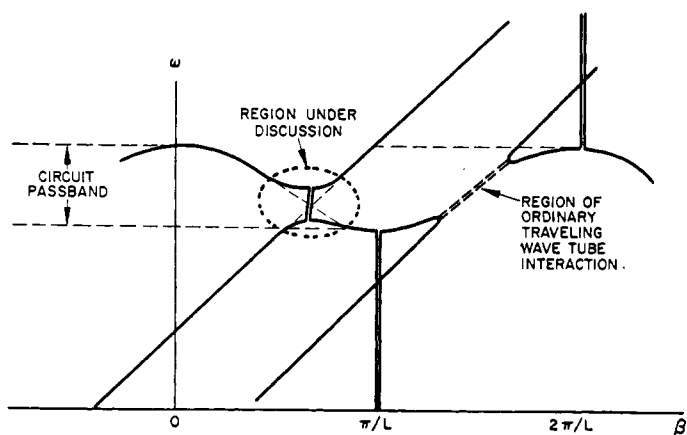


Fig. 7—Diagram of " ω versus β " for an electron beam coupled to a periodic slow wave circuit of fundamental period L . The fast space charge wave couples with the reverse circuit wave to produce an increasing and a decreasing wave.

A similar situation occurs when a fast space charge wave is coupled to a backward circuit wave, as illustrated by the ω vs β diagram of Fig. 7, where ω_p/ω has been taken somewhat larger than normal for purposes of illus-

TABLE I

Interacting Waves		v_{phase}	v_{power}	v_{group}	Resulting Device
	{ circuit wave slow space charge wave	same	opposite	same	TWT amplifier
	{ circuit wave fast space charge wave	same	same	same	TWT at Kompfner dip
	{ backward circuit wave slow space charge wave	same	same	opposite	BWO and BWA
	{ backward circuit wave fast space charge wave	same	opposite	opposite	apparently not useful
same beam	{ fast space charge wave slow space charge wave	same	opposite	same	velocity jump amplifier space charge wave amplifier (rippled wall)
two beams	{ fast space charge wave slow space charge wave	same	opposite	same	double stream amplifier
beams in opposite direction	{ fast space charge wave fast space charge wave	infinite	opposite	opposite	apparently not useful

tration. The power flow and group velocity of the fast space charge wave are in the $+z$ direction, and the power flow and group velocity of the backward circuit waves are in the $-z$ direction. When their phase velocities are approximately equal, coupling occurs. The backward circuit wave is likely to be excited by mismatch at the output of a traveling-wave tube, and in the absence of coupling with the fast space charge wave it will propagate unattenuated to the input end of the tube, unless an attenuator is provided. As a result of coupling with the fast space charge wave, the backward circuit wave is no longer unattenuated. Through the coupling two new waves result. According to the previous discussion, it is primarily the wave which decreases in the $-z$ direction which is excited by the mismatch.

Finally, the results of these arguments are summarized in Table I above. The results of Pierce are included.

DISCUSSION

It should be apparent to the reader that the above results can also be obtained from the more complete theory³⁻⁵, by making approximations which are appropriate to large space charge operation. The essential contribution of this paper is to apply the coupling of modes theory to get at these results, and to set down some of the results explicitly. In many tubes, one is justified in calculating their performance from a large space charge theory, in which case there is considerable simplification.

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