

\hat{Z} invariants at rational τ

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ABSTRACT: \hat{Z} invariants of 3-manifolds were introduced as series in $q = e^{2\pi i\tau}$ in order to categorify Witten-Reshetikhin-Turaev invariants corresponding to $\tau = 1/k$. However modularity properties suggest that all roots of unity are on the same footing. The main result of this paper is the expression connecting Reshetikhin-Turaev invariants with \hat{Z} invariants for $\tau \in \mathbb{Q}$. We present the reasoning leading to this conjecture and test it on various 3-manifolds.

KEYWORDS: Chern-Simons Theories, Quantum Groups, Supersymmetric Gauge Theory, Topological Field Theories

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1 Introduction and summary

The main goal of this paper is to explore the behaviour of \hat{Z} invariants of 3-manifolds at rational τ (in general $\tau \in \mathbb{H}$ — the upper half-plane). \hat{Z} invariants were introduced in [1–4] as series in $q = e^{2\pi i\tau}$ with integer coefficients in order to enable the categorification of Witten-Reshetikhin-Turaev (WRT) invariants of 3-manifolds. It turns out that, apart from the topological applications, \hat{Z} invariants are very interesting from the point of view of physics and number theory.

Physically \hat{Z} invariant is a 3d analogue of the elliptic genus introduced in [5]. More precisely it is a supersymmetrix index of 3d $\mathcal{N} = 2$ theory with 2d $\mathcal{N} = (0, 2)$ boundary condition studied first in [6]. Detailed analysis of this interpretation can be found in [1, 3], whereas [7] provides a lot of explicit results for various examples. \hat{Z} invariants are also related to 2d logarithmic conformal field theories [4] and newly proposed two-variable series for knot complements [8].

Due to their modular properties, \hat{Z} invariants are interesting from the point of view of number theory. A broad discussion of this subject can be found in [4]. For us the most

important are two aspects. Firstly, for many 3-manifolds \hat{Z} invariants can be expressed as a linear combination of false theta functions [2, 4, 9]. This fact plays an important role in explicit calculations in sections 4.2 and 4.3. An analogous property for WRT invariants was studied earlier in [10–16].

In order to understand the second aspect, let us make a step back to the relation between WRT invariants and \hat{Z} invariants for plumbed 3-manifolds [1–4]

$$\begin{aligned} \text{WRT}[M_3(\Gamma); 1/k] &= \lim_{q \rightarrow e^{\frac{2\pi i}{k}}} \frac{\sum_{a \in \text{Coker} M} e^{-2\pi i k(a, M^{-1}a)} \sum_{b \in 2\text{Coker} M + \delta} S_{ab} \hat{Z}_b}{2(q^{1/2} - q^{-1/2})}, \\ S_{ab} &= \frac{e^{-2\pi i(a, M^{-1}b)}}{|\det M|^{1/2}}, \end{aligned} \tag{1.1}$$

where M is the linking matrix of the plumbing graph Γ (for details see section 2.1). Equation (1.1) corresponds to $\tau = 1/k$. In this case there exists a well-known physical interpretation in the language of Chern-Simons theory, where $k \in \mathbb{N}$ is the quantum-corrected Chern-Simons level [17] (in the whole paper we restrict to the $SU(2)$ gauge group). However from the point of view of number theory $\tau = 1/k$ is conceptually on the same footing as all other rational numbers [10]. Therefore there arises a natural question (which is the main motivation of this work):

What happens with (1.1) for $\tau = r/s$?

Since for $\tau = r/s$ ($r, s \in \mathbb{Z}$) there is no Chern-Simons theory interpretation, we will refer to the left hand side as the Reshetikhin-Turaev (RT) invariant — their combinatorial definition using quantum group representation theory [18] works for all $\tau \in \mathbb{Q}$. The main result of this paper is the following expression connecting the RT invariant with the \hat{Z} invariant

$$\begin{aligned} \text{RT}[M_3(\Gamma); r/s] &= \lim_{q \rightarrow e^{2\pi i \frac{r}{s}}} \frac{\sum_{a \in \text{Coker}(rM)} e^{-2\pi i \frac{s}{r}(a, M^{-1}a)} \sum_{b \in 2\text{Coker} M + \delta} S_{ab} \hat{Z}_b}{2(q^{1/2} - q^{-1/2}) G(s, r)^L}, \\ S_{ab} &= \frac{e^{-2\pi i(a, M^{-1}b)}}{|\det M|^{1/2}}, \\ G(s, r) &= \sum_{c \in \mathbb{Z}_r} e^{2\pi i \frac{s}{r} c^2}, \end{aligned} \tag{1.2}$$

where values of the quadratic Gauss sum are discussed in section 3. We checked this formula in many examples and conjecture that it is true for all plumbed 3-manifolds. We expect that similar formula holds for all 3-manifolds, but in that situation obtaining $\tau \rightarrow r/s$ limit of \hat{Z} and testing is problematic.

The form of (1.2), especially the summation over $a \in \text{Coker}(rM)$, is quite surprising. Is $M \mapsto rM$ a purely computational phenomenon or does it have a topological interpretation? If the latter is true, should we view rM as the matrix defining a 3-manifold? What would be the relation to the initial one? We will come back to these questions in sections 3 and 5.

