DIELECTRIC PROPERTIES OF A LATTICE OF ANISOTROPIC PARTICLES

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Abstract

The dielectric properties of lattices composed of identical metallic or dielectric elements of various geometries, such as spheres, discs and strips have been investigated from a molecular point of view by Kock[(1)], Corkum[(2)] and others. These investigations have treated two cases, in both of which, the element size and spacings are small compared to the wave length. The first applies to the case where the spacings are large compared to element size and which therefore neglects interaction effects. The second treats interaction for the special case in which the lattice has the structural isotropy of a cubical array and for which the application of the Clausius-Mosotti relation is valid, when the elements are not too closely packed.

The main objective of this note will be to extend the treatment to general uniform lattice structures made of identically shaped and oriented particles of general constitutive characteristics. Thus, it will include the most general case of a uniform lattice with structural anisotropy and both element isotropy and anisotropy at the lattice points.

Introduction

It is known that if a uniform electric field of strength $\vec{E_0}$ is applied to an array of like elements (see Figure 1), a distribution of charge is established within each element. The total charge of each element remains zero but is redistributed in such a way that an additional field is created. A first approximation to this extra field is obtained by examining the charge distribution of an isolated element in a uniform field. This induced field can, for a generalized element, be considered as being approximately equivalent to a set of three dipoles respectively parallel to each of the coordinate axes. The resultant of these dipoles, the electric dipole moment vector, will be denoted by $\vec{p} = p_x \vec{a_x} + p_y \vec{a_y} + p_z \vec{a_z}$ where $\vec{a_x}$, $\vec{a_y}$ and $\vec{a_z}$ are the unit coordinate vectors. The magnitude of $\vec{p}$ is proportioned to the magnitude of the applied field $\vec{E_o}$.

$$\vec{p} = (\alpha) \cdot \vec{E_o} \tag{1}$$

The constant of proportionality (α) is the polarizability tensor and its value depends on the geometry and material of the element. The polarization $\vec{P}$ is the total dipole moment per unit volume, so that if there are N elements per unit volume

$$\vec{P} = N\,\vec{p} = N\,(\alpha) \cdot \vec{E_o} \tag{2}$$

The dielectric constant tensor (k_e), is related to the electric field strength and the polarization by the equation

$$\vec{D} = (\epsilon) \cdot \vec{E_o} = (\epsilon_o) \cdot \vec{E_o} + \vec{P} \tag{3}$$

where

$\vec{D}$ - the displacement vector

ϵ_o - permittivity of free space

ϵ - permittivity of the lattice medium ($\epsilon = \epsilon_o\, k_e$)

k_e - dielectric constant of the lattice medium.

Substitution for $\vec{p}$ from Equation (2) yields

$$\begin{aligned}(k_e) &= 1 + \frac{N(\alpha)}{\epsilon_o} \\ &= 1 + (x_e)\end{aligned} \tag{4}$$

In the foregoing, it was assumed that the field acting on each individual element in the presence of the others remained equal to the externally applied

