Frequency, Intensity, and Field Fluctuations in Laser Oscillators

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Abstract—**The basic differential equation governing laser noise is derived from classical and quantum mechanical considerations. Its linearized Van Der Pol form is used to derive the frequency, field, and intensity fluctuation spectra.**

I. Introduction and Preview

THE PROBLEM of the quantum mechanical limitations to the spectral purity of the laser output has occupied numerous investigators [1]–[5]. The theoretical analyses of this problem depend heavily on advanced quantum-field formalisms or, equivalently, on density-matrix treatments of the quantized field–atom system. These treatments have, for all practical purposes, put the understanding of this field beyond the reach of all but a small group of specialists.

The purpose of this analysis is to rederive all the known results concerning the laser noise spectrum from a semiclassical point of view. The laser is treated as an oscillator with a saturable gain which is driven by noise. The exact form of the gain saturation is of key importance in a laser noise analysis and considerable care is devoted to its formulation. This can be done rigorously by an electromagnetic derivation of the laser equation and by using Lamb's results for the induced atomic polarization [6]. The resulting differential equation (11) is of the Van Der Pol [7] form. The Van Der Pol equation has been used to treat oscillation problems by a number of investigators [3], [7], [8]. Below threshold it reduces to that of a linear oscillator. Well above threshold where the fluctuating components of the laser-field amplitude are small it can be linearized. The derivations of the linearized Van Der Pol equations (16) and (17) starting from electromagnetic and quantum-mechanical considerations is the main contribution of this paper. The rest of the paper involves the straightforward derivation of the laser-noise spectra from these equations.

The three types of laser spectra are illustrated by Fig. 1. They are, respectively: a) the laser frequency-fluctuation spectrum $W_{\delta\omega}(\Omega)$, b) the laser power-fluctuation spectrum $W_{\Delta P}(\Omega)$, and c) the laser-field spectrum $W_\epsilon(\omega)$. The first of these $W_{\delta\omega}(\Omega)$ has not, to our knowledge, been considered before. It is important for signal-to-noise considerations involving FM laser modulation since $W_{\delta\omega}(\Omega)$ appears as additive noise in the final demodulated output. It also plays a key role in the derivation of the so-called "phase-noise" spectrum $W_\epsilon(\omega)$.

The basic interrelationships of the three noise spectra are illustrated by Fig. 2.

In addition to the conventional parametric description of the laser spectra, we show that an equivalent descrip-

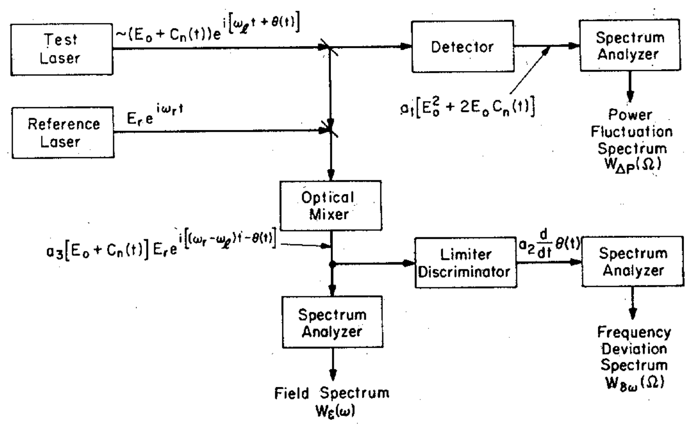

Fig. 1. A block diagram demonstrating a typical arrangement for measuring the various noise spectra of a laser oscillator.

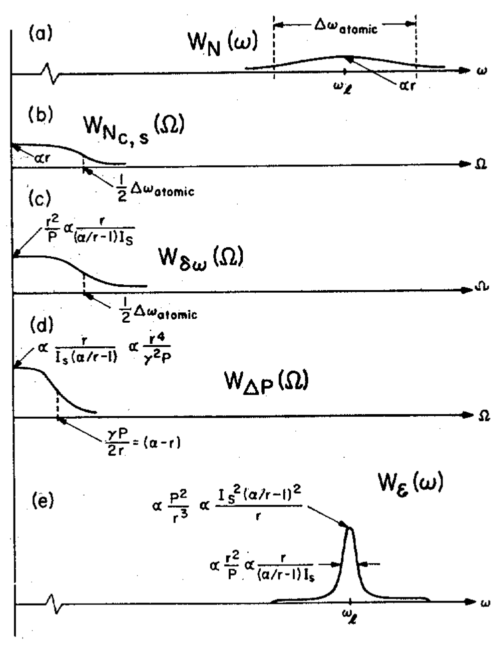

Fig. 2. The frequency plot of the various spectra. (a) The gain profile of the lasing medium. (b) The spectra of the in-phase ($W_{N_c}(\Omega)$) and quadrature ($W_{N_s}(\Omega)$) driving-noise amplitudes. (c) The spectrum of the frequency deviation. (d) The power fluctuation spectrum. (e) The laser-field spectrum.

