A NOTE CONCERNING A GYROELECTRIC MEDIUM

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Abstract

The fact that a homogeneous electron gas when immersed in a uniform magnetostatic field becomes electrically anisotropic, i.e., gyroelectric, is placed in evidence. The permeability of the gas remains equal to that of free space, but its dielectric constant is transformed to a dyadic or tensor upon application of the magnetostatic field. The properties of the dielectric tensor are such that a plane electromagnetic wave propagating through such a medium undergoes a Faraday rotation. This rotation is the dual of the Faraday rotation produced by gyromagnetic media.

The dielectric tensor of the electron gas is deduced and the Faraday rotation constant is calculated.

Introduction

It is well known that an idealized electron gas has a permeability μ_o and a dielectric constant $\varepsilon = \varepsilon_o (1 - \omega_o^2/\omega^2)$ where μ_o and ε_o are the permeability and dielectric constant of free space, ω_o is the plasma frequency, and ω is the frequency of the wave propagating through the gas.[1]

However, when a uniform magnetostatic field is applied, the gas becomes electrically anisotropic. This anisotropy is due to the fact that when an electromagnetic wave propagates through such a medium the convection current of electrons has not the same direction as the electric vector of the wave. Consequently in such a medium $\underline{B} = \mu_o \underline{H}$ and $\underline{D} = (\varepsilon) \cdot \underline{E}$ where (ε) is the dielectric tensor.

The calculation of the components of the dielectric tensor consists of first calculating the velocity of the electrons in terms of the electric vector $\underline{E}$ of the wave propagating through the medium, and then, from a knowledge of this velocity, deducing the convection current of the electrons. By adding to this convection current the free space displacement current of the wave, the total current density is obtained. The total current density is thought of as a displacement current in an anisotropic dielectric, and thus the components of (ε) are calculated.

And from the components of (ε) expressed in terms of ω, ω_o, and ω_g where ω_g is the gyrofrequency, in a straightforward manner the properties of a plane electromagnetic wave traveling through such a medium in an arbitrary direction with respect to the magnetostatic field are determined.(2)

The direction of propagation of importance to microwave applications is the one in which the wave travels parallel to the applied magnetostatic field because it is in this direction that the gas produces a Faraday rotation of the traveling wave. The Faraday rotation factor is easily determined from the fact that the two waves which travel in this parallel direction are circularly polarized in opposite senses and have unequal velocities. And hence a superposition of the two yields a "linearly" polarized wave which rotates about the direction of the applied magnetostatic field.

And, of course, for propagation perpendicular to the direction of the applied magnetostatic field the Cotton-Mouton effect(3) is exhibited.

Since an electron gas with applied magnetic field produces Faraday rotation, it can be used as a nonreciprocal microwave circuit component.[4] This aspect of the problem will be discussed elsewhere in detail.

Dielectric Tensor of an Electron Gas with Applied Magnetostatic Field.

We deduce the dielectric tensor of a medium consisting of a homogeneous electron gas immersed in a magnetostatic field $\underline{B}_o$. When a plane electromagnetic wave whose electric and magnetic vectors are respectively $\underline{E}$ and $\underline{H}$ travels through the medium, each electron is subjected to a force $\underline{F}$ which depends on the electronic charge q and the electronic velocity $\underline{v}$ according to the well-known relation of Lorentz:

$$\underline{F} = q\,\underline{E} + q\,\mu_o\,\underline{v} \times \underline{H} + q\,\underline{v} \times \underline{B}_o \tag{1}$$

wherein μ_o is the permeability of free space. Applying Newton's law to each electron of mass m and neglecting the second term on the right side of (1) we obtain

$$m\,\frac{d}{dt}\underline{v} = q\,\underline{E} + q\,\underline{v} \times \underline{B}_o \tag{2}$$

In this equation $\underline{v}$ and $\underline{E}$ are real vector functions of space and time, and $\underline{B}_o$ is spatially uniform and independent of time.