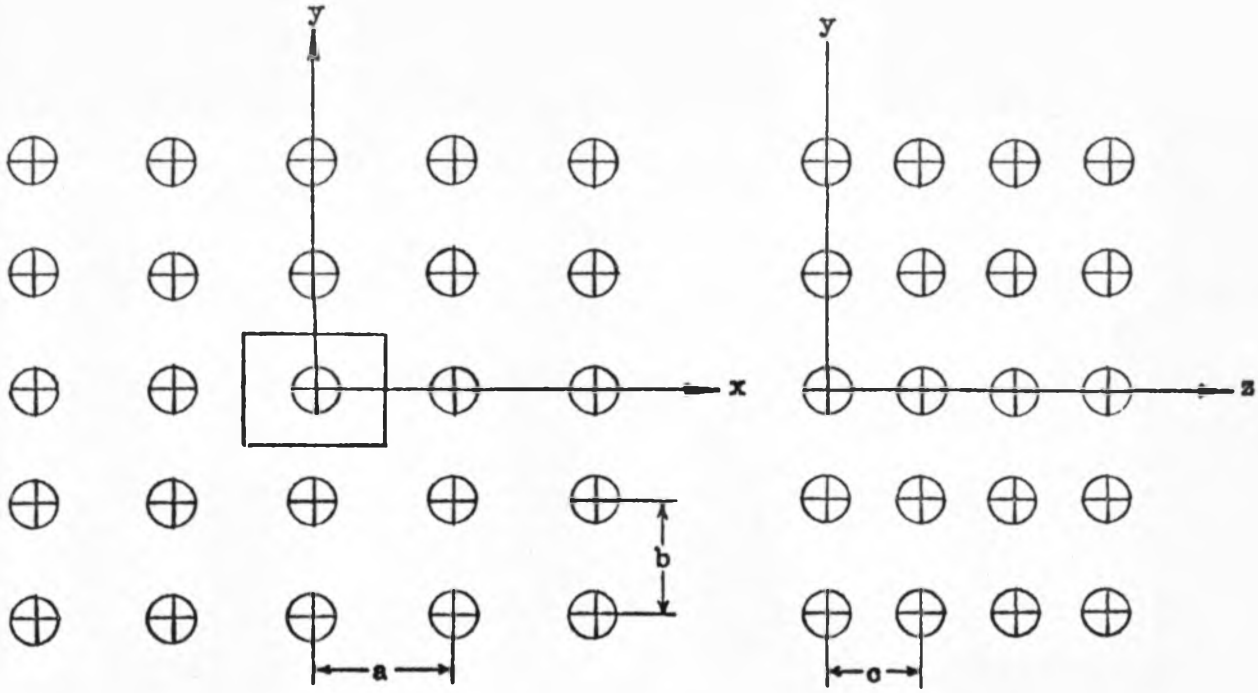


Figure 1 - A Tetragonal Array.

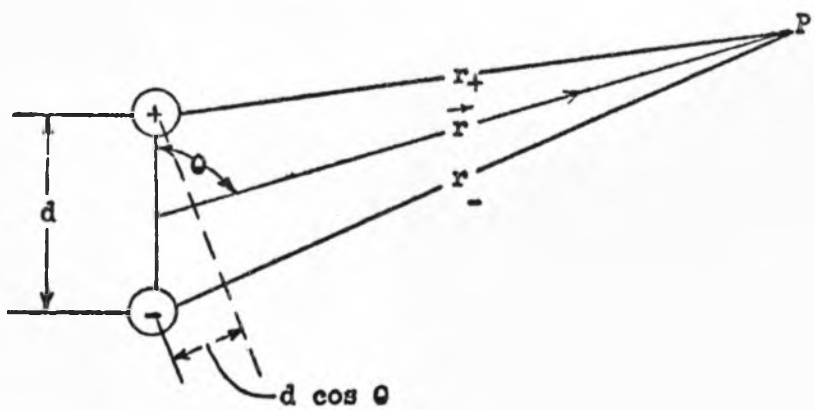


Figure 2 - Dipole and Distant Point P.

field and that the charge distribution of each element corresponds to that of a set of dipoles. The first restriction will be removed in the next section by including the contributions of all elements in the exciting field of each element. The second restriction implies that the elements of the array are not too closely packed and this will be assumed to be the case in the following. That is, the packing is so close that interactions must be considered, but not so close that the resulting field distortion can no longer be described by simple dipole fields.

Section I - <u>Evaluation of the Dielectric Constant of an Array of Isotropic Elements</u>.

The method used to take interaction into consideration is a standard one used in the molecular theory of solids and consists simply of calculating the additional field acting on each element because of the dipoles induced in each of the other elements. To consider the interaction field $\vec{E}_s$, consider a dipole with charges $\pm q$ separated by a distance d (see Figure 2). The potential φ at a distance r from its center is given by

$$\varphi = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{r_+} - \frac{1}{r_-}\right)$$

For $r \gg d$, $r_- - r_+ \simeq d \cos\theta$ $r_- r_+ = r^2$ and hence

$$\varphi = \frac{q\,d}{4\pi\epsilon_0 r^2}\cos\theta = \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3} \qquad (5)$$

$$(q\,d = p)$$

The field at a distance $\vec{r}$ from this dipole is

$$\vec{E} = -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\left(\frac{\vec{p}\cdot\vec{r}}{r^3}\right)$$

$$= -\frac{1}{4\pi\epsilon_0}\left[\frac{\vec{p}}{r^3} - \frac{3(\vec{p}\cdot\vec{r})\vec{r}}{r^5}\right] \qquad (6)$$

