Internal Modulation in Multimode Laser Oscillators*†

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Multimode laser oscillation in the presence of internal resonator modulation is discussed. Modulation of the dielectric constant (ϵ) is shown to lead to frequency modulation of the laser light while modulation of the losses (or gain) leads to amplitude modulation. The results are discussed with particular reference to recent experiments and proposals involving modulation inside laser resonators. The mathematical formalism is that of normal-mode parametric interactions.

I. INTRODUCTION

A NUMBER of recent experiments and proposals[1–9] involve a modulation of some parameter within a laser resonator.

The modulation schemes can take various forms and lead, consequently, to different experimental results. From the point of view of the laser optical resonator, however, we can classify most of these schemes into two broad categories: (1) Modulation (in time and space) of the dielectric constant ϵ.[10] We shall refer to these as "reactive," or "parametric, modulation." (2) Modulation of the loss or gain of the laser medium—to be lumped under the term "loss modulation."

We present below some of the pertinent results of a rather lengthy analysis[11] which are necessary for the interpretation of current experimental work.

The starting point of view is similar to that adopted by Louisell *et al.*[12] The mode spectrum of the resonator is describable by an orthonormal set of electric and magnetic vector functions $\mathbf{E}_a(\mathbf{r})$ and $\mathbf{H}_a(\mathbf{r})$, where a is the running mode index number. Ordinarily the total field in the resonator can be expressed as $\mathbf{E}(\mathbf{r},t) = \sum_a A_a \cos(\omega_a t)\mathbf{E}_a(\mathbf{r})$ with arbitrary A_a. Once we allow ϵ(or μ) or σ to vary with time the situation is altered. We can still expand the field in terms of $\mathbf{E}_a(\mathbf{r})$ and $\mathbf{H}_a(\mathbf{r})$, but the multiplying coefficients must be time dependent. The nature of the time dependence is determined by the type of parameter modulation and its solution is the main concern of this paper.

In physical terms the time variation of the, otherwise constant, coefficients corresponds to energy exchange between various modes at different frequencies. This exchange has a number of interesting consequences, some of which will be treated below.

The total electric and magnetic fields are written as

$$\mathbf{E}(\mathbf{r},t) = -\sum_a \epsilon^{-\frac{1}{2}} p_a(t)\mathbf{E}_a(\mathbf{r}), \tag{1}$$

$$\mathbf{H}(\mathbf{r},t) = \sum_a \mu^{-\frac{1}{2}} \omega_a q_a(t)\mathbf{H}_a(\mathbf{r}). \tag{2}$$

$\mathbf{E}_a(\mathbf{r})$ and $\mathbf{H}_a(\mathbf{r})$ are the orthonormal vector functions for the ath resonator mode. ω_a is the (radian) resonance frequency of the mode and $p_a(t)$ and $q_a(t)$ are the time-dependent coefficients described above. In the absence of modulation these coefficients correspond to harmonic oscillation at a frequency $\omega_a/2\pi$.

The analysis is facilitated greatly by transforming to a "normal-mode" set of coordinates $c_a(t)$ and its complex conjugate $c_a^*(t)$ defined by[11]

$$c_a(t) = [1/(2\omega_a)^{\frac{1}{2}}][\omega_a q_a(t) + j p_a(t)], \tag{3}$$

to which we will refer as the mode amplitudes. These contain all the phase and amplitude information of the individual modes. Another convenient feature of the mode amplitudes is that the expression for the total energy in the resonator takes the simple form

$$E_{\text{total}} = \sum_a \omega_a c_a c_a^*, \tag{4}$$

so that $c_a c_a^*$ is proportional to the number of photons in the ath mode (which is given by $c_a c_a^*/\hbar$).

II. REACTIVE MODULATION

The reactive modulation is introduced by allowing the dielectric constant to vary in time and space as

$$\epsilon(\mathbf{r},t) = \epsilon_0 + \epsilon(\mathbf{r})\cos(\omega_m t + \phi), \tag{5}$$

assuming that we are dealing with an infinite mode spectrum of equal frequency spacing it can be shown,[11,12] by substituting Eqs. (1), (2), (3), and (5) in Maxwell equations, that sustained, nonfluctuating, interaction between modes can only take place when the modula-

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[10] Modulation of μ is treated in the same way as that of ϵ and will, consequently, not be discussed separately.

