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LETTER TO THE EDITOR

Propagation and stability of optical pulses in a diffractive dispersive non-linear medium

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Abstract. Propagation and stability of light pulses under the combined influence of the optical Kerr effect, dispersion and diffraction are investigated by adopting a variational procedure. In particular, it is found that 'light bullets', i.e. radially symmetric pulses propagating without distortion, are not necessarily unstable under perturbations which do not maintain radial symmetry.

The self-induced non-linear contribution to the refractive index n_0 of a transparent medium ($n = n_0 + n_2 I$, I being the instantaneous optical intensity of the propagating field) is known as the optical Kerr effect. It can give rise to two distinct phenomena, that is a longitudinal and a transverse one, respectively termed as 'self phase modulation' and 'self focusing'. The first is able to modify the temporal shape of the pulse, either shortening or broadening it according to the dispersive properties of the medium at the carrier frequency; in particular, in a single mode optical fibre, where diffraction can be neglected under usual operating conditions, the non-linear shortening of the pulse can exactly balance the linear dispersive broadening and give rise to an 'envelope soliton' which propagates undistorted. The second affects the transverse spatial characteristics of the pulse, in view of the self-guiding properties of the beam associated with the presence of a refractive index larger (whenever $n_2 > 0$) at its centre than at the periphery; in particular, in an unbounded, non-dispersive medium, self focusing can exactly compensate for the effect of diffraction and produce a beam propagating with no variation in its transverse shape. The above two situations were first investigated in the pioneering papers of Hasegawa and Tappert [1] and of Chiao *et al* [2], respectively, while a considerable amount of work has been devoted to the stability properties under perturbations of the equilibrium situation in which exact balance occurs [3–6].

More recently, propagation of optical pulses under the simultaneous influence of dispersion, diffraction, self focusing and self phase modulation has attracted a good deal of attention [7–10]. In particular, the possibility has been recognized of generating pulses ('light bullets') which, held together by the presence of non-linearity, propagate without changing their temporal and spatial shape [9]. The stability analysis, both numerical [11] and analytical [9,10], shows that they are unstable, that is that any small deviation from the equilibrium condition leads either to their collapse in space and

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time or to their divergence. While this is the case for radially symmetric fields, that is fields depending (apart from the propagation coordinate z) on the combination $\rho^2 = \xi^2 + \eta^2 + \tau^2$, where ξ, η and τ are respectively the transverse coordinates and the temporal one (suitably normalized), it is no longer true for the more general situation of non-spherical deviations from equilibrium (i.e. separately depending on $\xi^2 + \eta^2$ and τ^2), as we will show in the following. This will be done by generalizing a Lagrangian approach which has been already successfully employed to describe non-linear propagation problems [10,12,13].

The wave equation describing paraxial propagation along the z direction reads

$$\mathrm{i}\left(\frac{\partial}{\partial z} + \frac{1}{V}\frac{\partial}{\partial t}\right)E + \frac{1}{2k}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)E - \frac{1}{2A}\frac{\partial^2}{\partial t^2}E + k\frac{n_2}{n_0}|E|^2E = 0 \tag{1}$$

where $E(x, y, z, t)\exp(\mathrm{i}kz - \mathrm{i}\omega t)$ is the electric field and $V = 1/(\mathrm{d}k/\mathrm{d}\omega)$ and $A = 1/(\mathrm{d}^2k/\mathrm{d}\omega^2)$ are respectively the group velocity and the group-velocity dispersion at the carrier frequency ω. In the anomalous dispersion regime (i.e. $A < 0$), equation (1) can be recast in the dimensionless form [9]

$$\mathrm{i}\frac{\partial u}{\partial \zeta} + \frac{1}{2}\left(\frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2} + \frac{\partial^2}{\partial \tau^2}\right)u + |u|^2u = 0 \tag{2}$$

where $\tau = (t - z/V)(-kA)^{1/2}$, $(\xi, \eta, \zeta) = (kx, ky, kz)$ and $u = (n_2/n_0)^{1/2}E$.