This can be expanded in rectangular coordinates as

$$E_x = \frac{1}{4\pi\epsilon_o}\left(\frac{3x^2 - r^2}{r^5} p_x + \frac{3xy}{r^5} p_y + \frac{3xz}{r^5} p_z\right)$$

$$E_y = \frac{1}{4\pi\epsilon_o}\left(\frac{3xy}{r^5} p_x + \frac{3y^2 - r^2}{r^5} p_y + \frac{3yz}{r^5} p_z\right)$$

$$E_z = \frac{1}{4\pi\epsilon_o}\left(\frac{3xz}{r^5} p_x + \frac{3yz}{r^5} p_y + \frac{3z^2 - r^2}{r^5} p_z\right)$$

It follows that each component of $\vec{E}$ is a linear function of the components of $\vec{p}$, the coefficients being functions of the components x, y, z of the radius vector $r = \sqrt{x^2 + y^2 + z^2}$. The interaction field $\vec{E_s}$ due to an infinite array of elements can be computed by considering the element at the origin of coordinates removed (see Figure 1), the array being otherwise unchanged, and the contributions of the infinite number of dipole elements summed up at the origin. The interaction field $\vec{E_s}$ is therefore given by

$$\vec{E_s} = (T) \cdot \vec{p} \tag{7}$$

where (T) is a symmetric tensor whose matrix is

$$(T) = \frac{1}{4\pi\epsilon_o}\begin{bmatrix} \sum\limits_{\mu} \frac{3x^2 - r^2}{r^5} & \sum\limits_{\mu} \frac{3xy}{r^5} & \sum\limits_{\mu} \frac{3xz}{r^5} \\ \sum\limits_{\mu} \frac{3xy}{r^5} & \sum\limits_{\mu} \frac{3y^2 - r^2}{r^5} & \sum\limits_{\mu} \frac{3yz}{r^5} \\ \sum\limits_{\mu} \frac{3xz}{r^5} & \sum\limits_{\mu} \frac{3yz}{r^5} & \sum\limits_{\mu} \frac{3z^2 - r^2}{r^5} \end{bmatrix}$$

μ is the summation index and the sums are extended over all elements within the array except the single element at the center of coordinates. For the case when the elements themselves are isotropic the induced dipole moment vector will be given by:

$$\vec{p} = \alpha\ (\vec{E}_o + \vec{E}_s)$$
$$= \alpha\ \left[(T) \cdot \vec{p} + \vec{E}_o \right] \tag{8}$$

where α as mentioned earlier is the electric polarizability of the isotropic elements and is a scalar quantity. If Equation (8) is solved for $\vec{p}$, there results:

$$\vec{p} = - \left[(T) - \left(\frac{1}{\alpha}\right) \right]^{-1} \cdot \vec{E}_o \tag{9}$$

where $\left(\frac{1}{\alpha}\right)$ is the matrix $\frac{1}{\alpha}$. Let R denote the matrix $(T) - \left(\frac{1}{\alpha}\right)$ then,

$$\vec{p} = \left(R^{-1}\right) \cdot \vec{E}_o \tag{10}$$

where

$$(R) = \begin{bmatrix} T_{xx} - \frac{1}{\alpha} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} - \frac{1}{\alpha} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} - \frac{1}{\alpha} \end{bmatrix}$$

Substitution of the value of $\vec{p}$ in the following equation

$$(\epsilon) \cdot \vec{E}_o = \epsilon_o \vec{E}_o + N \vec{p} \tag{11}$$

yields

$$(k_e) - (1) = - \frac{N}{\epsilon_o} \left(R^{-1}\right) \tag{12}$$

where (1) is the unitary matrix. The inverse of the matrix (R) is given by

$$\left(R^{-1}\right) = \frac{\left(R^{ji}\right)}{\det (R)}$$

The matrix $\left(R^{ji}\right)$ is formed by arranging the cofactors of the elements R_{ij} in a matrix array and then transposing the rows and columns of the resulting matrix. From Equation (12) the elements of the dielectric constants tensor are given by the following expressions

$$k_{exx}-1 = -\frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{yy}-1 & T_{yz} \\ T_{zy} & T_{zz}-\frac{1}{a} \end{vmatrix}}{\det(R)} \qquad k_{exy} = \frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{xy} & T_{xz} \\ T_{zy} & T_{zz}-\frac{1}{a} \end{vmatrix}}{\det(R)}$$