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tion frequency $\omega_m/2\pi$ is nearly equal to some multiple of the basic mode spacing. Under these conditions the mode amplitudes obey the recurrence equation

$$\frac{dc_a^*}{dt}=[j\omega_a-\sigma_0(\omega_a)/2\epsilon_0]c_a^*+j(\kappa/2)e^{-j(\omega_m t+\phi)}c_{a+1}^*+j(\kappa/2)e^{+j(\omega_m t+\phi)}c_{a-1}^*, \quad (6)$$

where $\sigma_0(\omega)$ is the frequency-dependent conductivity of the medium, which is negative for amplifying media, and κ is the parametric coupling coefficient given by

$$\kappa=(\omega_a/2\epsilon_0)\int_v \epsilon(\mathbf{r})\mathbf{E}_a(\mathbf{r})\cdot\mathbf{E}_{a+1}(\mathbf{r})dv,$$

$$\omega_m=\omega_{a+1}-\omega_a\ll\omega_a. \quad (7)$$

The frequency requirement mentioned above takes the form $\omega_m=\omega_{a+1}-\omega_a$. This assures conservation of energy.

It should be noted that Eq. (6) when specialized for three frequencies only, i.e., ω_1, ω_2, and $\omega_m=\omega_1-\omega_2$ describes the case of parametric up-conversion from ω_2 to ω_1. The same formalism yields the equations describing parametric amplification (or oscillation) which takes place when $\omega_m=\omega_1+\omega_2$. These cases have been treated in Ref. 12 and will not be discussed further here. We will return to Eq. (6) since our main interest in this paper is in cases where $\omega_m\ll\omega_a$. This is the case when ω_m corresponds, for instance, to a microwave signal and the ω_a's to the equispaced longitudinal mode resonances of an optical resonator. Equation (6) can be solved exactly when the losses (or gains) are neglected or assumed equal for all the modes.

For the case of no losses the solution is[11]

$$c_n^*(t)=c_0^*(0)\exp\{j[(\omega_0+n\omega_m)t+n(\phi+\pi/2)]\}J_n(\kappa t), \quad (8)$$

which has the form of an FM signal with a carrier frequency $\omega_0/2\pi$ and a modulation index κt.

A consequence of (8) is that if at $t=0$ only one mode ($n=0$) oscillates, parametric pumping; as in (5), will cause a transfer of energy into new frequencies belonging to modes with indices $n\neq 0$. The growth of these other modes is given by (8). Note that for $\kappa t\ll 1$ the transfer is limited to the $n=\pm 1$ modes, i.e., to the two modes separated from the oscillating $n=0$ mode by the modulation frequency $\omega_m/2\pi$.

Parametric frequency up-conversion and frequency modulation of a laser oscillator based on these ideas has been demonstrated by Peterson and Yariv.[9] Electro-optic modulation of ϵ at 8.9 Gc/sec was used to transfer energy from the 6328-Å oscillation of a He–Ne gas laser to a new frequency shifted upwards or downwards by 8.9 Gc/sec.

The frequency modulated signal described by (8) is one in which the modulation index is time dependent. A steady-state FM laser oscillation is described in Sec. IV.

III. LOSS MODULATION

Recent experiments by Gürs and Müller,[1] and by Hargrove, Fork, and Pollack[8] employed loss modulation within a laser resonator.

The formalism for this case is similar to that of reactive modulation. The loss modulation is introduced by allowing the conductivity σ to vary with space and time as

$$\sigma(\mathbf{r},t)=\sigma_0+\sigma_1(\mathbf{r})\cos(\omega_m t+\phi). \quad (9)$$

Using the same assumptions which were employed in deriving Eq. (6) we get

$$dc_a^*/dt=(j\omega_a-\sigma_0/2\epsilon_0)c_a^*-(\kappa/2)e^{-j(\omega_m t+\phi)}c_{a+1}^*-(\kappa/2)e^{+j(\omega_m t+\phi)}c_{a-1}^*, \quad (10)$$

where κ is given by

$$\kappa=(1/2\epsilon_0)\int_v \sigma_1(\mathbf{r})\mathbf{E}_a(\mathbf{r})\cdot\mathbf{E}_{a+1}(\mathbf{r})dv. \quad (11)$$

