datum voltage of node α, then

$$v_\beta = \sum_\alpha T_{\alpha\beta} v_\alpha. \quad \textbf{(1)}$$

Forming voltages into vector v and transmittances into the square matrix T, we write for (1)

$$v = T_t v, \quad (2)$$

T_t being the transpose of T. Equivalently, denoting by I the unit matrix,

$$(T_t - I)v = 0. \quad (3)$$

This equation solves the network. Note that v contains the inputs as well as the driven voltages.

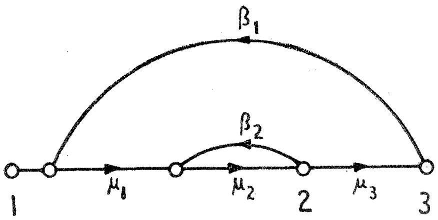

Fig. 1.

As a specific example consider Fig. 1, where the hollow circles denote mixing points. In order to analyze this situation without the use of additional rules, a sufficient number of nodes to break all loops must be identified. In our example one node is required in addition to the input terminal. By inspection,

$$T - I = \begin{Vmatrix} 0 & \mu_1\mu_2 \\ 0 & \beta_2\mu_2 + \mu_3\beta_1\mu_1\mu_2 - 1 \end{Vmatrix}.$$

Denoting cofactors by superscripts we find for the through transmittance H_{13},

$$-\mu_3 H_{12} = \mu_3 \frac{V_2}{V_1} = \mu_3 \frac{(T_t - I)^{12}}{(T_t - I)^{11}} = \mu_3 \frac{(T - I)^{21}}{(T - I)^{11}} = \frac{\mu_1\mu_2\mu_3}{1 - \mu_2\beta_2 - \mu_1\mu_2\mu_3\beta_1}.$$

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On the Coupling Coefficients in the "Coupled-Mode" Theory*

The "coupled-mode" theory has proved itself to be an important tool in the analysis of energy exchange phenomena between traveling waves. In its original form[1] it is capable of yielding important qualitative results. To extend its range of usefulness into the quantitative domain, one needs to evaluate the coupling coefficients which govern the energy exchange. This is done in this paper where we treat the "small coupling" case. We assume that in obtaining the coupling coefficients for the small coupling case we may use for the different physical observables their values in the absence of coupling. This procedure is analogous to that used in evaluating the Q of a cavity or the attenuation constant of a waveguide, for the small loss case, where the loss-free field solutions are used instead of the actual solutions in the presence of losses, and is a type of perturbation theory formulated on physical grounds.

Using Haus'[2] formulation of Pierce's coupled-mode theory we write for the system of differential equations obeyed by the mode amplitudes

$$\frac{d[A]}{dz} = [C][A]. \quad (1)$$

$[A]$ is a column matrix, whose individual components $A_i(z)$ are normalized such that $\pm A_i A_i^*$ is the power carried by the ith mode in the $+z$ direction, the z direction being taken as the direction of propagation. $[c]$ is a square matrix whose determination is the subject of this paper.

The condition of power conservation yields:

$$C_{ki}^* = \mp C_{ik} \qquad k \neq i \quad (2)$$

where the upper and lower signs apply when modes i and k carry power in the same or opposite directions, respectively. Using (1) and (2) we get:

$$C_{mn} = \frac{\frac{d}{dz}(A_m A_m^*)}{2\,\mathrm{Re}\,(A_m^* A_n)} \quad (3)$$

for C_{mn} real.

$$C_{mn} = j\,\frac{\frac{d}{dz}(A_m A_m^*)}{\mathrm{Im}\,(A_m A_n^*)} \quad (4)$$

for C_{mn} imaginary.

A_m and A_n have to be defined in terms of the physical observables (fields, current, etc.) in such a way that either $Re(A_m A_n^*)$ is proportional to the distance rate of change of power in mode m (or n), in which case (3) applies or Im $(A_m A_n^*)$ is proportional to the power rate of change, in which case (4) applies. The proportionality constant is real in both cases.