The plan of this paper is as follows. Section 2 contains the necessary preparations, focusing on plumbed 3-manifolds and an expression for RT invariant independent of (1.2).

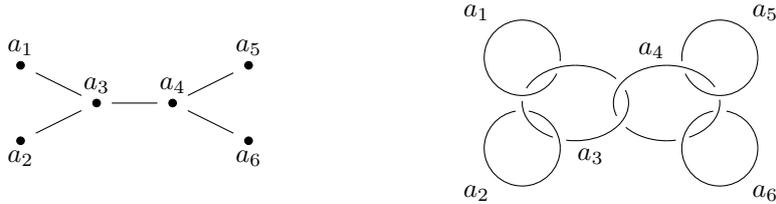


Figure 1. An example of a plumbing graph Γ (left) and the associated link of unknots (right) denoted as $\mathcal{L}(\Gamma)$. Each vertex label corresponds to the framing of the respective link. The manifold $M_3(\Gamma)$ can be constructed by performing a Dehn surgery on $\mathcal{L}(\Gamma)$.

In section 3 we derive and discuss our main result — the formula (1.2). Tests on various examples are presented in section 4. Finally, section 5 is devoted to the future directions.

Remark: soon after this paper appeared on arXiv, an independent approach to \hat{Z} invariants at rational τ was presented in [19].

2 Prerequisites

2.1 Plumbed 3-manifolds

In this paper we focus on a very large class of 3-manifolds corresponding to decorated graphs which, for simplicity, are assumed to be connected. For a given graph Γ we can obtain the associated plumbed 3-manifold $M_3(\Gamma)$ by performing a Dehn surgery on $\mathcal{L}(\Gamma)$ — the corresponding link of framed unknots (see figure 1). We are mainly interested in Seifert fibrations over S^2 which correspond to star-shaped graphs and are denoted by $M(b; \{b_i/a_i\}_i)$, where $b, b_i, a_i \in \mathbb{Z}$. Among them there is a special class of Brieskorn homology spheres. They are defined as the intersection of the complex unit sphere with the hypersurface $z_1^{p_1} + z_2^{p_2} + z_3^{p_3} = 0$ (p_1, p_2, p_3 are coprime integers) and denoted by $\Sigma(p_1, p_2, p_3)$.

Let us denote the set of vertices of Γ by V and the set of edges by E . $L = |V|$ is equal to the number of components of $\mathcal{L}(\Gamma)$. We can encode the information given by the plumbing graph in a convenient way by the following $L \times L$ matrix

$$M_{v_1, v_2} = \begin{cases} 1 & v_1 \text{ and } v_2 \text{ connected by the edge,} \\ a_v & v_1 = v_2 = v \text{ (framing of the link } v), \\ 0 & \text{else.} \end{cases} \quad v_i \in V \cong \{1, \dots, L\} \quad (2.1)$$

From the link perspective M is the linking matrix of $\mathcal{L}(\Gamma)$. The cokernel of M is equal (setwise) to the first homology group of $M_3(\Gamma)$

$$H_1(M_3(\Gamma), \mathbb{Z}) \cong \text{Coker } M = \mathbb{Z}^L / M\mathbb{Z}^L. \quad (2.2)$$

The number of elements in each set is given by $\det M$.

2.2 Formula for RT invariants

In appendix A of [3] the reasoning leading to equation (1.1) starts from the following formula for the WRT invariant of a plumbed 3-manifold $M_3(\Gamma)$

$$\begin{aligned} \text{WRT}[M_3(\Gamma); 1/k] &= \frac{F[\Gamma; 1/k]}{F[+1\bullet; 1/k]^{b_+} F[-1\bullet; 1/k]^{b_-}}, \\ F[\Gamma; 1/k] &= \sum_{n \in \{1, \dots, k-1\}^L} \prod_{v \in V} q^{\frac{a_v(n_v^2-1)}{4}} \left(q^{\frac{n_v}{2}} - q^{-\frac{n_v}{2}} \right)^{2-\deg(v)} \\ &\quad \times \frac{\prod_{(v_1, v_2) \in E} \left(q^{\frac{n_{v_1} n_{v_2}}{2}} - q^{-\frac{n_{v_1} n_{v_2}}{2}} \right)}{(q^{1/2} - q^{-1/2})^{L+1}} \Bigg|_{q=e^{\frac{2\pi i}{k}}}, \end{aligned} \quad (2.3)$$

where b_+ and b_- are the number of positive and negative eigenvalues of the matrix M . The symbol $\pm 1\bullet$ denotes the plumbing graph with one vertex corresponding to the unknot with ± 1 framing. In this paper we always assume

$$q = e^{2\pi i \tau} \quad (2.4)$$

and the WRT invariant corresponds to $\tau = 1/k$.