Manuscript received October 23, 1973.
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tion can be found using the saturation intensity, the cavity losses, and the pumping ratio.

II. Derivation of the Van Der Pol Equations

The total field in the laser resonator is expanded in Slater modes [9]

$$\mathbf{E}(\mathbf{r}, t) = \sum_a -\frac{1}{\sqrt{\epsilon}} p_a(t)\mathbf{E}_a(\mathbf{r}) \tag{1}$$

$$\mathbf{H}(\mathbf{r}, t) = \sum_a \frac{1}{\sqrt{\mu}} \omega_a q_a(t)\mathbf{H}_a(\mathbf{r}) \tag{2}$$

with

$$\nabla \times \mathbf{H}_a = k_a\mathbf{E}_a, \qquad \nabla \times \mathbf{E}_a = k_a\mathbf{H}_a$$

$$\int_v (\mathbf{E}_a \times \mathbf{E}_b)\, dv = \delta_{ab}, \qquad k_a = \omega_a\sqrt{\mu\epsilon}. \tag{3}$$

Substitute (1) and (2) into Maxwell's equation

$$\nabla \times \mathbf{H} = \sigma\mathbf{E} + \epsilon\frac{\partial \mathbf{E}}{\partial t} + \frac{\partial}{\partial t}\mathbf{P}_{\text{res}}(\mathbf{r}, t). \tag{4}$$

Multiply by $\mathbf{E}_l$ and integrate over the cavity volume. The result is

$$\omega_l^2 q_l = -\frac{\sigma}{\epsilon} p_l - \dot{p}_l + \frac{1}{\sqrt{\epsilon}}\frac{\partial}{\partial t}\int_{v_c} \mathbf{P}_{\text{res}}(\mathbf{r}, t)\cdot\mathbf{E}_l(\mathbf{r})\, dv \tag{5}$$

where $\mathbf{P}_{\text{res}}$ $(\mathbf{r}, t)$ is the medium polarization due to the resonant laser transition, and σ is the effective medium conductivity. From the Maxwell equation $\nabla \times \mathbf{E} = -(\partial\mathbf{B}/\partial t)$ we obtain

$$p_l = \dot{q}_l \tag{6}$$

which when used in the derivative of (5) results in the oscillator equation

$$\omega_l^2 p_l + \frac{d^2 p_l}{dt^2} + \frac{\sigma}{\epsilon}\frac{dp_l}{dt} = \frac{1}{\sqrt{\epsilon}}\frac{\partial^2}{\partial t^2}\int_{v_c} \mathbf{P}_{\text{res}}(\mathbf{r}, t)\cdot\mathbf{E}_l(\mathbf{r})\, dv. \tag{7}$$

The next step is to use Lamb's self-consistent treatment of the laser oscillator [6]. Early in this analysis Lamb expresses the dependence of the induced polarization at ω_l on the inducing electric field of the same frequency up to third order in the field amplitude. This relationship can be summarized by writing the *local* dependence between the induced polarization $P_{\text{res}}(\mathbf{r}, t)$ and the inducing electric field

$$E_l(t) = E_0(t)\cos\omega_l t \tag{8}$$

as

$$\begin{aligned} P_{\text{res}}(\mathbf{r}, t) = {} & [\chi_R^{(1)}E_0(t) + \chi_R^{(3)}E_0^3(t)]\cos\omega_l t \\ & + [\chi_I^{(1)}E_0(t) + \chi_I^{(3)}E_0^3(t)]\sin\omega_l t \end{aligned} \tag{9}$$

where $E_0(t)$ is a "slowly" varying real function, and the R and I subscripts denote, respectively, the real (in-phase) and imaginary (quadrature) susceptibilities.

It can be shown by a tedious, but straightforward substitution that by assuming a dependence in the form of (9), (7) assumes a form

$$\frac{\partial^2 \mathcal{E}_l}{\partial t^2} + (r - \alpha)\frac{d\mathcal{E}_l}{dt} + \frac{\gamma}{3}\frac{d\mathcal{E}_l^3}{dt} + \omega_l^2\mathcal{E} = 0 \tag{10}$$

provided $\chi_R^{(1)} = 0$, $\chi_R^{(3)} = 0$ which is true when the oscillation frequency ω_l is equal to the atomic transition frequency ω_{atomic}. $\mathcal{E}_l(t)$ is a normalized mode amplitude which is proportional to $p_l(t)$. The normalization constant and the dimensions of $\mathcal{E}_l$ are chosen without loss of generality in such a way that $\overline{\mathcal{E}_l^2(t)}$ is the average mode energy (here the bar denotes time averaging over a time long compared to $2\pi/\omega_l$). The parameters r, α, and γ are thus determined. This will be discussed in detail later in this paper.