$$k_{exz} = -\frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{xy} & T_{xz} \\ T_{yy}-\frac{1}{a} & T_{yz} \end{vmatrix}}{\det(R)} \qquad k_{eyx} = \frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{yx} & T_{yz} \\ T_{zx} & T_{zz}-\frac{1}{a} \end{vmatrix}}{\det(R)}$$

$$k_{eyy}-1 = -\frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{xx}-\frac{1}{a} & T_{xz} \\ T_{zx} & T_{zz}-\frac{1}{a} \end{vmatrix}}{\det(R)} \qquad k_{eyz} = \frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{xx}-\frac{1}{a} & T_{xz} \\ T_{yx} & T_{yz} \end{vmatrix}}{\det(R)} \tag{13}$$

$$k_{ezx} = -\frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{yx} & T_{yy}-\frac{1}{a} \\ T_{zx} & T_{zy} \end{vmatrix}}{\det(R)} \qquad k_{ezy} = \frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{xx}-\frac{1}{a} & T_{xy} \\ T_{zx} & T_{zy} \end{vmatrix}}{\det(R)}$$

$$k_{ezz}-1 = \frac{N}{\epsilon_o}\,\frac{\begin{vmatrix} T_{xx}-\frac{1}{a} & T_{xy} \\ T_{yx} & T_{yy}-\frac{1}{a} \end{vmatrix}}{\det(R)}$$

Since the elements were assumed to be isotropic, it is important to observe that the tensor nature of the dielectric constant is entirely due to the geometry of the lattice medium.

The Tetragonal Lattice

For a tetragonal array, it can easily be shown that the elements T_{xy}, T_{xz}, T_{yx}, T_{yz}, T_{zx}, T_{zy} are zero and the dielectric constant reduces to a diagonal tensor of the form

$$(k_e) = (1) - \frac{N}{\epsilon_o} \begin{bmatrix} \frac{1}{T_{xx} - \frac{1}{\alpha}} & 0 & 0 \\ 0 & \frac{1}{T_{yy} - \frac{1}{\alpha}} & 0 \\ 0 & 0 & \frac{1}{T_{zz} - \frac{1}{\alpha}} \end{bmatrix} \tag{14}$$

and therefore

$$k_{exx} = 1 + \frac{N\alpha/\epsilon_o}{1 - \alpha\, T_{xx}}, \qquad k_{eyy} = 1 + \frac{N\alpha/\epsilon_o}{1 - \alpha\, T_{yy}}, \qquad k_{ezz} = 1 + \frac{N\alpha/\epsilon_o}{1 - \alpha\, T_{zz}} \tag{15}$$

In which, for the tetragonal array

$$T_{xx} = \frac{1}{4\pi\epsilon_o} \sum\sum\sum \frac{2m_1a^2 - b^2m_2{}^2 - c^2m_3{}^2}{(m_1{}^2a^2 + m_2{}^2b^2 + m_3{}^2c^2)^{5/2}} \tag{16}$$

where a, b and c are the interelemental spacings as shown in Figure (1) and m_1, m_2 and m_3 are any integers. Unfortunately this series is only conditionally convergent and different values can be obtained for its sum depending on the order in which the summations are performed. Lorentz[3] has used a convenient scheme by which he replaces the collection of dipoles by a continuous dielectric and then removing a sphere at the origin. This series will be considered again later in the discussion and at present the special case of a cubical lattice where $a = b = c$ will be considered in detail.

The Cubical Lattice.

For a cubical lattice T_{xx} becomes:

$$T_{xx} = \frac{1}{a^3\, 4\pi\epsilon_o} \sum{}' \sum{}' \sum{}' \frac{2m_1^2 - m_2^2 - m_3^2}{(m_1^2 + m_2^2 + m_3^2)^{5/2}}$$

where a is the interelemental spacing. This can be written as

$$T_{xx} = \frac{1}{4\pi\epsilon_o a^3} \left\{ 2\sum_{m_1=1}^{\infty} \frac{2}{m_1^3} - 2\sum_{m_2=1}^{\infty} \frac{1}{m_2^3} - 2\sum_{m_3=1}^{\infty} \frac{1}{m_3^3} \right\} +$$

$$+ \frac{2}{\pi\epsilon_o a^3} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \frac{2m_1^2 - m_2^2 - m_3^2}{(m_1^2 + m_2^2 + m_3^2)^{5/2}} \tag{17}$$