Our first task is to solve (2) for the velocity $\underline{v}$. To do this we differentiate (2) with respect to time,

$$m\,\frac{d^2}{dt^2}\,\underline{v} = q\,\frac{d}{dt}\underline{E} + q\,\frac{d}{dt}\underline{v} \times \underline{B}_o\,, \tag{3}$$

and then postmultiply vectorially by $\underline{B}_o$,

$$m \frac{d^2}{dt^2} \underline{v} \times \underline{B}_o = q \frac{d}{dt} \underline{E} \times \underline{B}_o + q(\frac{d}{dt} \underline{v} \times \underline{B}_o) \times \underline{B}_o \tag{4}$$

Multiplying (4) by q/m^2 and using a well-known vector identity to transform the right side, we get

$$\frac{q}{m} \frac{d^2}{dt^2} \underline{v} \times \underline{B}_o = \frac{q^2}{m^2} \frac{d}{dt} \underline{E} \times \underline{B}_o + \frac{q^2}{m^2}(\frac{d}{dt} \underline{v} \cdot \underline{B}_o) \underline{B}_o - \frac{q^2}{m^2} (\underline{B}_o \cdot \underline{B}_o) \frac{d}{dt} \underline{v} . \tag{5}$$

Multiplying (2) scalarly by $\underline{B}_o$ we get

$$m \frac{d}{dt} \underline{v} \cdot \underline{B}_o = q \underline{E} \cdot \underline{B}_o , \tag{6}$$

from which it follows after multiplication by $\underline{B}_o q^2/m^3$ that

$$\frac{q^2}{m^2} (\frac{d}{dt} \underline{v} \cdot \underline{B}_o) \underline{B}_o = \frac{q^3}{m^3} (\underline{E} \cdot \underline{B}_o) \underline{B}_o . \tag{7}$$

And operating on (2) with $\frac{1}{m} \frac{\partial^2}{\partial t^2}$ we get

$$\frac{d^3}{dt^3} \underline{v} = \frac{q}{m} \frac{d^2}{dt^2} \underline{E} + \frac{q}{m} \frac{d^2}{dt^2} \underline{v} \times \underline{B}_o . \tag{8}$$

We note that the second term on the right side of (8) is identical to the term on the left side of (5). And the second term on the right side of (5) is identical to the left side of (7). With these observations it is clear that (5), (7), and (8) yield

$$\frac{d^3}{dt^3} \underline{v} + (\frac{q}{m} \underline{B}_o)^2 \frac{d}{dt} \underline{v} = \frac{q}{m} \frac{d^2}{dt^2} \underline{E} + \frac{q^2}{m^2} \frac{d}{dt} \underline{E} \times \underline{B}_o + \frac{q^3}{m^3} (\underline{E} \cdot \underline{B}_o) \underline{B}_o . \tag{9}$$

Since no term in (9) contains a product of time-dependent functions, we are free to restrict the time dependence to $e^{-i\omega t}$ by replacing $\partial/\partial t$ by $-i\omega$, etc. Thus (9) becomes

$$\underline{v}\left\{i\omega^3 - i\omega\left(\frac{q}{m}\underline{B}_o\right)^2\right\} = -\omega^2\frac{q}{m}\underline{E} - i\omega\frac{q^2}{m^2}\underline{E}\times\underline{B}_o + \frac{q^3}{m^3}(\underline{E}\cdot\underline{B}_o)\underline{B}_o \ . \qquad (10)$$

The term $(\frac{q}{m}\underline{B}_o)^2$ is a scalar and equal to the square of the gyrofrequency ω_g, i.e.,

$$\omega_g^2 = \left(\frac{q}{m}\underline{B}_o\right)\cdot\left(\frac{q}{m}\underline{B}_o\right) . \qquad (11)$$