Equation (2.3) comes from the quantum group construction [18] where all roots of unity are on the same footing. More formally, formula (2.3) transforms equivariantly under the Galois group $\text{Gal}\left(\mathbb{Q}(e^{2\pi i \frac{\tau}{s}})/\mathbb{Q}\right)$ [10, 20] and in consequence its generalisation to $\tau = r/s$ is given by substitution $q = e^{2\pi i \frac{\tau}{s}}$

$$\begin{aligned} \text{RT}[M_3(\Gamma); r/s] &= \frac{F[\Gamma; r/s]}{F[+1\bullet; r/s]^{b_+} F[-1\bullet; r/s]^{b_-}}, \\ F[\Gamma; r/s] &= \sum_{n \in \{1, \dots, s-1\}^L} \prod_{v \in V} q^{\frac{a_v(n_v^2-1)}{4}} \left(q^{\frac{n_v}{2}} - q^{-\frac{n_v}{2}} \right)^{2-\deg(v)} \\ &\quad \times \frac{\prod_{(v_1, v_2) \in E} \left(q^{\frac{n_{v_1} n_{v_2}}{2}} - q^{-\frac{n_{v_1} n_{v_2}}{2}} \right)}{(q^{1/2} - q^{-1/2})^{L+1}} \Bigg|_{q=e^{2\pi i \frac{r}{s}}}. \end{aligned} \quad (2.5)$$

We will use this formula in many examples in section 4, but it is interesting on its own.

According to Turaev construction [21] we can associate a modular tensor category (MTC) to the 3d topological quantum field theory. The MTC comes equipped with modular S and T matrices which capture the structure of the topological partition function. For the plumbed 3-manifold this relation reads (see [22, 23] for more details)

$$Z_{\text{top}}[M_3(\Gamma)] = \sum_n \prod_{v \in V} (T_{n_v n_v})^{a_v} (S_{0n_v})^{2-\deg(v)} \prod_{(v_1, v_2) \in E} S_{n_{v_1} n_{v_2}}. \quad (2.6)$$

Comparing (2.5) with (2.6) we can see that the expression for F matches the structure of

Z_{top} for

$$\begin{aligned}
 T_{mn} &= \delta_{m,n} q^{\frac{n^2-1}{4}}, \\
 S_{0n} &= \frac{1}{i\sqrt{2s}} \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}} \right), \\
 S_{mn} &= \frac{1}{i\sqrt{2s}} \left(q^{\frac{nm}{2}} - q^{-\frac{nm}{2}} \right), \\
 q &= e^{2\pi i \frac{r}{s}}.
 \end{aligned}
 \tag{2.7}$$

This is a projective representation of $\text{SL}(2, \mathbb{Z})$, where the phase factor is an integer multiple of $1/8$. In order to restore $(ST)^3 = \pm 1$ we have to rescale T

$$T_{mn} \mapsto \delta_{m,n} q^{\frac{2n^2-s}{8}}. \tag{2.8}$$

The condition $S^2 = \pm 1$ is ensured by the normalisation factor $\frac{1}{i\sqrt{2s}}$ which cancels out in (2.5).

Another important observation is the invariance of formula (2.5) under $r \mapsto r + ns$ symmetry ($n \in \mathbb{Z}$). It is equivalent to the multiplication of every q by $e^{2\pi i n} = 1$. The $r \mapsto r + ns$ symmetry helps to solve the problem of choosing the branch of the complex root which arises in the context of RT invariants (see section 3.1).

3 Main conjecture

3.1 RT invariants from \hat{Z} invariants

The reasoning leading to our main conjecture follows the appendix A of [3], which starts from expression (2.3) and, in the crucial step, uses the Gauss sum reciprocity formula

$$\begin{aligned}
 \sum_{n \in \mathbb{Z}^L / 2k\mathbb{Z}^L} \exp \left[\frac{\pi i}{2k} (n, Mn) + \frac{\pi i}{k} (l, n) \right] \\
 = \frac{e^{\pi i \sigma} (2k)^{L/2}}{|\det M|^{1/2}} \sum_{a \in \mathbb{Z}^L / M\mathbb{Z}^L} \exp \left[-2\pi i k \left(a + \frac{l}{2k}, M^{-1} \left[a + \frac{l}{2k} \right] \right) \right],
 \end{aligned}
 \tag{3.1}$$

where $l \in \mathbb{Z}^L$, (\cdot, \cdot) is the standard pairing on \mathbb{Z}^L and $\sigma = b_+ - b_-$ is the signature of the linking matrix M . The final result is the relation between the WRT invariant and the \hat{Z} invariant for $\tau = 1/k$

$$\begin{aligned}
 \text{WRT}[M_3(\Gamma); 1/k] &= \lim_{q \rightarrow e^{\frac{2\pi i}{k}}} \frac{\sum_{a \in \text{Coker } M} e^{-2\pi i k (a, M^{-1} a)} \sum_{b \in 2\text{Coker } M + \delta} S_{ab} \hat{Z}_b}{2 (q^{1/2} - q^{-1/2})}, \\
 S_{ab} &= \frac{e^{-2\pi i (a, M^{-1} b)}}{|\det M|^{1/2}},
 \end{aligned}
 \tag{3.2}$$

where $\delta \in \mathbb{Z}^L / 2\mathbb{Z}^L$ and $\delta_v \equiv \deg v \pmod{2}$.