For the case when only the mode $n=0$ is excited (with unity amplitude) at $t=0$, the solution of (10) is

$$c_n^*(t)=I_n(\kappa t)\exp\{j[(\omega_0+n\omega_m)t+n(\pi+\phi)]\}\times\exp[-(\sigma_0/2\epsilon_0)t], \quad (12)$$

where $I_n(\kappa t)$ is the hyperbolic Bessel function of order n and argument κt. In the limit of $\kappa t\sim\infty$, Eq. (12) becomes

$$c_n^*(t)\underset{\kappa t\to\infty}{\longrightarrow}(2\pi\kappa t)^{-\frac{1}{2}}\exp\{j[(\omega_0+n\omega_m)t+n(\pi+\phi)]\}\times\exp[\kappa-(\sigma_0/2\epsilon_0)]t. \quad (13)$$

The factor $n\phi$ in Eq. (13) is of no physical significance and can be removed by shifting the time origin by $(\pi+\phi)/\omega_m$. The result is

$$c_n^*(t)\underset{\kappa t\to\infty}{\longrightarrow}(2\pi\kappa t)^{-\frac{1}{2}}\exp\{j[(\omega_0+n\omega_m)t-(\omega_0/\omega_m)(\pi+\phi)]\}\times\exp[\kappa-(\sigma_0/2\epsilon_0)]t. \quad (14)$$

In a laser oscillator $\sigma_0(\omega)<0$ and the waves are growing with time. This growth will, in real laser, be arrested by gain saturation. The most significant feature of (14), from the point of view of the present discussion, is the fact that all the modes are "locked" into an oscillation with the *same phase* and, as a result of the constant gain assumption, with the same amplitude.

The implications of the phase and amplitude locking can be deduced from a consideration of the net instantaneous field $C_n(t)$ resulting from phase locking, as in Eq. (14), of n equally spaced (in frequency) and equal amplitude modes

$$C_n(t)=\sum_{a=-\frac{1}{2}(n-1)}^{\frac{1}{2}(n-1)}e^{j(\omega_0+a\omega_m)t}=e^{j\omega_0 t}\frac{2\sin(n\omega_m t/2)\sin(\omega_m t/2)}{1-\cos\omega_m t}. \quad (15)$$

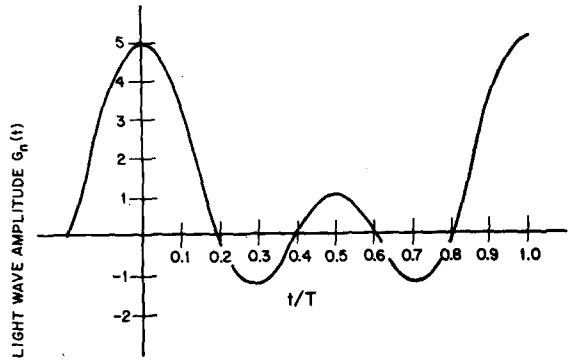

FIG. 1. A plot of one period of the light envelope function $G_n(t)$ resulting from phase locking of five ($n=5$) equal-amplitude waves of frequency spacing T^{-1}.

The envelope function $G_n(t)=2\sin(n\omega_m t/2)\sin(\omega_m t/2)(1-\cos\omega_m t)^{-1}$ is plotted in Fig. 1 for $n=5$ (five modes). Some of the properties of $G_n(t)$ are:

(1) For large n the oscillation envelope $G_n(t)$ approaches the form of a pulse train with a period $T=2\pi/\omega_m$ where $\omega_m/2\pi$ is the frequency of the modulation signal (which is equal to some multiple of the longitudinal mode spacing $C/2L$).

(2) The pulse width (to the first zero) is $\tau=T/n$.

(3) The peak field amplitude is equal to n times the individual mode amplitudes.

(4) The peak power is n times the average power.