The first three single index series cancel each other and T_{xx} becomes

$$T_{xx} = \frac{2}{\pi\epsilon_o a^3} \sum_{m_1=1}^{\infty} \sum_{m_2=1}^{\infty} \sum_{m_3=1}^{\infty} \frac{2m_1^2 - m_2^2 - m_3^2}{(m_1^2 + m_2^2 + m_3^2)^{5/2}} \tag{18}$$

To evaluate the above series, the Poisson transformation is used. For a triple series this transformation can be written as

$$\sum_{a}^{\infty} \sum_{b}^{\infty} \sum_{c}^{\infty} f(m_1,m_2,m_3) = -\frac{f(abc)}{8} + \frac{1}{2} \left\{ \sum_{m_2=b}^{\infty} \sum_{m_3=c}^{\infty} f(a,m_2,m_3) + \right.$$

$$\left. + \sum_{m_1=a}^{\infty} \sum_{m_3=c}^{\infty} f(m_1,b,m_3) + \sum_{m_1=a}^{\infty} \sum_{m_2=b}^{\infty} f(m_1,m_2,c) \right\} + \tag{19}$$

$$+ \int_a^{\infty} \int_b^{\infty} \int_c^{\infty} f(\alpha\, \beta\, \gamma)\, d\alpha\, d\beta\, d\gamma +$$

$$8 \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} \sum_{n_3=1}^{\infty} \int_a^{\infty} \int_b^{\infty} \int_c^{\infty} f(\alpha\beta\gamma) \cos 2\pi n_1\alpha \cos 2\pi n_2\beta \cos 2\pi n_3\gamma d\alpha d\beta d\gamma$$

The symmetry of the expression causes the first term and the double index series to cancel. The triple summation of the Poisson transformation formula represents corrections to the trapezoidal rule and is negligible in this case, because the expression for T_{xx} varies slowly with respect to m_1 , m_2 and m_3 . Hence,

$$T_{xx} = \frac{2}{\pi \epsilon_o a^3} \int_{1/2}^{\infty} \int_{1/2}^{\infty} \int_{1/2}^{\infty} \frac{2\alpha^2 - \beta^2 - \gamma^2}{(\alpha^2 + \beta^2 + \gamma^2)^{5/2}} d\alpha d\beta d\gamma \tag{20}$$

Carrying on the integration with respect to α ,

$$T_{xx} = \frac{2}{\pi \epsilon_o a^3} \int_{1/2}^{\infty} \int_{1/2}^{\infty} \left[\frac{-\alpha}{(\alpha^2 + \beta^2 + \gamma^2)^{3/2}} \right]_{1/2}^{\infty} d\beta d\gamma$$

$$= \frac{1}{\pi \epsilon_o a^3} \int_{1/2}^{\infty} \int_{1/2}^{\infty} \frac{d\beta d\gamma}{(1/4 + \beta^2 + \gamma^2)^{3/2}} \tag{21}$$

Using the formula,

$$\int \frac{dx}{(a + bx^2)^{m+1}} = \frac{1}{2ma} \frac{x}{(a + bx^2)^m} + \frac{2m - 1}{2ma} \int \frac{dx}{(a + bx^2)^m}$$

Equation (21) becomes

$$T_{xx} = \frac{1}{\pi \epsilon_o a^3} \int_{1/2}^{\infty} \left[\frac{1}{(1/4 + \gamma^2)} \frac{\beta}{(1 + \beta^2 + \gamma^2)^{1/2}} \right]_{1/2}^{\infty} d\gamma$$

$$= \frac{1}{\pi \epsilon_0 a^3} \int_{1/2}^{\infty} \left[\frac{1}{1/4 + \gamma^2} \quad \frac{1/2}{(1/4 + \gamma^2)(1/2 + \gamma^2)^{1/2}} \right] d\gamma \qquad (22)$$

Making the substitution

$$1/4 + \gamma^2 = u^2$$

in the second term of the integral, there results

$$T_{xx} = \frac{1}{\pi \epsilon_0 a^3} \left[\int_{1/2}^{\infty} \frac{1}{1/4 + \gamma^2} d\gamma - 1/2 \int_{1/2}^{\infty} \frac{du}{u(u^2 - 1/16)^{1/2}} \right]$$

$$= \frac{1}{\pi \epsilon_0 a^3} \left\{ 2 \tan^{-1} \frac{\gamma}{1/2} \Bigg|_{1/2}^{\infty} - \cos^{-1} \frac{1/4}{u^2} \Bigg|_{1/\sqrt{2}}^{\infty} \right\} \qquad (23)$$

$$= \frac{1}{\pi \epsilon_0 a^3} \frac{\pi}{3} = \frac{N}{3\epsilon_0} \qquad \left(\text{since } N = \frac{1}{a^3} \right)$$