We would like to have an analogous derivation for $\tau = r/s$, so we start from equation (2.5) and follow all the steps of the appendix A. The crucial one is again the Gauss sum reciprocity formula. In order to deal with $\tau = r/s$ we have to rescale the formula,

which is equivalent to considering (3.1) for $\tilde{M} = rM$ and $\tilde{l} = rl$ (we also write s instead of k). We obtain

$$\begin{aligned} \sum_{n \in \mathbb{Z}^L / 2s\mathbb{Z}^L} \exp \left[\frac{\pi i}{2s} (n, rMn) + \frac{\pi i}{s} (rl, n) \right] & \quad (3.3) \\ &= \frac{e^{\pi i \sigma (2s/r)^{L/2}}}{|\det M|^{1/2}} \sum_{a \in \mathbb{Z}^L / rM\mathbb{Z}^L} \exp \left[-2\pi i s \left(a + \frac{rl}{2s}, (rM)^{-1} \left[a + \frac{rl}{2s} \right] \right) \right], \end{aligned}$$

which leads to our main conjecture

$$\begin{aligned} \text{RT}[M_3(\Gamma); r/s] &= \lim_{q \rightarrow e^{2\pi i \frac{r}{s}}} \frac{\sum_{a \in \text{Coker}(rM)} e^{-2\pi i \frac{s}{r} (a, M^{-1}a)} \sum_{b \in 2\text{Coker}M + \delta} S_{ab} \hat{Z}_b}{2 (q^{1/2} - q^{-1/2}) G(s, r)^L}, \\ S_{ab} &= \frac{e^{-2\pi i (a, M^{-1}b)}}{|\det M|^{1/2}}, \\ G(s, r) &= \sum_{c \in \mathbb{Z}_r} e^{2\pi i \frac{s}{r} c^2} = \begin{cases} \sqrt{r} \left(\frac{s}{r} \right) & r \equiv 1 \pmod{4} \\ i\sqrt{r} \left(\frac{s}{r} \right) & r \equiv 3 \pmod{4} \end{cases} \end{aligned} \quad (3.4)$$

where $\left(\frac{s}{r} \right)$ is the Jacobi symbol. If we want to use above formula for even r , we have to choose another representant of the $r \sim r + ns$ equivalence class to avoid dividing by $G(s, r) = 0$ (in fact this happens only for $r \equiv 2 \pmod{4}$ but it is more convenient to treat all even r the same). This problem is a reflection of the fact that for some choices of roots of $q = e^{2\pi i \frac{r}{s}}$ (for $\text{SU}(2)$ we deal with 4 values of $q^{1/4}$) we have $F[\pm 1 \bullet; r/s] = 0$. A detailed discussion of the vanishing denominator in the RT invariants can be found in [24, 25].

There are two important differences between (3.4) and (3.2). The first one is in the summation range — $\text{Coker}(rM)$ has r^L more elements than $\text{Coker}M$. On the other hand we have $G(s, r)^L$ in denominator which scales as $r^{L/2}$ and “compensates” this growth. For $r = 1$ equation (3.4) reduces to (3.2) which provides the first consistency check.

3.2 Rational τ limit of \hat{Z} invariants

For some simple 3-manifolds such as lens spaces $L(p, 1)$ the $\tau \rightarrow r/s$ limit of the \hat{Z} invariant is very easy to obtain (see section 4.1), however these are exceptions rather than the rule. Fortunately for many 3-manifolds (e.g. Seifert manifolds with 3 singular fibers) the \hat{Z} invariant can be expressed as a linear combination of false theta functions defined as

$$\begin{aligned} \Psi_{m, \alpha} &= \sum_{n=0}^{\infty} \psi_{2m, \alpha}(n) q^{\frac{n^2}{4m}} = \sum_{n=0}^{\infty} \psi_{2m, \alpha}(n) e^{\frac{\pi i \tau n^2}{2m}}, \\ \psi_{2m, \alpha}(n) &= \begin{cases} \pm 1 & n \equiv \pm \alpha \pmod{2m}, \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (3.5)$$

In this case the calculation of $\lim_{\tau \rightarrow r/s} \hat{Z}$ is more difficult, but still possible. In [10, 26] we find that

$$\lim_{\tau \rightarrow r/s} \Psi_{m, \alpha} = \sum_{n=0}^{ms} \psi_{2m, \alpha}(n) \left(1 - \frac{1}{ms} \right) e^{\frac{\pi i r n^2}{2ms}}. \quad (3.6)$$

Since this result is an essential tool in section 4, it serves also as the guiding rule in choosing examples for testing our main conjecture.

3.3 Conventions

Before moving to examples let us discuss some conventional issues.

In many papers, e.g. [1, 2, 4], the normalisation of the RT invariant (or the WRT invariant for $\tau = 1/k$) is different. In our notation

$$\text{RT}[S^3; r/s] = 1, \tag{3.7}$$

whereas there

$$\text{RT}_{\text{CS}}[S^2 \times S^1; r/s] = 1. \tag{3.8}$$

We write RT_{CS} because this notation is based on the value of the Chern-Simons partition function for $r/s = 1/k$ (many authors write Z_{CS} instead of RT_{CS} but we want to avoid the confusion with \hat{Z}). The relation between these two conventions is given by

$$\text{RT}[M_3(\Gamma); r/s] = \frac{i\sqrt{2s}}{q^{1/2} - q^{-1/2}} \text{RT}_{\text{CS}}[M_3(\Gamma); r/s]. \tag{3.9}$$