Without loss of generality we introduce a rectangular coordinate system so oriented that $\underline{B}_o$ lies along the z-axis. If we denote the unit vectors along the coordinates axis by $\underline{a}_x$, $\underline{a}_y$, and $\underline{a}_z$, we can write $\underline{B}_o = \underline{a}_z B_o$, $\underline{E}\cdot\underline{B}_o = E_z B_o$, and $\underline{E}\times\underline{B}_o = \underline{a}_x E_y B_o - \underline{a}_y E_x B_o$. And if we define the plasma frequency ω_o by

$$\omega_o^2 = \frac{Nq^2}{m\varepsilon_o} \qquad (12)$$

where N is the number of electrons per unit volume, and ε_o is the dielectric constant of vacuum, we can write (10) in the following form:

$$Nq\,\underline{v} = -i\omega\left\{-\frac{\varepsilon_o\omega_o^2}{\omega^2-\omega_g^2}\underline{E} + \frac{\varepsilon_o\omega_o^2\omega_g}{i\omega(\omega^2-\omega_g^2)}(\underline{a}_x E_y - \underline{a}_y E_x) + \frac{\varepsilon_o\omega_o^2\omega_g^2}{\omega^2(\omega^2-\omega_g^2)}\underline{a}_z E_z\right\} \qquad (13)$$

$Nq\,\underline{v}$ is the convection current and to it we must add the displacement current $-i\omega\,\varepsilon_o\underline{E}$ in order to obtain the total current $\underline{J}$, i.e.,

$$\underline{J} = -i\omega\,\varepsilon_o\underline{E} + Nq\,\underline{v} \ . \qquad (14)$$

The x, y, and z components of $\underline{J}$ are easily obtained from (13) and (14). They are

$$J_x = -i\omega \left[\mathcal{E}_o \left(1 - \frac{\omega_o^2}{\omega^2 - \omega_g^2}\right) E_x - i\, \mathcal{E}_o \frac{\omega_o^2\, \omega_g}{\omega(\omega^2 - \omega_g^2)} E_y \right]$$

$$J_y = -i\omega \left[i\, \mathcal{E}_o \frac{\omega_o^2\, \omega_g}{\omega(\omega^2 - \omega_g^2)} E_x + \mathcal{E}_o \left(1 - \frac{\omega_o^2}{\omega^2 - \omega_g^2}\right) E_y \right] \tag{15}$$

$$J_z = -i\omega \left[\mathcal{E}_o \left(1 - \frac{\omega_o^2}{\omega^2}\right) E_z \right]$$

It is suggestive to write (15) in the following manner:

$$J_x = -i\omega\, \mathcal{E}_{xx} E_x - i\omega\, \mathcal{E}_{xy} E_y$$

$$J_y = -i\omega\, \mathcal{E}_{yx} E_x - i\omega\, \mathcal{E}_{yy} E_y \tag{16}$$

$$J_z = -i\omega\, \mathcal{E}_{zz} E_z$$

where

$$\mathcal{E}_{xx} = \mathcal{E}_o \left(1 - \frac{\omega_o^2}{\omega^2 - \omega_g^2}\right) = \mathcal{E}_{yy}$$

$$\mathcal{E}_{xy} = -i\, \mathcal{E}_o \frac{\omega_o^2\, \omega_g}{\omega(\omega^2 - \omega_g^2)} = -\mathcal{E}_{yx} \tag{17}$$

$$\mathcal{E}_{zz} = \mathcal{E}_o \left(1 - \frac{\omega_o^2}{\omega^2}\right)$$

We can express (16) as a tensor equation:

$$\underline{J} = -i\omega(\mathcal{E}) \cdot \underline{E} \tag{18}$$

where $(\mathcal{E})$ is the dielectric tensor whose matrix is

$$(\varepsilon) = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \qquad (19)$$

It is important to note that $\varepsilon_{xx} = \varepsilon_{yy}$ and $\varepsilon_{xy} = -\varepsilon_{yx}$ and that the diagonal components are purely real whereas the off-diagonal components are purely imaginary. To place this in evidence we let

$$\varepsilon_{xy} = i\,\varepsilon_{xy}' \quad \text{and} \quad \varepsilon_{yx} = -i\,\varepsilon_{yx}' \qquad (20)$$

where ε_{xy}' and ε_{yx}' are purely real. Consequently (19) becomes

$$(\varepsilon) = \begin{pmatrix} \varepsilon_{xx} & i\varepsilon_{xy}' & 0 \\ -i\varepsilon_{yx}' & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \qquad (21)$$

where

$$\varepsilon_{xy}' = -\varepsilon_o \frac{\omega_o^2\ \omega_g}{\omega(\omega^2 - \omega_g^2)} = \varepsilon_{yx}' \,. \qquad (22)$$

When the applied magnetostatic field vanishes $(B_o \rightarrow 0)$ we see that $\varepsilon_{xy}' \rightarrow 0$, $\varepsilon_{yx}' \rightarrow 0$, and the diagonal terms become equal to $\varepsilon_o(1 - \omega_o^2/\omega^2)$. The electron gas is isotropic when $B_o = 0$.