We account for the random polarization of the inverted atomic population and that due to thermal fields by introducing a random stationary "force" $N(t)$

$$\frac{\partial^2 \mathcal{E}_l}{\partial t^2} + (r - \alpha)\frac{d\mathcal{E}_l}{dt} + \frac{\gamma}{3}\frac{d\mathcal{E}_l^3}{dt} + \omega_l^2\mathcal{E}_l = N(t) \tag{11}$$

$$\omega_l \simeq \omega_{\text{atomic}}.$$

We note that $r = \sigma/\epsilon$ is the rate of intensity decay. The saturable gain of the laser medium is accounted for by the term $\alpha - \gamma\epsilon^2$ in (11). We also note that $\alpha \propto \omega_l\chi_I^{(1)}(\nu)$ and $\gamma \propto \omega_l\chi_I^{(3)}(\nu)$. By ignoring the frequency dependence of α and γ in (11) we are limiting, at the outset, any results derived from the following analysis to spectral regions comparable to the laser gain linewidth $\Delta\omega_{\text{atomic}}$ [i.e., the width of $\chi_I(\nu)$]. Since the various spectral widths which characterize the laser spectrum turn out, as summarized in Fig. 2, to be considerably smaller than $\Delta\omega_{\text{atomic}}$, the neglect of the frequency dependence of α and γ is justified. A crucial point to the validity of the representation of the laser oscillation by (11) is that the saturation term has to be taken as $d\mathcal{E}_l^3/dt$ rather than $\overline{\mathcal{E}_l^2(t)}(d\mathcal{E}_l/dt)$ where the bar denotes time averaging.

Equation (11) is of the Van Der Pol form [7]. The nonlinear term involving $\mathcal{E}_l^3$ plays a key role in determining the spectral properties of the laser field. Without it the spectrum of $\mathcal{E}(t)$ will correspond simply to that of $N(t)$ going through a linear resonant filter with a Lorentzian bandwidth $(r - \alpha)$. This is essentially the case below threshold where the cubic term can be neglected. Above threshold, however, the nonlinear term determines the steady-state oscillation level as well as the nature of the laser "noise."

The Van Der Pol equation in the form of (11) has been considered in some detail by Caughey [7].

The field quantity $\mathcal{E}_l(t)$ is chosen without loss of generality in such a way that $\overline{\mathcal{E}_l^2(t)}$ is the average mode energy. The parameters r, α, γ, as well as the dimension of $N(t)$ are thus determined. These will be considered in detail later in this paper.

The laser field $\mathcal{E}_l(t)$ is taken in the form

$$\mathcal{E}_l(t) = E_0\cos\omega_l t + C_n(t)\cos\omega_l t + S_n(t)\sin\omega_l t \tag{12}$$

where $C_n(t)$ and $S_n(t)$ are the "slowly" varying fluctuation amplitudes. We will also assume in what follows that the laser is sufficiently above threshold so that $E_0^2 \gg C_n^2$, S_n^2. We substitute (12) into (11). In expanding the term $\gamma(d\mathcal{E}_l^3/dt)$ in (11) we discard nonsynchronous terms involving $\sin(3\omega_l t)$ and $\cos(3\omega_l t)$. We also neglect the terms which are proportional to $S_n^2E_0$, $C_n^2E_0$, S_n^3, and C_n^3, keeping only those proportional to E_0^3, $E_0^2S_n$, $E_0^2C_n$. In addition, we ignore $\ddot{C}_n$, $\ddot{S}_n$ but keep $\omega_l\dot{C}_n$ and $\omega_l\dot{S}_n$ consistent with our "slow" behavior assumption. The result after substantial algebra is

$$\begin{aligned}&\Big[-2\omega_l\dot{C}_n + (r-\alpha)(-\omega_lE_0 - \omega_lC_n + \dot{S}_n)\\ &\quad + \gamma\Big(-\frac{\omega_lE_0^3}{2} - \frac{3}{2}\omega_lE_0^2C_n\Big)\Big]\sin\omega_lt\\ &\quad + [2\omega_l\dot{S}_n + (r-\alpha)\omega_lS_n + (r-\alpha)\dot{C}_n + \tfrac{1}{2}\omega_l\gamma E_0^2S_n]\\ &\quad \cdot\cos\omega_lt = N(t). \qquad (13)\end{aligned}$$

Next we express the random "force" $N(t)$ as

$$N(t) = N_c(t)\cos\omega_lt + N_s(t)\sin\omega_lt \qquad (14)$$

where $N_c(t)$ and $N_s(t)$ are slowly varying random Gaussian fields. Equating cosine and sine terms on both sides of (13) we notice the existence inside the square brackets of terms which do not involve C_n or S_n. Equating their sum to zero gives

$$E_0^2 = \frac{4}{\gamma}(\alpha - r) \qquad (15)$$

for the average intensity.