Substitution of Equation (23) in Equation (15) yields:

$$k_{exx} = 1 + \frac{N\alpha/\epsilon_0}{1 - \frac{\alpha N}{3\epsilon_0}}$$

The expression for k_{exx} is often written

$$\frac{k_{exx} - 1}{k_{exx} + 2} = \frac{N\alpha}{3\epsilon_0} \qquad (24)$$

and is known as the Clausius-Mosotti formula. This formula has been used to determine the microwave properties of artificial dielectrics when the dimensions and spacings of the elements are small with respect to wave length and the geometry of the obstacles possesses a three dimensional symmetry which permits the evaluation of averaged constitutive parameters. It is important to realize that Equation (24) is valid only when the lattice is cubical, otherwise the application of the Clausius-Mosotti relation will be an approximation[4].

Evaluation of the Tensor Components

The usefulness of Equation (12) for the determination of the dielectric constants of lattices of general shape depends on the tractability of the summations expressing the tensor components. However, for any infinite uniform lattice, it is possible to approximate the summation by the following artifice. Consider the lattice to be a cubical one except in the neighborhood of the origin of coordinates at which point the interaction field $\vec{E}_s$ is being computed. The physical justification for this is that, for a central region sufficiently large, the value of the summation of the elements outside does not appreciably depend on small deviations from a cubical pattern. Furthermore, when the array is cubical, the contribution of the interaction field $\vec{E}_s$ is zero within a finite sphere or cube around the origin. Therefore if we excise the central non-cubical region and substitute for it a cubical one, we have added no contribution to the remainder of the array. As a consequence of this reasoning, it is clear that any uniform lattice can be replaced by an infinite cubical lattice with interelemental spacing $\frac{1}{\sqrt[3]{N}}$, plus a finite spherical or cubical region with the original lattice pattern and centered around the point where the interaction field is being evaluated (see Figure 3). Therefore the summations expressed by the components of the structural anisotropy tensor (T) will consist of two parts, (a) the contribution of the infinite cubical array to be denoted by T'_{ij} and (b) the contribution of the finite region with the original lattice pattern, to be denoted by T''_{ij} . The values of primed quantities can be expressed by

$$T'_{ij} = \frac{N}{3\epsilon_0}\, \delta(i - j) \qquad \left[\text{see Equations (16) - (23)}\right] \quad \left.\begin{matrix} i \\ j \end{matrix}\right\} = x,y,z$$