Wave Propagation in Tensor Dielectric Medium

Now we investigate the propagation of a plane electromagnetic wave through a medium whose permeability is μ_o and whose dielectric constant is (ε). As shown in Figure 1, $\underline{n}$ denotes a unit vector in the direction of wave propagation. We let $\underline{k}$ denote the vector propagation constant, and hence $\underline{k} = \underline{n}\frac{\omega}{v}$ where v is the phase velocity. Also we let $\underline{r}$ be a position vector, i.e., $\underline{r} = \underline{a}_x x + \underline{a}_y y + \underline{a}_z z$. The electric vector of a plane wave traveling in the direction $\underline{n}$ has the form

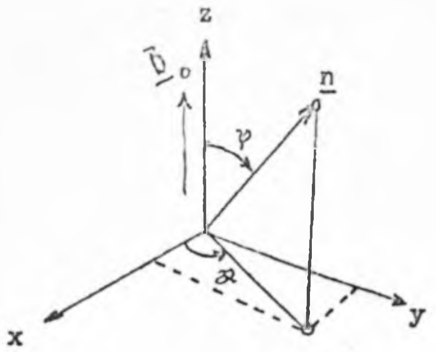

Fig. 1. Arbitrary direction $\underline{n}$ of wave propagation in electron gas with applied magnetic field B_o.

$$\underline{E} = \underline{E}_o \, e^{i\underline{k}\cdot\underline{r}} \, e^{-i\omega t} \tag{23}$$

where $\underline{E}_o$ is a constant. To determine the equation that $\underline{E}$ must satisfy, we note that the two curl equations of Maxwell are

$$\nabla \times \underline{H} = -i\,\omega(\varepsilon)\cdot\underline{E} \tag{24}$$

and

$$\nabla \times \underline{E} = i\omega\mu_o\,\underline{H} \tag{25}$$

where (ε) is the dielectric tensor and the product $(\varepsilon)\cdot\underline{E}$ is the displacement vector, i.e., $\underline{D} = (\varepsilon)\cdot\underline{E}$ or

$$\begin{pmatrix} D_x \\ D_y \\ D_z \end{pmatrix} = \begin{pmatrix} \varepsilon_{xx} & \varepsilon_{xy} & 0 \\ \varepsilon_{yx} & \varepsilon_{yy} & 0 \\ 0 & 0 & \varepsilon_{zz} \end{pmatrix} \cdot \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} \tag{26}$$

It follows from (24) and (25) that

$$\nabla \times \nabla \times \underline{E} = \omega^2 \mu_o (\mathcal{E}) \cdot \underline{E} \tag{27}$$

Since

$$\nabla \times \nabla \times \underline{E}_o e^{i\underline{k}\cdot\underline{r}} = -\underline{k}(\underline{k}\cdot\underline{E}_o e^{i\underline{k}\cdot\underline{r}}) + \underline{k}\cdot\underline{k}\,\underline{E}_o e^{i\underline{k}\cdot\underline{r}} ,$$

we obtain upon substituting (23) into (27)

$$\underline{k}\cdot\underline{k}\,\underline{E}_o e^{i\underline{k}\cdot\underline{r}} - \underline{k}(\underline{k}\cdot\underline{E}_o e^{i\underline{k}\cdot\underline{r}}) = \omega^2 \mu_o (\mathcal{E})\cdot\underline{E}_o e^{i\underline{k}\cdot\underline{r}} . \tag{28}$$