If we consider cylindrical symmetric solutions of the kind $u(\rho, \tau)$, where $\rho = (\xi^2 + \eta^2)^{1/2}$, an appropriate field Lagrangian density $\mathscr{L}$, from which equation (2) can be deduced as a Euler–Lagrange equation corresponding to a vanishing variation [14]

$$\delta \int \mathscr{L}\, \mathrm{d}\rho\, \mathrm{d}\tau\, \mathrm{d}\zeta = 0 \tag{3}$$

reads

$$\begin{aligned}\mathscr{L} &= \rho\left[\frac{1}{2\mathrm{i}}\left(u^*\frac{\partial u}{\partial \zeta} - u\frac{\partial u^*}{\partial \zeta}\right) + \frac{1}{2}\left|\frac{\partial u}{\partial \rho}\right|^2 + \frac{1}{2}\left|\frac{\partial u}{\partial \tau}\right|^2 - \tfrac{1}{2}|u|^4\right] \\ &= \rho|u|^2\frac{\partial \varphi}{\partial \zeta} + \tfrac{1}{2}\rho\left[\left(\frac{\partial |u|}{\partial \rho}\right)^2 + |u|^2\left(\frac{\partial \varphi}{\partial \rho}\right)^2 + \left(\frac{\partial |u|}{\partial \tau}\right)^2 + |u|^2\left(\frac{\partial \varphi}{\partial \tau}\right)^2 - |u|^4\right]\end{aligned} \tag{4}$$

having set $u = |u|\exp(\mathrm{i}\varphi)$.

In the frame of the Ritz optimization procedure, we look for an approximate analytical solution to be found within a set of suitably chosen trial functions of the form

$$u(\rho, \tau, \zeta) = \frac{M_0^{1/2}(\zeta)}{\sigma(\zeta)\mu^{1/2}(\zeta)} F^{1/2}\left[\left(\frac{\rho^2}{\sigma^2} + \frac{\tau^2}{\mu^2}\right)^{1/2}\right]\exp[\mathrm{i}a_0(\zeta) + \mathrm{i}a_1(\zeta)\rho^2 + \mathrm{i}b_1(\zeta)\tau^2] \tag{5}$$

where F is a well behaved prescribed function. If we insert equation (5) into equation (3), the variational principle reduces to

$$\delta \int_{-\infty}^{+\infty} \mathscr{L}_r(\zeta)\, \mathrm{d}\zeta = 0 \tag{6}$$

with

$$\mathscr{L}_r(\zeta) = \int_{-\infty}^{+\infty} \mathrm{d}\tau \int_0^{+\infty} \mathrm{d}\rho\, \mathscr{L}(\rho, \tau, \zeta) = \dot{a}_0 M_0 + \alpha \dot{a}_1 \sigma^2 M_0 + \tfrac{1}{2}\alpha \dot{b}_1 \mu^2 M_0$$
$$+ \tfrac{1}{8}\beta M_0/\sigma^2 + \tfrac{1}{16}\beta M_0/\mu^2 + 2\alpha a_1^2 \sigma^2 M_0 + \alpha b_1^2 \mu^2 M_0 - \gamma M_0^2/2\sigma^2\mu \tag{7}$$

where the dot stands for the ζ derivative,

$$\alpha = \frac{4}{3}\int_0^{+\infty} w^4 F(w)\,\mathrm{d}w \qquad \beta = \frac{4}{3}\int_0^{+\infty} w^2 \frac{(\mathrm{d}F/\mathrm{d}w)^2}{F}\,\mathrm{d}w$$
$$\gamma = 2\int_0^{+\infty} w^2 F^2(w)\,\mathrm{d}w \tag{8}$$

and we have assumed the normalization condition

$$2\int_0^{+\infty} w^2 F(w)\,\mathrm{d}w = 1. \tag{9}$$

Accordingly, the exact field Lagrangian $\mathscr{L}$ is substituted by the reduced Lagrangian $\mathscr{L}_r$ whose Euler–Lagrange equations for the ζ-dependent parameters M_0, σ, μ, a_0, a_1, b_1 read [14]

$$\dot{M}_0 = 0 \text{ (energy conservation)} \qquad \dot{\sigma} = 2\sigma a_1 \qquad \dot{\mu} = 2\mu b_1 \tag{10}$$

$$\dot{a}_0 + \alpha \dot{a}_1 \sigma^2 + \tfrac{1}{2}\alpha \dot{b}_1 \mu^2 + \tfrac{1}{8}\beta/\sigma^2 + \tfrac{1}{16}\beta/\mu^2 + 2\alpha\sigma^2 a_1^2 + \alpha\mu^2 b_1^2 - \gamma M_0/\sigma^2\mu = 0 \tag{11}$$

$$\alpha \dot{a}_1 \sigma - \tfrac{1}{8}\beta/\sigma^3 + 2\alpha a_1^2 \sigma + \tfrac{1}{2}\gamma M_0/\sigma^3\mu = 0 \tag{12}$$

$$\alpha \dot{b}_1 \mu - \tfrac{1}{8}\beta/\mu^3 + 2\alpha b_1^2 \mu + \tfrac{1}{2}\gamma M_0/\sigma^2\mu^2 = 0. \tag{13}$$