The second issue is related to the \mathbb{Z}_2 symmetry group acting on $\text{Coker}M \cong H_1(M_3(\Gamma), \mathbb{Z})$ by $a \mapsto -a$. Since (3.4) is invariant under this transformation and $\hat{Z}_a = \hat{Z}_{-a}$ we could write

$$\begin{aligned} \text{RT}[M_3(\Gamma); r/s] &= \lim_{q \rightarrow e^{2\pi i \frac{r}{s}}} \frac{\sum_{a \in \text{Coker}(rM)/\mathbb{Z}_2} e^{-2\pi i \frac{s}{r}(a, M^{-1}a)} \sum_{b \in (2\text{Coker}M + \delta)/\mathbb{Z}_2} S'_{ab} \hat{Z}'_b}{2(q^{1/2} - q^{-1/2}) G(s, r)^L}, \\ S'_{ab} &= \frac{\sum_{a' \in \{\mathbb{Z}_2\text{-orbit of } a\}} e^{-2\pi i(a', M^{-1}b)}}{|\det M|^{1/2}}, \\ \hat{Z}'_b &= |\mathbb{Z}_2\text{-orbit of } b| \hat{Z}_b. \end{aligned} \tag{3.10}$$

This convention is often called *folded* whereas ours — *unfolded*. The former is present in [1–4], we use the latter because it is inconvenient to divide $\text{Coker}(rM)$ by \mathbb{Z}_2 for every considered r . We would like to stress that because of that our \hat{Z}_b differs from the folded one (denoted by \hat{Z}'_b) by the factor of 2 if b is not a fixed point of \mathbb{Z}_2 symmetry. Moreover, some papers use different numeration of \hat{Z}_b . Detailed discussion of this issue can be found in [3].

4 Examples

In this section we test our main conjecture (3.4) by comparing it to (2.5) on various examples. All computations are done numerically using **Mathematica**.

4.1 Lens spaces $L(p, 1)$

For the lens space $L(p, 1)$ the plumbing graph Γ is given by



In consequence $L = 1$, $M = [-p]$, $\text{Coker}(rM) = \mathbb{Z}_{rp}$, and $2\text{Coker}M + \delta = 2\mathbb{Z}_p$. However, only for three $b \in 2\mathbb{Z}_p$ the invariant \hat{Z}_b is non-zero [1]

$$\hat{Z}_0 = -2q^{\frac{p-3}{4}}, \quad \hat{Z}_{-2} = \hat{Z}_2 = q^{\frac{p-3}{4}} q^{\frac{1}{p}}. \tag{4.1}$$

Therefore the formula (3.4) reduces to

$$\text{RT}[L(p, 1); r/s] = \lim_{q \rightarrow e^{2\pi i \frac{r}{s}}} \frac{\sum_{a \in \mathbb{Z}_{rp}} e^{2\pi i \frac{a^2}{r}} \sum_{b \in \{-2, 0, 2\}} e^{2\pi i \frac{ab}{p}} \hat{Z}_b}{2\sqrt{p} (q^{1/2} - q^{-1/2}) G(s, r)}. \tag{4.2}$$

On the other hand we can use (2.5) to write

$$\text{RT}[L(p, 1); r/s] = \frac{\sum_{n \in \{1, \dots, s-1\}} q^{(-p)\frac{n^2-1}{4}} \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right)^2}{\sum_{n \in \{1, \dots, s-1\}} q^{(-1)\frac{n^2-1}{4}} \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right)^2} \Bigg|_{q=e^{2\pi i \frac{r}{s}}}. \tag{4.3}$$

Using *Mathematica* we checked that (4.2) and (4.3) give the same result. We compared both formulas for $p = 3, 5, 7, 9, 11$ and r/s up to $16/17$.

4.2 Brieskorn spheres

Brieskorn homology spheres $\Sigma(p_1, p_2, p_3)$ are interesting examples, because in their case $\text{Coker}M = \{0\}$ so we have only one invariant $\hat{Z}_b = \hat{Z}_\delta$ and the RT invariant is equal (up to normalisation) to \hat{Z}_δ [4]

$$\text{RT}[\Sigma(p_1, p_2, p_3); r/s] = \lim_{q \rightarrow e^{2\pi i \frac{r}{s}}} \frac{\hat{Z}_\delta}{2 (q^{1/2} - q^{-1/2})}. \tag{4.4}$$

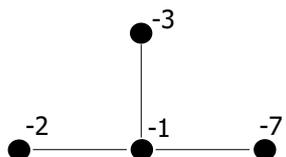
For $r = 1$ this statement immediately follows from (3.2). However $\hat{Z}_\delta/2 (q^{1/2} - q^{-1/2})$ is defined for all $\tau \in \mathbb{H}$ (q inside unit disk) with well-defined limits at all rational τ , so in this case there is no difference between $r = 1$ and other integers. Comparing (4.4) with (3.4) we can see that

$$\frac{\sum_{a \in \text{Coker}(rM)} e^{-2\pi i \frac{a}{r}} S_{a\delta}}{G(s, r)^L} = 1, \tag{4.5}$$

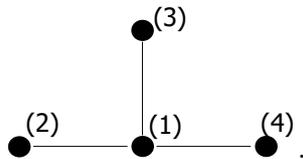
which we numerically checked using *Mathematica*.