Canceling out the exponential factor, recalling that $\underline{k} = \underline{n}\,\omega/v$, $c^2 = 1/\sqrt{\mu_o \mathcal{E}_o}$, and $\underline{n}\cdot\underline{n} = 1$, we obtain

$$\underline{E}_o - \underline{n}(\underline{n}\cdot\underline{E}_o) = \frac{v^2}{c^2}\,\frac{(\mathcal{E})\cdot\underline{E}_o}{\mathcal{E}_o} . \tag{29}$$

Without loss of generality we rotate the coordinate system about the z-axis so that $\theta = \pi/2$ and $\underline{n}$ lies in the yz plane. Therefore, in (29) $\underline{n} = \underline{a}_y \sin\phi + \underline{a}_z \cos\phi$, $\underline{n}\cdot\underline{E}_o = E_{oy}\sin\phi + E_{oz}\cos\phi$, and the x,y,z components of the vector equation (29) are

$$E_{ox}\cdot\left(1 - \frac{v^2}{c^2}\frac{\mathcal{E}_{xx}}{\mathcal{E}_o}\right) - E_{oy}\cdot\left(\frac{v^2}{c^2}\frac{\mathcal{E}_{xy}}{\mathcal{E}_o}\right) + 0 = 0 \tag{30}$$

$$E_{ox}\cdot\left(-\frac{v^2}{c^2}\frac{\mathcal{E}_{yx}}{\mathcal{E}_o}\right) + E_{oy}\left(\cos^2\phi - \frac{v^2}{c^2}\frac{\mathcal{E}_{yy}}{\mathcal{E}_o}\right) + E_{oz}\cdot(-\cos\phi\sin\phi) = 0$$

$$0 + E_{oy}\cdot(-\cos\phi\sin\phi) + E_{oz}\cdot\left(\sin^2\phi - \frac{v^2}{c^2}\frac{\mathcal{E}_{zz}}{\mathcal{E}_o}\right) = 0 .$$

Since these three simultaneous equations are homogeneous, for them to yield a non-trivial solution it is necessary that the determinant Δ of the coefficients vanish:

$$\Delta = \begin{vmatrix} (1 - \frac{v^2}{c^2}\frac{\varepsilon_{xx}}{\varepsilon_o}) & (-\frac{v^2}{c^2}\frac{\varepsilon_{xy}}{\varepsilon_o}) & 0 \\ (-\frac{v^2}{c^2}\frac{\varepsilon_{yx}}{\varepsilon_o}) & (\cos^2\phi - \frac{v^2}{c^2}\frac{\varepsilon_{yy}}{\varepsilon_o}) & (-\sin\phi\cos\phi) \\ 0 & (-\sin\phi\cos\phi) & (\sin^2\phi - \frac{v^2}{c^2}\frac{\varepsilon_{zz}}{\varepsilon_o}) \end{vmatrix} = 0. \tag{31}$$

In expanding this determinant we find it convenient to introduce ε_1, ε_2, and ε_3 defined by

$$\varepsilon_1 = \frac{\varepsilon_{xx}}{\varepsilon_o} - i\frac{\varepsilon_{xy}}{\varepsilon_o},$$
$$\varepsilon_2 = \frac{\varepsilon_{xx}}{\varepsilon_o} + i\frac{\varepsilon_{xy}}{\varepsilon_o}, \tag{32}$$
$$\varepsilon_3 = \frac{\varepsilon_{zz}}{\varepsilon_o}.$$

With a little algebraic manipulation (31) leads to

$$\tan^2\phi = -\frac{(\frac{v^2}{c^2} - \frac{1}{\varepsilon_1})(\frac{v^2}{c^2} - \frac{1}{\varepsilon_2})}{(\frac{v^2}{c^2} - \frac{1}{\varepsilon_3})(\frac{v^2}{c^2} - \frac{1}{2}[\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}])}. \tag{33}$$

This equation is of the second degree in $\frac{v^2}{c^2}$ and always has two real roots.