From equations (10), (12) and (13) we obtain

$$\ddot{\sigma} = \tfrac{1}{4}\beta/\alpha\sigma^3 - \gamma M_0/\alpha\sigma^3\mu \qquad \ddot{\mu} = \tfrac{1}{4}\beta/\alpha\mu^3 - \gamma M_0/\alpha\sigma^2\mu^2. \tag{14}$$

While equations (14) are not derivable from a potential, this turns out to be possible after performing the change of variable $v = \mu/2^{1/2}$, which allows the resulting equations to be written as $\ddot{\sigma} = -\partial V/\partial\sigma$, $\ddot{v} = -\partial V/\partial v$, where (see figure 1)

$$V(\sigma, v) = \tfrac{1}{8}\beta/\alpha\sigma^2 - \gamma M_0/2\sqrt{2}\alpha\sigma^2 v + \tfrac{1}{32}\beta/\alpha v^2. \tag{15}$$

We now look for self-supporting fields, that is pulses of the form of equation (5) with σ and μ independent from ζ. The corresponding equilibrium values σ_{eq} and v_{eq} are obtained through the vanishing of the first partial derivatives of V, which yields $\sigma_{eq} = 2^{1/2}v_{eq} = \mu_{eq} = 4\gamma M_0/\beta$. In turn, the constancy of σ and μ entails the vanishing of a_1 and b_1 (see the second and third parts of equation (10)), while the equality of σ_{eq} and μ_{eq} implies (see equation (5)) radially symmetric solutions of the kind found in [9].

The stability analysis, i.e. the field behaviour for small displacements of σ and v from their equilibrium value, is carried out in a straightforward way by evaluating the second partial derivative of V, which furnishes

$$\frac{\partial^2 V}{\partial\sigma^2} = 0 \qquad \frac{\partial^2 V}{\partial v^2} = \frac{\beta}{4\alpha\sigma_{eq}^4} > 0 \tag{16}$$

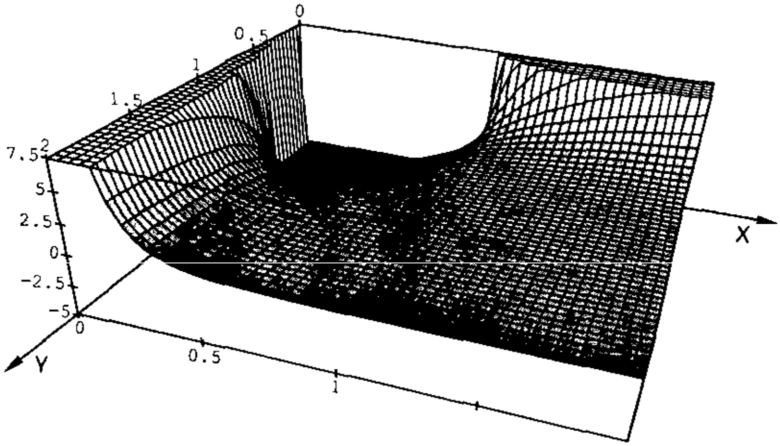

Figure 1. Tridimensional plot of the normalized potential $V(\sigma, \nu)/[\beta^3/4(4\gamma M_0)^2\alpha] = 1/X^2 - 1/X^2Y + 1/4Y^2$ where $(X, Y) = (\beta/2^{1/2}2\gamma M_0)(\sigma, \nu)$.

both derivatives being evaluated for $\sigma = \sigma_{eq}$, $\nu = \nu_{eq}$. Equations (16) imply that the field is in a state of indifferent equilibrium under perturbation of its transverse width, while it is in stable equilibrium under perturbation of its temporal width. These are noteworthy results, since a perturbation preserving radial symmetry, that is along the line $\sigma = \mu$, yields unstable equilibrium (collapse or divergence of the pulse), as found in [9,10] or, directly, through our approach; this can be seen by evaluating the second derivative of V along the line $\sigma = 2^{1/2}\nu = \mu$, thus obtaining the negative value $-\frac{3}{8}\beta/\alpha\sigma_{eq}^4$.