Using (14) and (15) in (13) and equating cosine and sine terms separately gives, assuming $\omega_l \gg d/dt$,

$$\frac{dS_n}{dt} = \frac{N_c(t)}{2\omega_l} \qquad (16)$$

$$\frac{dC_n}{dt} + \frac{\gamma E_0^2}{4}C_n = \frac{N_s(t)}{2\omega_l}. \qquad (17)$$

Subject to the limitations previously mentioned, these remarkably simple equations describe the statistical nature of the laser field. These, together with (15), are our key working equations. They display a basic qualitative difference between the behavior of the quadrature component of the laser fluctuation field S_n and the in-phase component C_n. While the latter acts as a spring with a "spring constant" proportional to the intensity E_0^2, the first is formally equivalent to the displacement of a free particle with *no* restoring force. The intensity fluctuations spectrum, determined by C_n, will consequently become fainter and broader with increasing frequency while the frequency spectrum is flat. The detailed behavior of the various spectra is considered below.

The Spectrum of the Random Driving Force

From (16) and (17) it follows that the laser spectrum is determined by the spectra of the random driving forces $N_s(t)$ and $N_c(t)$ as defined by (14). Specifically, we need to know the spectral density functions $W_{N_c}(\omega)$ and $W_{N_s}(\omega)$ of $N_c(t)$ and $N_s(t)$.

We start by recognizing that (11) holds for a passive, i.e., unpumped resonator, where $\alpha = 0$ and $\gamma\mathcal{E}^2 \ll r$. In this case (11) becomes

$$\frac{\partial^2\mathcal{E}}{\partial t^2} + r\frac{\partial\mathcal{E}}{\partial t} + \omega_l^2\mathcal{E} = N(t). \qquad (18)$$

Taking the Fourier transform of (18) and solving for $\mathcal{E}(\omega)$ results in

$$\mathcal{E}(\omega) = \frac{N(\omega)}{\omega_l^2 - \omega^2 + i\omega r}. \qquad (19)$$

The average energy in the resonator mode is thus

$$\begin{aligned}E_{\text{mode}} = \overline{\mathcal{E}^2(t)} &= \lim_{T\to\infty}\frac{2\pi}{T}\int_{-\infty}^{\infty}|\mathcal{E}_T(\omega)|^2\,d\omega\\ &= \lim_{T\to\infty}\frac{4\pi}{T}\int_0^{\infty}\frac{|N_T(\omega)|^2\,d\omega}{|\omega_l^2-\omega^2+i\omega r|^2}. \qquad (20)\end{aligned}$$

Where we used $\mathcal{E}(\omega) = \mathcal{E}^*(-\omega)$ to limit the integration to real frequencies. The symbol T represents an integration time. Since in typical optical laser resonators $\omega_l > 10^6 r$, we replace the last term in (20) by

$$\frac{4\pi}{T}\left(\frac{1}{2\omega_l}\right)^2\int_0^{\infty}\frac{|N_T(\omega)|^2\,d\omega}{(\omega-\omega_l)^2 + \left(\frac{r}{2}\right)^2}. \qquad (21)$$

Assuming that $N_T(\omega)$ is nearly constant over a region of width r centered on ω_l, the integration yields

$$E_{\text{mode}} = \frac{\pi}{r}\left(\frac{1}{2\omega_l}\right)^2 W_N(\omega_l) \qquad (22)$$

where

$$W_N(\omega) \equiv \lim_{T\to\infty}\frac{4\pi}{T}|N_T(\omega)|^2 \qquad (23)$$

is the spectral density function of $N(t)$.[1]

The mode energy as given by (22) is equal to the thermal excitation energy of the mode so that

$$\frac{\pi}{r}\left(\frac{1}{2\omega_l}\right)^2 W_N(\omega_l) = \hbar\omega_l\left[\frac{1}{e^{\hbar\omega_l/kT}-1}+\frac{1}{2}\right] \equiv \hbar\omega_l n_{\text{th}} \qquad (24)$$

so that

$$W_N(\omega_l) = \frac{4r\omega_l^2}{\pi}\hbar\omega_l n_{\text{th}}. \qquad (25)$$

The last expression gives the thermal contribution to the spectral density function of the random driving force. The effect of an inverted population is to add a term proportional to the number N_2 of spontaneously emitting atoms. This term is similar to (25) except that here we

[1] The definition (23) of $W_N(\omega)$ and *all* our subsequent use of spectral densities is consistent with

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T|N(t)|^2\,dt = \int_0^{\infty}W_N(\omega)d\omega.$$

replace r by $-\alpha$, the atomic "loss" rate, and replace $e^{(\hbar\omega/kT)}$ by $(g_2/g_1)(N_1/N_2)$ [10] where N_1 and N_2 are the populations of the lower and upper laser levels, respectively, and g_1 and g_2 are the level degeneracies.