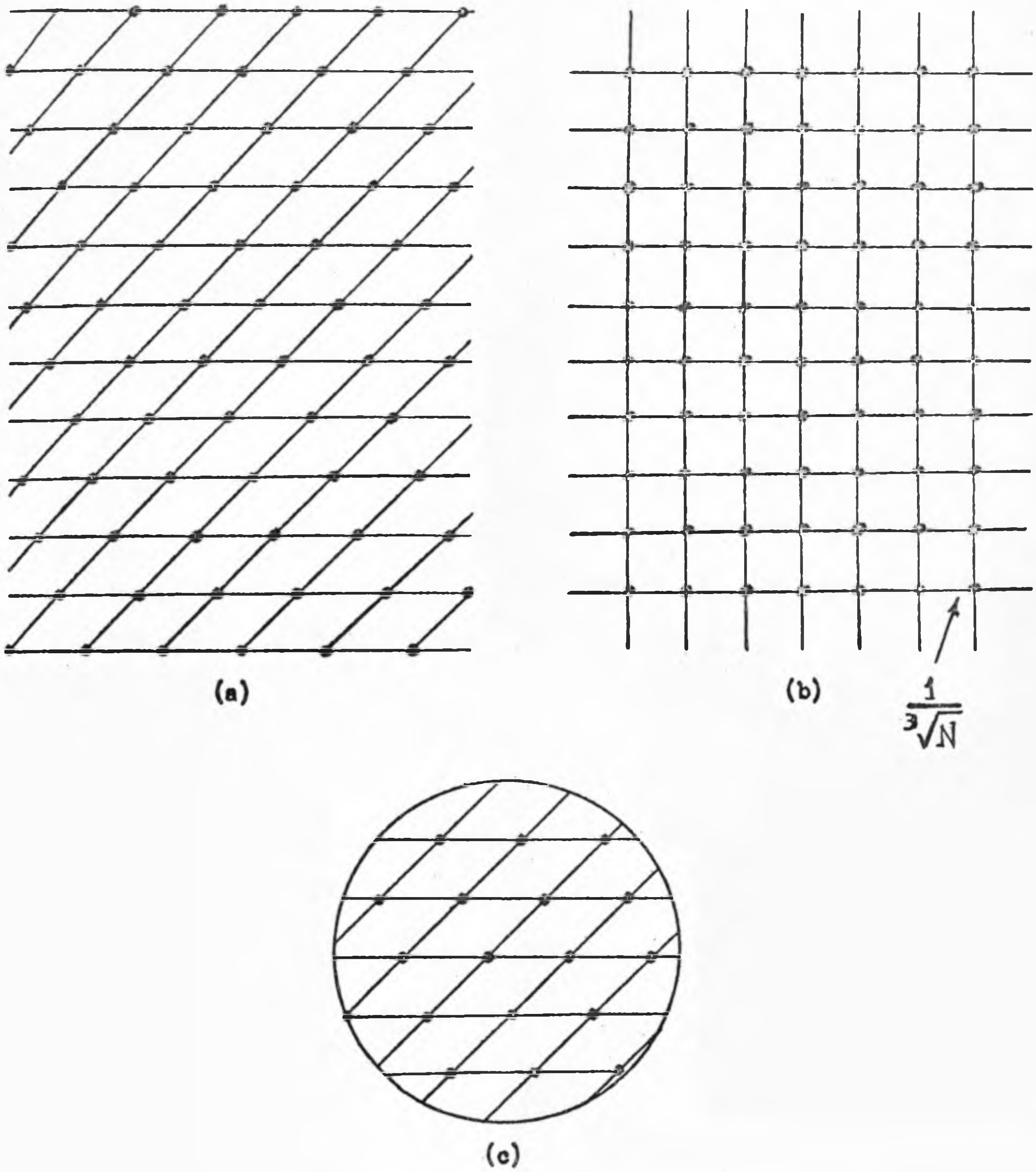


Figure 3 (a) An Infinite Uniform Lattice

= (b) An Infinite Cubical Lattice

+ (c) A Finite Region of the Lattice Shown in (a).

where $\delta(i - j)$ is the delta function. To evaluate T''_{ij} the summation must be carried on term by term within the finite sphere or cube centered around the origin. It should be noted that this finite region can be surprisingly small and still render the approximation valid, since the greatest contribution to the interaction field find their source in elements very near the origin[5]. For a tetragonal array $a = b = 5$ units and $c = 1$ unit, the values of T''_{xx} for different trial radii are:

$$4\pi\epsilon_0 T''_{xx} = -1.182152 \quad , \quad r = \sqrt[3]{1200} \text{ units}$$

$$4\pi\epsilon_0 T''_{xx} = -1.181886 \quad , \quad r = \sqrt[3]{1875} \text{ units}$$

For a cube

$$4\pi\epsilon_0 T''_{xx} = -1.181838 \qquad \text{side 30 units}$$

It should also be noted that

$$T''_{xx} + T''_{yy} + T''_{zz} = \frac{1}{4\pi\epsilon_0} \sum_{\mu} \frac{3x^2 - r^2}{r^5} + \frac{3y^2 - r^2}{r^5} + \frac{3z^2 - r^2}{r^5} = 0 \tag{25}$$

An application of this method will be illustrated in Section II, Example 2.

Section II - Evaluation of the Dielectric Constant of an Array of Anisotropic Elements.

In the previous section, it was shown that for a general lattice geometry, the dielectric constant is a tensor which reduces to a diagonal tensor for a tetragonal array and to a scalar in the case of a cubical arrangement. The expressions derived are valid only for the case where the lattice elements themselves are isotropic. When the assumption of element isotropy is removed, however, α , (the polarizability of these elements) becomes itself a tensor. The expression for the dielectric tensor components derived in the previous section will now be generalized to include the anisotropy of the elements.