The previous analysis can be as well applied to a bidimensional situation, that is a field independent from one of the transverse dimensions, e.g. η. The Lagrangian reads in this case

$$\mathscr{L} = \frac{1}{2i}\left(u^*\frac{\partial u}{\partial \zeta} - u\frac{\partial u^*}{\partial \zeta}\right) + \frac{1}{2}\left|\frac{\partial u}{\partial \xi}\right|^2 + \frac{1}{2}\left|\frac{\partial u}{\partial \tau}\right|^2 - \tfrac{1}{2}|u|^4 \tag{17}$$

while the trial functions have the form

$$u(\xi, \tau, \zeta) = \frac{M_0^{1/2}(\zeta)}{\sigma^{1/2}(\zeta)\mu^{1/2}(\zeta)} F^{1/2}\left[\left(\frac{\xi^2}{\sigma^2} + \frac{\tau^2}{\mu^2}\right)^{1/2}\right] \exp[ia_0(\zeta) + ia_1(\zeta)\xi^2 + ib_1(\zeta)\tau^2]. \tag{18}$$

With the same procedure adopted in the three-dimensional case, we again obtain equation (10) while equations (11) and (14) are substituted by

$$\dot{a}_0 + \alpha'\dot{a}_1\sigma^2 + \alpha'\dot{b}_1\mu^2 + \tfrac{1}{8}\beta'/\sigma^2 + \tfrac{1}{8}\beta'/\mu^2 + 2\alpha'\sigma^2 a_1^2 + 2\alpha'\mu^2 b_1^2 - \gamma' M_0/\sigma\mu = 0 \tag{19}$$

$$\ddot{\sigma} = \frac{\beta'}{4\alpha'\sigma^3} - \frac{\gamma' M_0}{2\alpha'\sigma^2\mu} \qquad \ddot{\mu} = \frac{\beta'}{4\alpha'\mu^3} - \frac{\gamma' M_0}{2\alpha'\sigma\mu^2} \tag{20}$$

where we have assumed the normalization condition

$$2\pi\int_0^{+\infty} wF(w)\,\mathrm{d}w = 1 \tag{21}$$

and we have set

$$\alpha' = \pi \int_0^{+\infty} w^3 F(w)\,\mathrm{d}w \qquad \beta' = \pi \int_0^{+\infty} w \frac{(\mathrm{d}F/\mathrm{d}w)^2}{F}\,\mathrm{d}w$$
$$\gamma' = 2\pi \int_0^{+\infty} wF^2(w)\,\mathrm{d}w. \tag{22}$$

The potential V associated with equation (20) reads

$$V(\sigma,\mu) = \frac{\beta'}{8\alpha'\sigma^2} - \frac{\gamma' M_0}{2\alpha'\sigma\mu} + \frac{\beta'}{8\alpha'\mu^2} \tag{23}$$

and the vanishing of the first partial derivatives yields $\sigma = \mu$ and $M_0 = M_{eq} = \beta'/2\gamma'$. Thus, in the two-dimensional case self-supporting fields exist independently from the value of σ and μ, provided that they are equal and $M_0 = M_{eq}$. With this choice, the potential becomes

$$V(\sigma,\mu) = \frac{\beta'}{8\alpha'}\left(\frac{1}{\sigma} - \frac{1}{\mu}\right)^2 \tag{24}$$

so that the system is obviously stable under perturbations removing it from the equilibrium line $\sigma = \mu$. If, conversely, $\sigma = \mu$ but $M_0 \neq M_{eq}$, equation (20) furnishes

$$\sigma(\zeta) = \mu(\zeta) = \sigma(0)\left(1 + \frac{a\zeta^2}{\sigma^4(0)}\right)^{1/2} \tag{25}$$

with $a = (M_{eq} - M_0)\gamma'/2\alpha'$, so that one has collapse for $M_0 > M_{eq}$ and divergence for $M_0 < M_{eq}$.

We note that the coefficients α, β, γ and α', β', γ' depend on the trial function F. For example, if we set $F(x) = (2/\pi^{1/2})\exp(-w^2)$ (tridimensional case) or $F(x) = (1/\pi)\exp(-w^2)$ (bidimensional case), we have $\alpha = 1$, $\beta = 4$, $\gamma = 1/(2\pi)^{1/2}$ and $\alpha' = \frac{1}{2}$, $\beta' = 2$ and $\gamma' = 1/2\pi$. Nevertheless, our stability analysis provides a criterion not based on the choice of the trial function, and this assures its general validity.

In conclusion, we have employed a Lagrangian formalism which allows us to investigate the stability properties of light pulses propagating under the combined effects of dispersion, diffraction and the optical Kerr effect. Our analysis turns out to be more complete than those developed for pulses maintaining radial symmetry [9,10]. In fact, while one could heuristically infer the equilibrium to be inherently unstable [9], we have shown that this is not in general the case under perturbations driving the system away from radial symmetry ($\sigma = \mu$). This has relevant implications as far as the possibility of generating optical pulses which collapse simultaneously in space and time is concerned, since it requires a fine tuning of the temporal and spatial widths.

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