The inversion contribution to $W_N(\omega_l)$ is then

$$W_N(\omega_l)_{\text{atomic}} = \frac{4\alpha\omega_l^2}{\pi}\hbar\omega_l\left[\frac{N_2}{N_2 - N_1\dfrac{g_2}{g_1}}\right]. \tag{26}$$

Above threshold the population N_2 saturates at a value corresponding to $\alpha = r$ so that putting $\alpha = r$ in (26) and adding it to (25) results in

$$W_N(\omega_l) = \frac{4r\omega_l^2}{\pi}\hbar\omega_l\left[\frac{N_{2t}}{N_{2t} - N_{1t}\dfrac{g_2}{g_1}} + n_{\text{th}}\right]. \tag{27}$$

The frequency dependence of $W_N(\omega)$ can be recovered by noting that the spontaneous-emission contribution is proportional to the transition line-shape function $g(\omega)$ so that above threshold

$$W_N(\omega) = \frac{4r\omega_l^2}{\pi}\hbar\omega_l\left[\frac{N_{2t}}{N_{2t} - N_{1t}\dfrac{g_2}{g_1}}\frac{g(\omega)}{g(\omega_l)} + n_{\text{th}}\right]. \tag{28}$$

We will also need the spectral density functions of $N_c(t)$ and $N_s(t)$ as defined by (14). In a manner consistent with the last footnote these are[2]

$$W_{N_c}(\Omega) = W_{N_s}(\Omega) = 2W_N(\Omega + \omega_l). \tag{29}$$

Plots of $W_N(\omega)$, $W_{N_c}(\Omega)$, and $W_{N_s}(\Omega)$ are shown in Fig. 2(a) and (b).

III. The Spectral Density of the Laser Frequency

In this section we obtain an expression for the spectral density function $W_{\delta\omega}(\Omega)$ of the instantaneous frequency deviation $\delta\omega(t) \equiv \omega(t) - \omega_l$ of the laser. The spectrum $W_{\delta\omega}(\Omega)$ can be obtained experimentally by beating a test laser with a far stabler laser, feeding the difference frequency signal to a limiter–discriminator, and then performing a spectral analysis on the output. This was the method employed by Mannes and Siegman [11]. The resulting spectrum is of direct practical significance in determining the signal-to-noise ratio in frequency-modulated laser communication systems.

Starting with (12) and assuming $E_0 \gg C_n, S_n$, we can write the laser field as

$$\mathcal{E}(t) \approx [E_0 + C_n(t)]\cos[\omega_l t + \theta(t)] \tag{30}$$

where

$$\theta(t) \approx -\tan^{-1}\left(\frac{S_n(t)}{E_0}\right) \approx -\frac{S_n(t)}{E_0}. \tag{31}$$

[2] Here and in the following we reserve Ω to describe "low" frequencies, say, from 0 to 10^{13}, while ω is used to describe optical frequencies $\sim 10^{15}$.

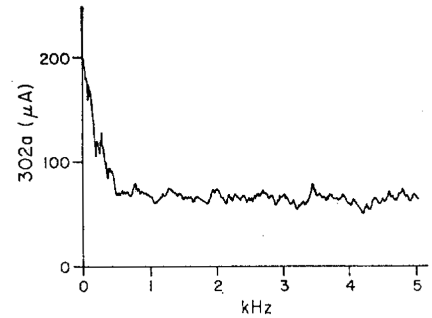

Fig. 3. The frequency-deviation spectrum $W_{\delta\omega}(\Omega)$ as observed by Mannes and Siegman [11].

The instantaneous phase is thus

$$\phi(t) = \omega_l t + \theta(t)$$

and the instantaneous frequency[3] is

$$\omega(t) \equiv \frac{d\phi}{dt} \approx \omega_l - \frac{\dot{S}_n(t)}{E_0}.$$

Using (16)

$$\delta\omega(t) \equiv \omega(t) - \omega_l = -\frac{1}{2\omega_l E_0}N_c(t)$$

so that

$$W_{\delta\omega}(\Omega) = \left(\frac{1}{2\omega_l E_0}\right)^2 W_{N_c}(\Omega). \tag{32}$$