When $\phi = 0$ propagation is along the z-axis and it follows from (17), (32), and (33) that

$$\frac{v^2}{c^2} = \frac{1}{\varepsilon_1} = \frac{1}{\frac{\varepsilon_{xx}}{\varepsilon_o} - i\frac{\varepsilon_{xy}}{\varepsilon_o}} = \frac{1}{1 - \frac{\omega_o^2}{\omega(\omega - \omega_g)}} \tag{34}$$

and

$$\frac{v^2}{c^2} = \frac{1}{\varepsilon_2} = \frac{1}{\frac{\varepsilon_{xx}}{\varepsilon_o} + i\,\frac{\varepsilon_{xy}}{\varepsilon_o}} = \frac{1}{1 - \frac{\omega_o^2}{\omega(\omega + \omega_g)}} \,. \tag{35}$$

$k = \frac{\omega}{v}$ $\quad$ $k = \frac{\omega}{c}\left(\frac{c}{v}\right)$

Therefore, the two propagation constants for waves traveling parallel to $\underline{B}_o$ are given by the following two expressions.

$$k'_o = \omega\sqrt{\mu_o\,\varepsilon_o}\,\sqrt{1 - \frac{\omega_o^2}{\omega(\omega - \omega_g)}} \tag{36}$$

and

$$k''_o = \omega\sqrt{\mu_o\,\varepsilon_o}\,\sqrt{1 - \frac{\omega_o^2}{\omega(\omega + \omega_g)}} \tag{37}$$

Moreover, when the propagation is along the y-axis, i.e., perpendicular to $\underline{B}_o$, ϕ is equal to $\pi/2$ and in this case we have

$$\frac{v^2}{c^2} = \frac{1}{\varepsilon_3} = \frac{\varepsilon_o}{\varepsilon_{zz}} = \frac{1}{1 - \omega_o^2/\omega^2} \tag{38}$$

and

$$\frac{v^2}{c^2} = \frac{1}{2}\left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\right) = \frac{1}{1 - \frac{\omega_o^2/\omega^2}{1 - \omega_g^2/(\omega^2 - \omega_o^2)}} \,. \tag{39}$$

The corresponding two propagation factors are

$$k'_{\pi/2} = \omega\sqrt{\mu_o\,\varepsilon_o}\,\sqrt{1 - \omega_o^2/\omega^2} = \omega\sqrt{\mu_o\varepsilon_o}\sqrt{\frac{\varepsilon_{zz}}{\varepsilon_o}} \tag{40}$$

and

$$k''_{\pi/2} = \omega\sqrt{\mu_o\,\varepsilon_o}\,\sqrt{1 - \frac{\omega_o^2/\omega^2}{1 - \omega_g^2/(\omega^2 - \omega_g^2)}} \,. \tag{41}$$

ω_o? Yes

Up to now we have deduced a formula (33) which allows us to compute the phase velocity of a wave propagating in an arbitrary direction with respect to the applied magnetostatic field. We shall continue by finding the components of $\underline{E}$ and $\underline{H}$ and thus determine the polarization properties of the medium.

Parallel Propagation

We have seen that when the direction of propagation is parallel to the applied magnetostatic field two independent waves are possible; one has a propagation constant k_o' and the other has a propagation constant k_o'', as given by (36) and (37). The two waves have no longitudinal electric field, i.e., $E_{oz} = 0$, as can be seen from the third of equations (30) upon setting $\phi = 0$. Nor do they have a longitudinal component of the magnetic field, i.e., $H_z = 0$, as can be verified by using (25) and recalling the fact that E_x and E_y are independent of x and y. Therefore, the two waves are purely TEM.

One of the waves is left-circularly polarized and the other is right-circularly polarized. Because they have different velocities, a superposition of the two yields a "linearly" polarized wave whose polarization direction rotates about the z-axis. This rotation is like the Faraday rotation of optically active substances.

Equations (30) for $\phi = 0$ reduce to

$$E_{ox}\left(1-\frac{v^2}{c^2}\frac{\varepsilon_{xx}}{\varepsilon_o}\right)-E_{oy}\left(\frac{v^2}{c^2}\frac{\varepsilon_{xy}}{\varepsilon_o}\right)=0$$

$$E_{ox}\left(-\frac{v^2}{c^2}\frac{\varepsilon_{yx}}{\varepsilon_o}\right)+E_{oy}\left(1-\frac{v^2}{c^2}\frac{\varepsilon_{yy}}{\varepsilon_o}\right)=0 \qquad (42)$$