4.2.1 $\Sigma(2, 3, 7)$

The graph $\Gamma_{\Sigma(2,3,7)}$ of the $\Sigma(2, 3, 7)$ Brieskorn sphere is given by



We number vertices in the following way (we do it for all 4-vertex graphs in this paper)



In consequence the linking matrix reads

$$M = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -7 \end{bmatrix}, \tag{4.6}$$

so $\det M = 1$ and $\text{Coker} M = \{0\}$. \hat{Z}_δ is given by [2]

$$\hat{Z}_\delta = q^{\frac{83}{168}} (\Psi_{42,1} - \Psi_{42,13} - \Psi_{42,29} + \Psi_{42,41}) \tag{4.7}$$

(There is a typo in [2], $q^{\frac{83}{168}}$ should be in numerator as in (4.7)). Therefore

$$\text{RT}[\Sigma(2, 3, 7); r/s] = \frac{\hat{Z}_\delta|_{\tau=r/s}}{4i \sin(\frac{\pi r}{s})}, \tag{4.8}$$

where

$$\hat{Z}_\delta|_{\tau=r/s} = e^{\frac{83}{84}\pi i \frac{r}{s}} \sum_{n=0}^{42s} (\psi_{84,1}(n) - \psi_{84,13}(n) - \psi_{84,29}(n) + \psi_{84,41}(n)) \left(1 - \frac{1}{42s}\right) e^{\frac{\pi i r}{84s} n^2} \tag{4.9}$$

was calculated by applying (3.6) to (4.7).

The formula (2.5) gives

$$\begin{aligned} \text{RT}[\Sigma(2, 3, 7); r/s] &= \frac{F[\Gamma_{\Sigma(2,3,7)}; r/s]}{F[-\mathbf{1}\bullet; r/s]^4}, \\ F[-\mathbf{1}\bullet; r/s] &= \frac{\sum_{n \in \{1, \dots, s-1\}} q^{(-1)\frac{n^2-1}{4}} \left(q^{\frac{n}{2}} - q^{-\frac{n}{2}}\right)^2}{(q^{1/2} - q^{-1/2})^2} \Bigg|_{q=e^{2\pi i \frac{r}{s}}}, \\ F[\Gamma_{\Sigma(2,3,7)}; r/s] &= \frac{\sum_{n \in \{1, \dots, s-1\}} \prod_{v \in V} T_{n_v n_v}^{a_v} S_{0 n_v}^{2-\text{deg}(v)} \prod_{(v_1, v_2) \in E} S_{n_{v_1} n_{v_2}}}{(q^{1/2} - q^{-1/2})^5} \Bigg|_{q=e^{2\pi i \frac{r}{s}}}, \end{aligned} \tag{4.10}$$

where¹

$$T_{n_v n_v}^{a_v} = q^{\frac{a_v(n_v^2-1)}{4}}, \quad a_v = \begin{cases} -1 & v = 1 \\ -2 & v = 2 \\ -3 & v = 3 \\ -7 & v = 4 \end{cases}$$

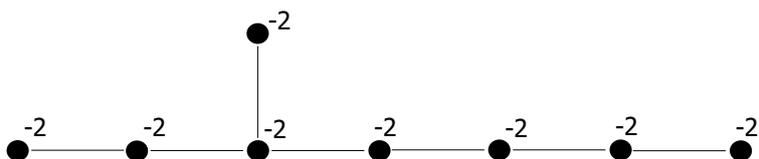
$$S_{n_{v_1} n_{v_2}} = q^{\frac{n_{v_1} n_{v_2}}{2}} - q^{-\frac{n_{v_1} n_{v_2}}{2}}, \quad (v_1, v_2) = (1, 2), (1, 3), (1, 4) \quad (4.11)$$

$$S_{0n_v}^{2-\text{deg}(v)} = \begin{cases} \left[q^{\frac{n_v}{2}} - q^{-\frac{n_v}{2}} \right]^{-1} & v = 1 \\ 1 & v = 2, 3, 4 \end{cases}$$

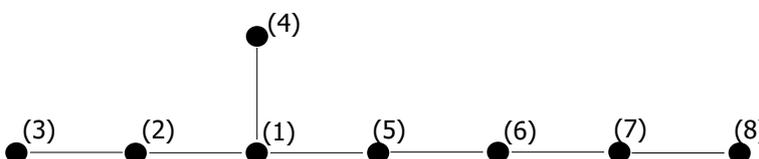
Using Mathematica we checked — for all r/s up to $12/13$ — that (4.8) and (4.10) give the same result.

4.2.2 Poncaré sphere

For the Poincaré sphere $\Sigma(2, 3, 5)$ we have the following plumbing graph $\Gamma_{\Sigma(2,3,5)}$



The numbering



leads to

$$M = \begin{bmatrix} -2 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}. \quad (4.12)$$

We have $\det M = 1$, $\text{Coker} M = \{0\}$ again and \hat{Z}_δ is given by [2]

$$\hat{Z}_\delta = q^{-\frac{181}{120}} \left[2q^{\frac{1}{120}} - (\Psi_{30,1} + \Psi_{30,11} + \Psi_{30,19} + \Psi_{30,29}) \right]. \quad (4.13)$$

¹For simplicity we do not include the $\frac{1}{i\sqrt{2s}}$ prefactor in formulas for S matrices in the whole section 4.