The average power output of the atomic system, including that going to losses, is

$$P = \frac{rE_0^2}{2} \tag{32a}$$

which when used in (32) together with (28) and (29) gives

$$\begin{aligned} W_{\delta\omega}(\Omega) &= \frac{r^2}{\pi P}\hbar\omega_l\left[\frac{N_2}{N_2 - N_1\dfrac{g_2}{g_1}}\frac{g(\Omega+\omega_l)}{g(\omega_l)} + n_{\text{th}}\right] \\ &= \frac{\gamma r}{2\pi(\alpha - r)}\hbar\omega_l\left[\frac{N_2}{N_2 - N_1\dfrac{g_2}{g_1}}\frac{g(\Omega+\omega_l)}{g(\omega_l)} + n_{\text{th}}\right]. \end{aligned} \tag{33}$$

The frequency spectrum is thus flat up to $\Omega \sim \Delta\omega_{\text{atomic}}$ where the first term in the square brackets begins to drop off. The much smaller thermal contribution represented by n_{th} persists to higher frequencies. It is plotted, qualitatively, in Fig. 2(c).

The inverse depedence of $W_{\delta\omega}(\Omega)$ on P and the flat nature of the spectrum were observed in the experiment of Mannes and Siegman [11]. A representative sample of their data is shown in Fig. 3.

[3] The concept of instantaneous frequency is valid when the interval between successive zero crossings of $\mathcal{E}(t)$ remains a constant during, at least, a number of optical periods.

We will show, later in this paper, that γ is inversely proportional to the saturation intensity I_S of the laser medium, so that to minimize the FM noise one needs a laser with a small saturation intensity. In addition, the external output coupling should be reduced (thus minimizing r) and the pumping α maximized.

IV. Power Fluctuations Spectrum

Here we are considering the spectrum of the fluctuations in the power output of the laser, i.e., of the quantity.

$$\Delta P \equiv P(t) - P_{\text{average}}.$$

From (12) and the relation

$$P(t) = r\overline{\mathcal{E}^2(t)}$$

one obtains well above threshold ($E_0 \gg C_n, S_n$)

$$\Delta P(t) \approx rE_0C_n(t) \tag{34}$$

so that

$$W_{\Delta P}(\Omega) = r^2E_0^2W_{C_n}(\Omega). \tag{35}$$

$W_{C_n}(\Omega)$ can be obtained starting with (17). Taking the Fourier transform of that equation and solving for $C_n(\Omega)$ gives

$$C_n(\Omega) = \frac{N_s(\Omega)}{2\omega_l(i\Omega + \frac{1}{4}\gamma E_0^2)} \tag{36}$$

so that

$$W_{C_n}(\Omega) = \frac{W_{N_s}(\Omega)}{4\omega_l^2[\Omega^2 + (\frac{1}{4}\gamma E_0^2)^2]} \tag{37}$$

and from (28), (29), and (35)

$$W_{\Delta P}(\Omega) = \frac{r^2E_0^2W_{N_s}(\Omega)}{4\omega_l^2[\Omega^2 + (\frac{1}{4}\gamma E_0^2)^2]} \tag{38}$$

$$= \frac{2r^3E_0^2\hbar\omega_l}{\pi[\Omega^2 + (\frac{1}{4}\gamma E_0^2)^2]}\left[\frac{N_{2t}}{N_{2t} - N_1\frac{g_2}{g_{1t}}}\frac{g(\omega_l + \Omega)}{g(\omega_l)} + n_{\text{th}}\right]$$

$$\approx \frac{2r^3E_0^2\hbar\omega_l}{\pi[\Omega^2 + (\frac{1}{4}\gamma E_0^2)^2]}\left[\frac{N_{2t}}{N_{2t} - N_{1t}\frac{g_2}{g_1}} + n_{\text{th}}\right] \tag{39}$$

for $1/4\gamma E_0^2 \ll \Delta\omega_{\text{atomic}}$.

A qualitative plot of $W_{\Delta P}(\Omega)$ is shown in Fig. 2(d). Of particular interest are the values $W_{\Delta P}$ (0) which in a given laser is inversely proportional to the power and the half-frequency

$$\Omega_{1/2} = \frac{\gamma P}{2r} = \alpha - r$$

where P is the average power output. We note from (11) that $\alpha - r$ corresponds to the value of the temporal rate of increase of the stored energy in the oscillator without saturation. We thus find that while the average power is saturated the full unsaturated gain is still available to suppress fluctuation in the power.

The conclusions of this section are in agreement with the discussion of Haus [8] and with the experiments of Mannes and Siegman [11].

V. The Field Spectrum

Here we consider the spectral density function $W_{\mathcal{E}}(\omega)$ of the full laser field

$$\mathcal{E}(t) = E_0 \cos \omega_l t + C_n(t) \cos \omega_l t + S_n(t) \sin \omega_l t. \tag{40}$$

This is the spectrum that will be displayed by an experiment in which, conceptually, the laser field passes through a tunable Fabry–Perot high-finesse etalon and the output intensity is plotted versus the tuning frequency. Because of the narrow spectrum involved here this result is achieved in practice by first beating the test laser against a stable reference laser, which is offset slightly in frequency, and then performing a spectral analysis on the resulting beat signal [11].