$$E_{oz}\left(-\frac{v^2}{c^2}\frac{\varepsilon_{zz}}{\varepsilon_o}\right)=0$$

where the two values of v^2/c^2 are given by (34) and (35). The third of these equations shows us that $E_{oz}=0$. When v^2/c^2 is given by (34), the first equation of (42) yields

$$\frac{E_{ox}}{E_{oy}}=\frac{\frac{v^2}{c^2}\frac{\varepsilon_{xy}}{\varepsilon_o}}{1-\frac{v^2}{c^2}\frac{\varepsilon_{xx}}{\varepsilon_o}}=\frac{\frac{1}{\frac{\varepsilon_{xx}}{\varepsilon_o}-i\frac{\varepsilon_{xy}}{\varepsilon_o}}\cdot\frac{\varepsilon_{xy}}{\varepsilon_o}}{1-\left[\frac{1}{\frac{\varepsilon_{xx}}{\varepsilon_o}-i\frac{\varepsilon_{xy}}{\varepsilon_o}}\right]\cdot\frac{\varepsilon_{xx}}{\varepsilon_o}}=i \qquad (43)$$

If we had used the second equation of (42) we would have obtained the same result. The corresponding propagation factor k_o' is given by (36). When v^2/c^2 is given by (35), we get in a similar way

$$\frac{E_{ox}}{E_{oy}}=-i \qquad (44)$$

Here the propagation factor is k_o'' as given by (37).

Therefore, the electric components of the two TEM waves propagating parallel to $\underline{B}_o$ are

$$E_x'=A\,e^{ik_o'z}\,e^{-i\omega t}$$
$$E_y'=-i\,A\,e^{ik_o'z}\,e^{-i\omega t} \qquad (45)$$

and

$$E_x'' = C\, e^{ik_o'' z}\, e^{-i\omega t}$$
$$E_y'' = i\, C\, e^{ik_o'' z}\, e^{-i\omega t} \qquad (46)$$

where A and C are amplitude constants. The corresponding magnetic field components are easily obtained by applying (25) to (45) and (46) thus

$$i\,\omega\,\mu_o\, H_x' = -A\, k_o'\, e^{ik_o' z}\, e^{-i\omega t}$$
$$i\,\omega\,\mu_o\, H_y' = i\, k_o'\, A\, e^{ik_o' z}\, e^{-i\omega t} \qquad (47)$$

and

$$i\,\omega\,\mu_o\, H_x'' = C\, k_o''\, e^{ik_o'' z}\, e^{-i\omega t}$$
$$i\,\omega\,\mu_o\, H_y'' = i\, C\, k_o''\, e^{ik_o'' z}\, e^{-i\omega t} \qquad (48)$$

These two waves (primed and double-primed) when cast into vector form are

$$\underline{E}' = \underline{a}_x E_x' + \underline{a}_y E_y' = (\underline{a}_x - i\, \underline{a}_y)\, A\, e^{ik_o' z}\, e^{-i\omega t} \qquad (49)$$

$$\underline{H}' = \underline{a}_x . H_x' + \underline{a}_y H_y' = (i\, \underline{a}_x + \underline{a}_y)\frac{k_o'}{\omega\mu_o}\, A\, e^{ik_o' z}\, e^{-i\omega t} \qquad (50)$$

and

$$E'' = \underline{a}_x E_x'' + \underline{a}_y E_y'' = (\underline{a}_x + i\, \underline{a}_y)\, C\, e^{ik_o'' z}\, e^{-i\omega t} \qquad (51)$$

$$\underline{H}'' = \underline{a}_x H_x'' + \underline{a}_y H_y'' = (-i\, \underline{a}_x + \underline{a}_y)\, \frac{k_o''}{\omega\,\mu_o}\, C\, e^{ik_o'' z}\, e^{-i\omega t} \qquad (52)$$