To illustrate the application of the theory we treat the cases of the traveling-wave tube and the double stream amplifier.

Traveling-Wave Tube

The coupled-modes are the "fast" and "slow" space-charge waves and the circuit-forward wave (the circuit-backward wave is assumed matched out).

Let:

$$A_n A_n^* = \frac{VV^*}{2K} = \frac{E_z E_z^*}{2\beta_c^2 K} \quad (5)$$

where

K = field-normalized circuit impedance,[3]
E_z = effective axial field,
β_c = free (uncoupled) propagation constant of the circuit. In its free state the beam carries RF kinetic power P_K

$$P_K = -\frac{Au_0}{2\eta}\,\mathrm{Re}\,(v_z i_z^*) \quad (6)$$

where

A = beam cross-section area,
μ_0 = beam dc velocity,
$\eta = |e/m|$ = charge to mass ratio of the electron,
v_z, i_z are the beam ac velocity and current density, respectively.

For a single space-charge wave (6) can be rewritten as:

$$P_K = A_m A_m^* = \frac{\omega_q V_0}{\omega I_0} I_z I_z^* \quad (7)$$

where:

ω_q, ω = beam reduced plasma frequency and frequency, respectively,
V_0, I_0 = beam dc voltage and current, respectively,
I_z = beam ac current.

In view of (6) and (7) we can write

$$A_m = \left(\frac{\omega_q V_0}{\omega I_0}\right)^{1/2} I_z \quad (8)$$

$$A_n = \frac{1}{(2K)^{1/2}\beta_c} E_z. \quad (9)$$

Employing Poynting's theorem we can show that the rate of change of circuit power is given by:

$$\frac{d}{dz}(A_n A_n^*) = -\frac{1}{2}\,\mathrm{Re}\,(E_z I_z^*)$$

using (8)–(10) and assuming

$$\beta_c \simeq \frac{\omega}{u_0} = \beta_e, \quad (10)$$

we get:

$$C_{mn}^2 = S^2 = \frac{\beta_e^3 C^3}{2\beta_q} \quad (11)$$

S being defined by (11).

Taking A_1 as the forward-circuit mode, A_2 and A_3 as the "fast" and "slow" space-charge wave, and assuming $C_{23} = 0$ and $e^{-\Gamma Z}$ propagation, (1) becomes

$$\begin{aligned} (-j\beta_c + \Gamma)A_1 + SA_2 + SA_3 &= 0 \\ -SA_1 + [-j(\beta_e - \beta_q) + \Gamma]A_2 &= 0 \\ SA_1 + [-j(\beta_e + \beta_q) + \Gamma]A_3 &= 0 \end{aligned} \quad \textbf{(12)}$$

resulting in the determinantal equation:

$$j\beta_e^2\beta_c - \beta_e^2\Gamma - j\beta_q^2\beta_c + \beta_q^2\Gamma - 2\beta_e\beta_c\Gamma - j\beta_c\Gamma^2 - 2j\beta_e\Gamma^2 + \Gamma^3 - 2j\beta_q S^2 = 0. \quad (13)$$

Eq. (13) can be solved directly for Γ. To cast the result in a more familiar form we adopt the convention:

$$\begin{aligned} -\Gamma &= -j\beta_e + \beta_e c\delta \\ \beta_c &= \beta_e(1 + cb) \\ \beta_q &= \beta_e(4QC^3)^{1/2}. \end{aligned}$$

Eq. (13) becomes the well-known equation for the traveling-wave tube[3]

$$(\delta^2 + 4QC)(-b + j\delta) = +1. \quad (14)$$

* Received by the IRE, May 26, 1958.

[1] J. R. Pierce, "Coupling of modes of propagation," *J. Appl. Phys.*, vol. 25, pp. 179–183; February, 1954.

[2] H. Haus, "Coupled Mode Theory," M.I.T., Cambridge, Mass., unpublished report.