The field $\mathcal{E}(t)$ can also be expressed, as in (30), by

$$\mathcal{E}(t) \approx [E_0 + C_n(t)] \cos [\omega_l t + \theta(t)]. \tag{41}$$

In the following we neglect the contribution to the field spectrum $W_{\mathcal{E}}(\omega)$ due to the term $C_n(t) \cos [\omega_l(t) + \theta(t)]$ relative to that of

$$f(t) = E_0 \cos [\omega_l t + \theta(t)]. \tag{42}$$

A simple consideration shows that the spectrum of the neglected term is broad (width $= 2\gamma E_0^2$), centered on ω_l and with a total area proportional to P^{-1}. The spectrum of (42), on the other hand, has a total area proportional to P and a width $\propto P^{-1}$. At $\omega = \omega_l$ the ratio of the spectra varies as P^3. The spectral density of the laser field is thus due mainly to the random nature of $\theta(t)$ in (42) and is consequently referred to, sometimes, as phase noise.

Returning to (41) we write it as

$$\mathcal{E}(t) \approx E_0 \cos [\omega_l t + \theta(t)]$$

$$= E_0 \text{ Re } [e^{i\omega_l t}e^{i\theta(t)}] = E_0 \text{ Re } [v(t)e^{i\omega_l t}]$$

where

$$v(t) \equiv e^{i\theta(t)}. \tag{43}$$

We first derive the spectrum of $v(t)$. According to Rowe [12] the autocorrelation function of $v(t)$ is [4]

$$C_v(\tau) \equiv \langle v(\tau)v^*(t + \tau)\rangle$$

$$= \exp\left[-2\int_0^\infty W_{\delta\omega}(\Omega)\left(\frac{\sin \Omega\tau/2}{\Omega/2\pi}\right)^2 d\Omega\right] \tag{44}$$

where $W_{\delta\omega}(\Omega)$ is given by (32) and (33). The last expression for $C_v(\tau)$ applies when the phase $\theta(t)$ is a random Gaussian process which is not, necessarily, stationary. Since $\theta(t) \approx -S_n(t)/E_0$, it follows from (16) that $\theta(t)$ is a Gaussian process being linearly related to the Gaussian force term $N_c(t)$.

[4] A factor of 2 difference between the exponent of (44) and the quoted result of Rowe [12] is due to our definition of $W_{\delta\omega}(\Omega)$ which is zero for $\Omega < 0$.

TABLE I

PARAMETERS DETERMINING THE NOISE PERFORMANCE OF SOME LOW-POWER LASERS

Laser	$V(m^3)$ $\sim (a \times b \times c)$	r(Hz)	$I_S = \dfrac{8\pi h\nu\, t_{spont}}{\lambda^2 g(\nu_0) t_2}$ (watts/m²)	$\Delta\nu_{laser}$ (Hz) and $\frac{1}{2} W_{\delta\omega}(0)$
CO_2 ($\lambda = 10.6\ \mu$, $\nu = 2.83 \times 10^{13}$) low pressure	10^{-5} ($0.5 \times 0.005 \times 0.005$)	($= -\frac{c}{L} \ln R$, $R = 0.99$) 6×10^6	800 ($\frac{t_2}{t_{spont}} \sim 3 \times 10^{-4}$)	1.5×10^{-3} at $\frac{\alpha}{r} = 2$
He - Ne ($\lambda = 0.6328\ \mu$, $\nu = 4.74 \times 10^{14}$)	4×10^{-7} ($0.5 \times 0.001 \times 0.001$)	($R = 0.995$) 3×10^6	2.9×10^4	7.7×10^{-3} at $\frac{\alpha}{r} = 2$
GaAs ($\lambda = 0.85\ \mu$, $\nu = 2.25 \times 10^{14}$)	1.5×10^{-15} $10^{-6} \times 50 \times 10^{-6} \times 300 \times 10^{-6}$	(r = Loss constant $\times$ c) 3×10^{11}	2.4×10^7	1.3×10^6 at $\frac{\alpha}{r} = 10$

Using (32) in (44) gives

$$C_v(\tau) = \exp\left(-\frac{\pi W_{N_c}(0)}{8\omega_l^2 E_0^2}|\tau|\right) \equiv e^{-S|\tau|} \tag{45}$$

$$\tau > \frac{1}{\Delta\omega_{\text{atomic}}}$$

where the limitation $\tau > \Delta\omega_{\text{atomic}}^{-1}$ enables us to treat $W_{\delta\omega}(\Omega)$ as a constant in the integration of (44).