In the case of a laser oscillator the number of oscillating modes is limited by the gain profile of width $\Delta\nu_D$ to $n\approx 2\Delta\nu_D LC^{-1}$. Applying a phase-locking signal at $f_m=C/2L$ results in a pulse width $\tau\approx\Delta\nu_D{}^{-1}$. For a visible gas laser with $\Delta\nu_D\approx 1.5$ Gc/sec the pulse width is $\tau\approx 6\times 10^{-10}$ sec.

The theoretical predictions summarized above are in substantial agreement with the reported experiments of Hargrove *et al.*[8]

In order to satisfy ourselves that the conclusions drawn from the analytical solution of Eq. (10) do not depend significantly on the assumed initial conditions or on the idealized model used, we resorted to a numerical (computer) solution. The problem solved consisted of 5 modes with a saturable Gaussian gain curve coupled through a fractional loss modulation of 0.3. Phase locking to within $\pm 0.5°$ and amplitude stability to within $\pm 0.01\%$ were reached in $2\kappa^{-1}$ sec *regardless* of the assumed initial amplitudes and phases.

IV. FREQUENCY PULLING AND STEADY-STATE MODULATION

Since in practice it is impossible to keep the modulation frequency equal exactly to the intermode spacing (or a multiple thereof), we investigate the case when the modulation frequency deviates slightly from its ideal value. We denote this deviation by $\Delta\omega$, i.e.,

$$\omega_{a+1}-\omega_a=\omega_m-\Delta\omega. \tag{16}$$

Neglecting losses (i.e., $\sigma_0=0$), defining $D_a{}^*(t)$ by

$$C_a{}^*(t)=D_a{}^*(t)\exp\{j[(\omega_a+a\Delta\omega)t-a\phi]\}, \tag{17}$$

and substituting in (6) yields

$$(dD_a{}^*/dt)+ja\Delta\omega D_a{}^*=j(\kappa/2)D_{a-1}{}^*+j(\kappa/2)D_{a+1}{}^*. \tag{18}$$

The steady-state solution of (18) is obtained by putting $dD_a{}^*/dt=0$. Equation (18) can then be identified, in a manner similar to that leading to (8), with a familiar Bessel recurrence equation whose solution is

$$D_a{}^*=J_a(\kappa/\Delta\omega)$$

which upon substitution in (17) gives

$$C_a{}^*(t)=J_a(\kappa/\Delta\omega)\exp\{j[(\omega_a+a\Delta\omega)t-a\phi]\}, \tag{19}$$
$$a=0,\pm 1,\pm 2,\cdots.$$

This corresponds to a steady-state FM laser oscillation with a modulation index $\kappa/\Delta\omega$. The resonance frequency of the ath mode has shifted from ω_a to $\omega_a+a\Delta\omega$. The new intermode frequency spacing is seen, using (16), to be equal to $\omega_m/2\pi$, so that deviation from resonance, $|\Delta\omega|\to 0$, causes a pulling of the laser frequencies. It should also be noted that the modulation phase ϕ is contained (as $a\phi$) in each mode. It can be recovered by mixing any two adjacent modes. This may prove significant in certain phase modulation applications. For practical purposes we may add that according to (7) κ has a maximum value of $\omega(\delta\epsilon/\epsilon)(d/2L)$, where ω is the laser radian frequency, d the distance over which $\delta\epsilon\neq 0$, and L is the resonator length.

If the frequency modulation is realized by applying a field $E_0\cos\omega_m t$ to an electro-optic crystal,[5,9] the perturbation in ϵ is $\delta\epsilon/2\epsilon=m^2\Gamma E_0/2$, where Γ is the electro-optic coefficient ($\sim 10^{-9}$, cm/V in KDP) and n is the index of refraction.