It is clear that these are two plane circularly polarized TEM waves. Their sum is

$$\underline{E} = \underline{E}' + \underline{E}'' = \underline{a}_x \left\{ A\, e^{ik_o' z}\, e^{-i\omega t} + C\, e^{ik_o'' z}\, e^{-i\omega t} \right\} + \underline{a}_y \left\{ -i\, A\, e^{ik_o' z}\, e^{-i\omega t} + i\, C\, e^{ik_o'' z}\, e^{-i\omega t} \right\}, \qquad (53)$$

$$\underline{H} = \underline{H}' + \underline{H}'' = \underline{a}_x \left\{ i \frac{k_o'}{\omega\mu_o} A e^{ik_o' z} e^{-i\omega t} - i \frac{k_o''}{\omega\mu_o} C e^{ik_o'' z} e^{-i\omega t} \right\} + \tag{54}$$

$$\underline{a}_y \left\{ \frac{k_o'}{\omega\mu_o} A e^{ik_o' z} e^{-i\omega t} + \frac{k_o''}{\omega\mu_o} C e^{ik_o'' z} e^{-i\omega t} \right\} .$$

We see that $\underline{E} \cdot \underline{H} = 0$ and hence $\underline{E}$ and $\underline{H}$ are perpendicular, as are $\underline{E}'$, $\underline{H}'$ and $\underline{E}''$, $\underline{H}''$. To study the polarization of the composite wave we consider the ratio E_y/E_x . From (53) we know that

$$\frac{E_x}{E_y} = \frac{A e^{ik_o' z} + C e^{ik_o'' z}}{-i A e^{ik_o' z} + i C e^{ik_o'' z}} = i \frac{1 + (C/A) e^{i(k_o'' - k_o') z}}{1 - (C/A) e^{i(k_o'' - k_o') z}} . \tag{55}$$

If we choose the amplitudes A and C to be equal, we get

$$\frac{E_x}{E_y} = \cot \left(\frac{k_o' - k_o''}{2} z \right) . \tag{56}$$

Therefore, the rotation of the resultant vector $\underline{E}$ about the z-axis per unit length of travel is equal to $(k_o' - k_o'')/2$. This rotation is called Faraday rotation and we denote it by τ, i.e.,

$$\tau = \frac{k_o' - k_o''}{2} \tag{57}$$

The rotation is either clockwise or counterclockwise depending on whether $k_o' > k_o''$ or $k_o' < k_o''$. Substituting (36) and (37) into (57) we get the Faraday rotation as a function of frequency:

$$\tau = \omega\sqrt{\mu_o \varepsilon_o} \left\{ \sqrt{1 - \frac{\omega_o^2}{\omega(\omega - \omega_g)}} - \sqrt{1 - \frac{\omega_o^2}{\omega(\omega + \omega_g)}} \right\} . \tag{58}$$

Porter, Phil. ... 40, 99 (1944)

Perpendicular Propagation

When the direction of propagation is along any direction in $z = 0$ plane, two independent waves are possible. One has a propagation constant $k'_{\pi/2}$ and the other, $k''_{\pi/2}$. However, in contradistinction to the case of parallel propagation, only one of the waves is TEM while the other is TM.

The TEM wave has an electric field component parallel to $\underline{B}_o$ and a magnetic field component perpendicular to $\underline{B}_o$ and $\underline{n}$. The propagation factor $k'_{\pi/2}$ of this wave is independent of B_o and hence is the same as that of a wave traveling through an electron gas without any applied magnetostatic field.

The TM wave has a magnetic component parallel to $\underline{B}_o$. Its propagation factor $k''_{\pi/2}$ does depend on B_o.

The TEM wave is linearly polarized parallel to $\underline{B}_o$. However, a superposition of the TEM wave and the TM yields a composite wave which is elliptically polarized. By this case wherein the rotation is due to a perpendicular magnetic field the reader will be reminded of the Cotton-Mouton effect of optically active substances.

References

(1) The reader is referred to Prof. W. O. Schumann's treatment of the subject in his book, "Elektrische Wellen", Munich, 1948, pp. 89-103.

(2) E. Madelung, "Die Mathematischen Hilfsmittel des Physikers", Berlin 1936, pp. 268-270.

(3) W. Voigt, "Magneto- und Elektrooptik, Leipzig 1908.

(4) L. Goldstein, M. Lampert, and J. Heney, Phys.Rev.28, 6, 956-957.