The spectral density of $v(t)$ is

$$W_v(\Omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_v(\tau) e^{-i\Omega\tau}\, d\tau \tag{46}$$

$$= \frac{W_{N_c}(0)}{2\omega_l^2 E_0^2} \frac{1}{\left\{\left(\frac{\pi}{8\omega_l^2 E_0^2} W_{N_c}(0)\right)^2 + \Omega^2\right\}}. \tag{47}$$

From (43) it follows that the spectrum of the laser field is related to that of $v(t)$ by

$$W_\varepsilon(\omega) = \frac{E_0^2}{4} W_v(\omega - \omega_l)$$

so that

$$W_\varepsilon(\omega) = \frac{W_{N_c}(0)}{16\omega_l^2} \frac{1}{\left[\left(\frac{\pi}{8\omega_l^2 E_0^2} W_{N_c}(0)\right)^2 + (\omega - \omega_l)^2\right]} \tag{48}$$

and using (15), (32a), and (39)

$$W_\varepsilon(\omega) = \frac{r\hbar\omega/2\pi\left[\dfrac{N_{2t}}{N_{2t} - N_{1t}\dfrac{g_2}{g_1}} + n_{\text{th}}\right]}{\left[\left[\dfrac{N_{2t}}{N_{2t} - N_{1t}\dfrac{g_2}{g_1}} + n_{\text{th}}\right]^2 \left(\dfrac{r^2}{2P}\hbar\omega\right)^2 + (\omega - \omega_l)^2\right]}. \tag{49}$$

The field spectrum is plotted in Fig. 2(e) and has a Lorentzian shape with a full width at half maximum of

$$\Delta\omega_l = \frac{\pi W_{N_c}(0)}{4\omega_l^2 E_0^2} \tag{50}$$

or from (49)

$$\Delta\nu_l \equiv \frac{\Delta\omega_l}{2\pi} = \frac{2\pi(\Delta\nu_c)^2 h\nu_l}{P}\left[\frac{N_{2t}}{N_{2t} - N_{1t}\dfrac{g_2}{g_1}} + n_{\text{th}}\right] \tag{51}$$

where $\Delta\nu_c = r/2\pi$ is the full linewidth of the passive resonator.

This is the result obtained originally by Schawlow and Townes [1]. The numerical factor in (51) depends on the statistical nature of the random driving function and the type of laser broadening. Our result applies to the case of a band-limited random driving-noise term $N(t)$ in the limit of homogeneous broadening [11].

By comparing (49) to (29) and (32) we obtain the following relationship between $\Delta\nu_l$ and $W_{\delta\omega}(0)$

$$\Delta\nu_l = \tfrac{1}{2} W_{\delta\omega}(0). \tag{52}$$

This relation between the two spectra was noted experimentally [11].

VI. DISCUSSION

The noise characteristics of the laser are summarized in Fig. 2, which also contains the dependence of the peak values and the widths of the various noise spectra. It has become customary to express the dependence of these parameters on the laser power P as indicated in the figure. An alternative approach and one which affords a better physical insight is to express them as a function of the (passive) resonator loss rate r, the pumping ratio (the factor by which threshold pumping is exceeded) α/r, and the laser-medium saturation intensity I_S where [13]

$$I_S = \frac{8\pi h\nu t_{\text{spont}}}{\lambda^2 g(\nu) t_2}. \tag{53}$$

Comparing the conventional expression [13] for the power emitted by the atomic system $P = (Vr/c)I_S[(\alpha/r) - 1]$ to our result

$$P = \frac{E_0^2 r}{2} = \frac{4r^2}{\gamma}\left(\frac{\alpha}{r} - 1\right) \tag{54}$$

we obtain

$$\frac{\gamma}{r} = \frac{2c}{VI_S} \tag{55}$$

thus relating the parameter γ introduced in (10) to the saturation intensity I_S and the mode volume V. Using (55) we expressed in Fig. 2 the key parameters of the laser spectra as a function of I_S, r, and α/r. The laser linewidth $\Delta\nu_l$, and according to (52) 1/2 $W_{\delta\omega}(0)$ are expressible in this fashion as

$$\Delta\nu_l = \tfrac{1}{2}W_{\delta\omega}(0) = \frac{2ch\nu}{\pi VI_S}\left[\frac{N_{2t}}{N_{2t} - N_{1t}\dfrac{g_2}{g_1}} + n_{\text{th}}\right]\frac{r}{\left(\dfrac{\alpha}{r} - 1\right)} \tag{56}$$

which affords a different insight to the parametric dependence of $\Delta\nu_l$ and $W_{\delta\omega}$.

A tabulation of typical values of these parameters in some lasers is given in Table I.

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