Therefore

$$\text{RT}[\Sigma(2, 3, 5); r/s] = \frac{\hat{Z}_\delta|_{\tau=r/s}}{4i \sin(\pi \frac{r}{s})}, \tag{4.14}$$

where

$$\hat{Z}_\delta|_{\tau=r/s} = e^{-\frac{181}{60}\pi i \frac{r}{s}} \left[2e^{\frac{\pi i r}{60s}} - \sum_{n=0}^{30s} (\psi_{60,1}(n) + \psi_{60,11}(n) + \psi_{60,19}(n) + \psi_{60,29}(n)) \left(1 - \frac{1}{30s}\right) e^{\frac{\pi i r}{60s} n^2} \right]. \tag{4.15}$$

On the other hand equation (2.5) leads to

$$\text{RT}[\Sigma(2, 3, 5); r/s] = \frac{F[\Gamma_{\Sigma(2,3,5)}; r/s]}{F[-\mathbf{1}\bullet; r/s]^8}, \tag{4.16}$$

$$F[\Gamma_{\Sigma(2,3,5)}; r/s] = \frac{\sum_{n \in \{1, \dots, s-1\}}^8 \prod_{v \in V} T_{n_v n_v}^{-2} S_{0n_v}^{2-\text{deg}(v)} \prod_{(v_1, v_2) \in E} S_{n_{v_1} n_{v_2}}}{(q^{1/2} - q^{-1/2})^9} \Bigg|_{q=e^{2\pi i \frac{r}{s}}},$$

where

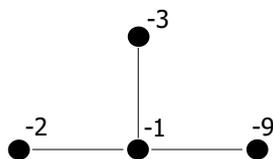
$$\begin{aligned} T_{n_v n_v}^{-2} &= q^{\frac{1-n_v^2}{2}}, \quad v = 1, 2, \dots, 8 \\ S_{n_{v_1} n_{v_2}} &= q^{\frac{n_{v_1} n_{v_2}}{2}} - q^{-\frac{n_{v_1} n_{v_2}}{2}}, \quad (v_1, v_2) = (1, 2), (2, 3), (1, 4), (1, 5), (5, 6), (6, 7), (7, 8) \\ S_{0n_v}^{2-\text{deg}(v)} &= \begin{cases} \left[q^{\frac{n_v}{2}} - q^{-\frac{n_v}{2}} \right]^{-1} & v = 1 \\ 1 & v = 2, 5, 6, 7 \\ q^{\frac{n_v}{2}} - q^{-\frac{n_v}{2}} & v = 3, 4, 8. \end{cases} \end{aligned} \tag{4.17}$$

We have used Mathematica to check that (4.14) and (4.16) give the same result. Having 8 vertices was much more involved for the computer so we stopped at $r/s = 8/9$.

4.3 Other Seifert manifolds

4.3.1 $M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9})$

The Seifert manifold $M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9})$ can be described by the plumbing graph $\Gamma_{M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9})}$



and the linking matrix

$$M = \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -9 \end{bmatrix}. \tag{4.18}$$

Therefore $\det M = 3$ and

$$\text{Coker}M = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 6 \end{bmatrix} \right\}. \tag{4.19}$$

We have

$$\delta = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \Rightarrow b \in 2\text{Coker}M + \delta = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \\ -3 \\ -13 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 3 \\ 13 \end{bmatrix} \right\} \tag{4.20}$$

and \hat{Z} invariants are given by [4]

$$\begin{aligned} \hat{Z}_{[1,-1,-1,-1]} &= q^{71/72} (\Psi_{18,1} + \Psi_{18,17}), \\ \hat{Z}_{[-3,1,3,13]} &= \hat{Z}_{[3,-1,-3,-13]} = -\frac{1}{2}q^{71/72} (\Psi_{18,5} + \Psi_{18,13}). \end{aligned} \tag{4.21}$$

We can use (3.6) to compute $\hat{Z}_b|_{\tau=r/s}$ and then (3.4) leads to

$$\begin{aligned} \text{RT} \left[M \left(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9} \right); \frac{r}{s} \right] &= \frac{\sum_{a \in \text{Coker}(rM)} e^{-2\pi i \frac{s}{r}(a, M^{-1}a)} \sum_{b \in 2\text{Coker}M + \delta} S_{ab} \hat{Z}_b|_{\tau=r/s}}{4i \sin(\pi \frac{r}{s}) G(s, r)^4}, \\ S_{ab} &= \frac{e^{-2\pi i(a, M^{-1}b)}}{\sqrt{3}}. \end{aligned} \tag{4.22}$$

In contrary to the Brieskorn spheres all terms are nontrivial.

On the other hand (2.5) gives

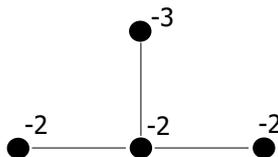
$$\begin{aligned} \text{RT} \left[M \left(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9} \right); \frac{r}{s} \right] &= \frac{F \left[\Gamma_{M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9})}; \frac{r}{s} \right]}{F[-1 \bullet; \frac{r}{s}]^4}, \\ F \left[\Gamma_{M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9})}; \frac{r}{s} \right] &= \frac{\sum_{n \in \{1, \dots, s-1\}^4} \prod_{v \in V} T_{n_v n_v}^{a_v} S_{0n_v}^{2-\deg(v)} \prod_{(v_1, v_2) \in E} S_{n_{v_1} n_{v_2}}}{(q^{1/2} - q^{-1/2})^5} \Bigg|_{q=e^{2\pi i \frac{r}{s}}}, \end{aligned} \tag{4.23}$$

where S and T matrices are the same as in (4.11) except $a_v = -9$ for $v = 4$.