[3] J. R. Pierce, "Traveling-Wave Tubes," D. Van Nostrand Co., Inc., New York, N. Y.; 1950.

It should be noted that in no place have we used results from other traveling-wave tube theories. We have merely adopted some of Pierce's conventional symbols in order to arrive at (14).

DOUBLE STREAM AMPLIFIER

The interaction is assumed to take place between the "slow" space-charge wave of the fast electron beam, called A_2, and the "fast" space-charge wave of the slow beam, denoted as A_1. The slow and fast electron beams have equal charge densities and have dc velocities u_{01} and u_{02}, respectively.

Using results of the kinetic power theorem in a way analogous to that leading to (8) we get:

$$A_1 = \left(\frac{u_{01}\omega_q}{2\eta\omega\,|\,\rho_0\,|\,A}\right)^{1/2} I_1, \qquad (15)$$

$$A_2 = \left(\frac{u_{02}\omega_q}{2\eta\omega\,|\,\rho_0\,|\,A}\right)^{1/2} I_2. \qquad (16)$$

In analogy with (10) we get:

$$\frac{d}{dz}(A_1 A_1^*) = -\operatorname{Re}(E_1 I_2^*). \qquad (17)$$

Using (15)–(17) in (4) plus the result:

$$E_1 = \frac{jI_1}{\omega\epsilon A}$$

leads to

$$C_{12} = -j\frac{\omega_p^2}{\omega_q(u_{01}u_{02})^{1/2}} \qquad (18)$$

and a determinantal equation whose solution, assuming $e^{-j\beta Z}$ propagation, is:

$$\beta_{1,2} = \frac{\beta_{01}+\beta_{02}}{2} \pm \left[\left(\frac{\beta_{01}-\beta_{02}}{\sqrt{2}}\right)^2 - |\,c_{12}\,|^2\right]^{1/2} \qquad (19)$$

where:

$$\beta_{01} = \frac{\omega}{u_{01}} - \frac{\omega_q}{u_0} = \frac{\omega}{u_{01}} - \beta_q$$

$$\beta_{02} = \frac{\omega}{u_{02}} + \frac{\omega_q}{u_0} = \frac{\omega}{u_{02}} + \beta_q.$$

Defining:

$$u_{01} = u_0(1-b)$$
$$u_{02} = u_0(1+b)$$

$$\beta_{1,2} = \beta_e(1+\delta_{1,2}), \qquad x = \frac{b\beta_e}{\beta_q}$$

and assuming

$$b^2 \ll 1, \qquad \frac{\beta_p}{\beta_e} < 1, \qquad \beta_p = \beta_q$$

leads to:

$$\delta_{1,2} = \pm\frac{\beta_p}{\beta_e}\left[(1+x^2) - (1+4x^2+4x)^{1/2}\right]^{1/2}. \qquad (20)$$

The "classical" small signal analysis yields:[4]

$$\delta_{1,2} = \pm\frac{\beta_p}{\beta_e}\left[(1+x^2) - (1+4x^2)^{1/2}\right]^{1/2}. \qquad (21)$$

[4] A. B. Haeff, "The electron-wave tube—a novel method of generation and amplification of microwave energy," PROC. IRE, vol. 37, pp. 4–10; January, 1949.

The difference between the two results is believed to be due to the failure of our two-mode picture to take into account the two extreme modes. Some of the differences are shown in Table I.

TABLE I

Feature	Small Signal (Haeff)	Coupled Modes (Two-Modes Approximation)
Maximum Gain at	$x=\frac{1}{2}\sqrt{3}$	$x=1.0$
Maximum Gain of	$\frac{1}{2}(\omega_p/\omega)$ nepers	ω_p/ω nepers
Gain Obtained for	$0<x<\sqrt{2}$	$0<x<2$

We are aware of the fact that this problem has been recently solved in a more formal manner by Haus of M.I.T. We believe that the approximate perturbation solution given here may still prove useful.