We can likewise investigate the effects of frequency deviation on mode coupling by loss modulation.[8] We assign to $\Delta\omega$ the same significance as that of Eq. (16), define $D_a{}^*(t)$ by

$$C_a{}^*(t)=D_a{}^*(t)\exp\{j[(\omega_a+a\Delta\omega)-a(\phi-\pi/2)]\}, \tag{20}$$

put $\sigma_0=0$, and substitute in Eq. (10). The steady-state ($dD_a{}^*/dt=0$) equation is

$$(2\Delta\omega a/\kappa)D_a{}^*=D_{a-1}{}^*-D_{a+1}{}^*, \tag{21}$$

whose solution is

$$D_a{}^*=I_a(\kappa/\Delta\omega), \tag{22}$$

where I_a is the hyperbolic Bessel function. In the limit of $\kappa\gg\Delta\omega$, $D_a{}^*$ becomes

$$D_a{}^*(t)\to[2\pi(\kappa/\Delta\omega)]^{-\frac{1}{2}}\exp(\kappa/\Delta\omega),\quad \kappa/\Delta\omega\gg 1. \tag{23}$$

In this case, again, a steady-state solution is obtained for off-resonance operation. In practice[8] this condition

is probably the one actually prevailing. Note that loss modulation, unlike ϵ modulation, tends to keep the modes oscillating with equal amplitudes. The relative phases are, as in (12), AM-like. The mode frequencies are again pulled so that their difference is equal to $\omega_m/2\pi$.

Complete derivations of the results quoted above and the results of numerical calculations will be published separately.[11]

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Stopping Cross Section of Low Atomic Number Materials for He^+, 65–180 keV

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Stopping cross sections were measured for H_1^+ in carbon and for He^+ in carbon, aluminum, and chromium over the energy range of 65 to 180 keV. Layers of the stopping materials were evaporated *in vacuo* onto a gold-plated quartz crystal and their areal densities were computed from the shift in the resonant frequency of the crystal. The energy lost by the ions in the stopping material was determined by measuring the energy of the scattered ions. Experimental results show a divergence from available theory above 80 keV.

INTRODUCTION

THERE are many areas of physics where a knowledge of stopping cross sections is important. In particular, they aid in the determination of the screened-interaction potential which is fundamental to the theoretical understanding of radiation damage and sputtering phenomena. Accurate stopping cross sections are also required for the proper interpretation of various nuclear reaction cross-section measurements. Finally, recent calculations[1] related to the differential energy loss suffered by low-energy ions have reached the point where detailed comparison with experiment is desirable.

The present experiments were undertaken to provide absolute stopping cross-section measurements for selected ion-medium combinations in the range of ion energy from 65 to 180 keV where data are relatively sparse or nonexistent.

EXPERIMENTAL PROCEDURE

The stopping cross section per atom, S_0 of a substance for a penetrating particle is defined by the equation

$$S_0 = -(1/N)(dE/dR),$$

where N is the number of stopping atoms per unit volume of the substance and dE is the energy lost by the particle in traversing a distance dR through the stopping substance.

To measure stopping cross sections directly, two quantities must be determined: $N\Delta R$, which is related to the mass per unit area of the stopping medium parallel to the path of the penetrating ions, and ΔE the amount of energy lost by the ions as they pass through this amount of material. In this experiment, ΔE was measured with a high-resolution (0.1%) electrostatic energy analyzer operated symmetrically. The quantity $N\Delta R$ was determined by noting the change in the fundamental frequency of a piezoelectric quartz crystal as a layer of the stopping material was evaporated onto one of its faces.

Apparatus

Positive ion beams were generated and mass separated in a 50- to 200-keV Cockcroft–Walton accelerator equipped with a Von Ardenne ion source and a 19.5° magnetic analyzer. The ions were collimated onto the target through a series of apertures that reduced the beam diameter to 1.27 mm (Fig. 1) and the current to about 0.2 μA. The target was located at the first focus of the 117° energy analyzer. Particles scattered by the target at an angle of 160° entered the analyzer through slit s_1 which was biased to -300 V to suppress secondary electrons. Ions selected by the analyzer were detected by a NaI photomultiplier combination behind exit slit s_2 located at the second focus.

The pressure in the target chamber was maintained at less than 5×10^{-6} Torr with cryogenically baffled diffusion pumps.

Targets

Targets were prepared on commercially available AT cut crystals that had a natural frequency of 7.5 Mc/sec. The crystals were quartz disks of 1.34 cm in diameter with a matte finish. Gold electrodes, approximately equal in thickness, were evaporated onto

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