We used `Mathematica` to check that (4.22) and (4.23) give the same result. Because of the necessity of calculating $\text{Coker}(rM)$ for each r it was easier to increase the parameter s and we stopped at $r/s = 7/30$.

4.3.2 $M(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2})$

The Seifert manifold $M(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2})$ has the following plumbing graph $\Gamma_{M(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2})}$



and linking matrix

$$M = \begin{bmatrix} -2 & 1 & 1 & 1 \\ 1 & -2 & 0 & 0 \\ 1 & 0 & -3 & 0 \\ 1 & 0 & 0 & -2 \end{bmatrix}. \quad (4.24)$$

Therefore $\det M = 8$, $\delta = [1, -1, -1, -1]$ and

$$b \in 2\text{Coker}M + \delta = \left\{ \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}, \pm \begin{bmatrix} 3 \\ -1 \\ -5 \\ -3 \end{bmatrix}, \pm \begin{bmatrix} 3 \\ -3 \\ -5 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -3 \\ -1 \\ -3 \end{bmatrix}, \pm \begin{bmatrix} 1 \\ -3 \\ -1 \\ -1 \end{bmatrix} \right\}. \quad (4.25)$$

\hat{Z} invariants are given by [4]

$$\begin{aligned} \hat{Z}_{[1,-1,-1,-1]} &= q^{-\frac{5}{12}} \left[2q^{\frac{1}{24}} - (\Psi_{6,1} + \Psi_{6,7}) \right] \\ \hat{Z}_{\pm[3,-1,-5,-3]} &= \hat{Z}_{\pm[3,-3,-5,-1]} = -\frac{1}{2} q^{-\frac{5}{12}} \Psi_{6,2} \\ \hat{Z}_{[3,-3,-1,-3]} &= -q^{-\frac{5}{12}} (\Psi_{6,1} + \Psi_{6,7}) \\ \hat{Z}_{\pm[1,-3,-1,-1]} &= q^{-\frac{5}{12}} \Psi_{6,4} \end{aligned} \quad (4.26)$$

Following the previous examples we use (3.6) to compute $\hat{Z}_b \Big|_{\tau=r/s}$ and then (3.4) to obtain

$$\begin{aligned} \text{RT} \left[M \left(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2} \right); \frac{r}{s} \right] &= \frac{\sum_{a \in \text{Coker}(rM)} e^{-2\pi i \frac{a}{r}(a, M^{-1}a)} \sum_{b \in 2\text{Coker}M + \delta} S_{ab} \hat{Z}_b \Big|_{\tau=r/s}}{4i \sin\left(\pi \frac{r}{s}\right) G(s, r)^4}, \\ S_{ab} &= \frac{e^{-2\pi i(a, M^{-1}b)}}{\sqrt{8}}. \end{aligned} \quad (4.27)$$

Similarly to $M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9})$ all terms in (4.27) are nontrivial.

Equation (2.5) leads to

$$\begin{aligned} \text{RT} \left[M \left(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2} \right); \frac{r}{s} \right] &= \frac{F \left[\Gamma_{M(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2})}; \frac{r}{s} \right]}{F[-1 \bullet; \frac{r}{s}]^4}, \\ F \left[\Gamma_{M(-2; \frac{1}{2}, \frac{1}{3}, \frac{1}{2})}; \frac{r}{s} \right] &= \frac{\sum_{n \in \{1, \dots, s-1\}^4} \prod_{v \in V} T_{n_v n_v}^{a_v} S_{0n_v}^{2-\deg(v)} \prod_{(v_1, v_2) \in E} S_{n_{v_1} n_{v_2}}}{(q^{1/2} - q^{-1/2})^5} \Bigg|_{q=e^{2\pi i \frac{r}{s}}}, \end{aligned} \quad (4.28)$$

where S and T matrices are the same as in (4.11) except

$$a_v = \begin{cases} -2 & v = 1, 2, 4 \\ -3 & v = 3 \end{cases} \quad (4.29)$$

Using **Mathematica** we checked that (4.27) and (4.28) give the same result. Similarly to $M(-1; \frac{1}{2}, \frac{1}{3}, \frac{1}{9})$ the necessity of calculating $\text{Coker}(rM)$ for each r made it easier to increase the parameter s (however in this case the cokernel is bigger) and we stopped at $r/s = 5/21$.

5 Open questions

The most interesting future direction seems to be the one towards the interpretation of our main conjecture. Do we really have another manifold associated to each r ? The manifold corresponding to the matrix rM is not an r -fold cover of the one corresponding to M and it is difficult to find another topologically reasonable candidate. Or maybe the interpretation should not involve another manifold? But what would the summation over $\text{Coker}(rM)$ mean in this case?

Another goals for future research are the proof of our main conjecture and an investigation of 3-manifolds that are not Seifert and — more generally — not plumbed.

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