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Effect of Beam Coupling Coefficient on Broad-Band Operation of Multicavity Klystrons*

To meet some of the requirements of present-day electronic systems, attention has recently been focused on the problem of broad-band operation of multicavity klystrons.[1–7] The major advantages of using broad-band multicavity klystrons at high-power levels (in comparison with a TWT operating over the same bandwidth) presumably are 1) no backward wave is present (as the multicavity-klystron structure is not bilateral) and the attenuator problem (which is present in a TWT) may be eliminated, and 2) possible better efficiency in the case of a multicavity klystron than in a TWT. This statement is based on the fact that under synchronously tuned conditions, a typical high-power klystron can have an efficiency in the neighborhood of 40 per cent.

In a discussion of broad-band operation of multicavity klystrons, one comes across the question: "Under what conditions is the broad-band operation of multicavity klystrons profitable?"

A suitable starting point for a discussion of the above problem is the relation

$$V^{(0)} = MZ^{(0)}i \qquad (1)$$

where M and $Z^{(0)}$ are the beam coupling coefficient and shunt impedance of the output cavity, respectively, and i is the ac current in the beam at the entrance to the output cavity. We note that the maximum voltage across the output gap $V_{\max}^{(0)}$ which we can best use is the dc voltage of the beam V_0; this is so because, for any value of $V^{(0)}$ larger than the dc beam voltage V_0, electrons would be reflected at the output gap and they will travel toward the input. It is also known that $i_{\max}$ (the maximum ac current in the beam at the output gap) is approximately equal to the dc current (for instance under ballistic assumptions $i_{\max} = 1.16I$).

Hence, under the optimum conditions for power transfer from the output cavity, we may write

$$V^{(0)} \approx V_0 \approx MZ^{(0)}I \qquad (2)$$

or

$$Z^{(0)} \approx Z_0 \qquad (2a)$$

when

$$M \approx 1$$
$$Z_0 = \frac{V_0}{I},$$

Z_0 being the dc beam impedance, and I the dc current of the beam at input.

We may also write

$$Z^{(0)} = (R/Q)_0 Q_L^{(0)} \qquad (3)$$

where $(R/Q)_0$ and $Q_L^{(0)}$ are the values of R/Q and Q of the output cavity under loaded conditions. From (3) we note that for optimum performance $Q_L^{(0)}$ is specified when once the beam voltage V_0 and dc beam current I are specified. In practice, however, the relation $Z^{(0)} \approx Z_0$ is not always satisfied. Many klystron engineers choose values of $Q_L^{(0)}$ based on experience which are at variance with the above simple relation. Actually the shunt impedance across the output gap is made several times the value of beam impedance Z_0 at higher frequencies. We note that as $Z^{(0)}$ goes up the value of $Q_L^{(0)}$ goes up or equivalently $Q_E^{(0)}$ goes up ($Q_E^{(0)}$ being the external Q of the output cavity).

Since RF power is extracted from the output cavity, it is clear that the same (the output cavity) determines the bandwidth over which efficient power transfer can be effected.

We will now discuss the problem as follows. The bunched beam as it enters the output gap exhibits a velocity distribution which in turn alters the value of the beam coupling coefficient considerably. The usual value of beam coupling coefficient M for a *gridless gap* is given by

* Received by the IRE, May 19, 1958.

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[2] L. D. Smullin and A. Bers, "Stagger Tuned Multicavity Klystron," Conference on Electron Tube Res., Berkeley, Calif.; 1957.

[3] S. V. Yadavalli, "Application of the potential analog in multicavity klystron design and operation," PROC. IRE, vol. 45, pp. 1286–1287; September, 1957. Also presented in collaboration with C. A. Arnold and J. P. Lindley at the Conference on Electron Tube Res., 1957.

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[6] P. G. R. King, "A five per cent bandwidth 2.5 mw s-band klystron," *SERL Tech. J.*, vol. 8, p. 29; January, 1958.

[7] H. T. Curnow, "Factors influencing the design of multicavity klystrons," *SERL Tech. J.*, vol. 8, p. 42; January, 1958.