When α is a tensor, Equation (8) becomes

$$\vec{p} = (\alpha) \left[(T) \cdot \vec{p} + \vec{E}_o \right] \tag{26}$$

where

$$(\alpha) = \begin{bmatrix} \alpha_{xx} & \alpha_{xy} & \alpha_{xz} \\ \alpha_{yx} & \alpha_{yy} & \alpha_{yz} \\ \alpha_{zx} & \alpha_{zy} & \alpha_{zz} \end{bmatrix}$$

Solving for $\vec{p}$, yields

$$\vec{p} = - \left[(\alpha)\ (T) - (1) \right]^{-1} (\alpha) \cdot \vec{E}_o$$

Using the abbreviations $(\alpha)\ (T) - (1) = (S)$, and substitution of the value of $\vec{p}$ in Equation (11) yields

$$(k_e) - (1) = - \frac{N}{\epsilon_o} \left(S^{-1} \right) (\alpha) \tag{27}$$

Example 1 - A Cubical Array of Ferrite Spheres.

Element anisotropy can be realized in two ways which may be described as element anisotropy due to material and element isotropy due to geometry.

An example of the former is the case of anisotropic ferrite or gaseous elements of any shape immersed in a magnetostatic field while an example of the latter is the case of metallic or dielectric objects of non-spherical shape. There can, therefore, in the most general case be three orders of anisotropy in a lattice—structural anisotropy at the lattice level, geometrical anisotropy at the lattice element level and material anisotropy of the elements on a molecular level.

The particular case of ferrite spheres in a cubical array will now be considered because of the great interest shown in recent years in propagation through generalized ferrite loaded regions.

Polder[6] has shown that a ferromagnetic medium which is homogeneously magnetized to saturation by a magnetostatic field is characterized by a tensor permeability. That is, the radio frequency magnetic field intensity $\vec{h}$ and flux density $\vec{b}$ are related by

$$\vec{b} = (Q) \cdot \vec{h} \tag{28}$$

where (Q) is the permeability tensor and has the form:

$$(Q) = \mu_o \begin{bmatrix} \mu & -jK & 0 \\ jK & \mu & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

when the magnetostatic field is along z-direction. The quantities

$$\mu = \mu' - j\mu''$$
$$K = K' - jK''$$

are complex. However, many ferrites exist for which the magnetic losses are extremely small provided the orienting field within the body is kept small so that the frequency of the wave propagating in the ferrite does not approach ferromagnetic resonance frequency. In the derivation here, we shall assume lossless ferrites. Equations which give μ and K, in terms of the applied magnetic field and the fundamental atomic constants are given by Hogan[7]. μ_o is the permeability of free space.

To determine the polarizability tensor of an isolated ferrite sphere, it will be assumed that the diameter of the sphere is small compared with the wavelength of the electromagnetic field outside as well as inside the sphere, in which case the external field can be considered uniform over the volume of the sphere. Therefore, the term $\frac{dD}{dt}$ is dropped from Maxwell's equations and the problem may be treated in a quasistatic way. Consequently the problem is reduced to the determination of the potential φ satisfying the following boundary conditions

(1).

$$\nabla\varphi^2 = 0 \qquad \text{outside the ferrite sphere}$$
$$\nabla \cdot \left[(\mu) \cdot \nabla\varphi\right] = 0 \qquad \text{inside the ferrite sphere} \tag{29}$$

(2).

$$\varphi^+ = \varphi^- \qquad \text{across the surface of the sphere} \tag{30}$$

(3).

$$\mu_o\left(\frac{\partial\varphi}{\partial r}\right)^+ = \vec{i}_r \cdot \left[(\mu) \cdot \left(\nabla\varphi\right)\right]^- \tag{31}$$

where φ^- is the potential within the sphere and φ^+, the potential outside the sphere. $\vec{i}_r$ is the unit vector along r-direction. Assume an incident magnetic field $\vec{H}_o = H_x\vec{a}_x + H_y\vec{a}_y$ where $\vec{a}_x$ and $\vec{a}_y$ are the unit coordinate vectors. Then at large distances from the ferrite sphere the magnetic field must reduce to $\vec{H}_o$ by virtue of the fact that the dipole fields vanish at large distances. On the basis of the solution for the magnetostatic potential of a dielectric sphere in a uniform magnetic field, it can be assumed that φ^- and φ^+ have the following forms:

$$\varphi^+ = - H_x\, x + b_1 \frac{x}{r^3} - H_y\, y + b_2 \frac{y}{r^3} \tag{32}$$

and

$$\varphi^- = c_1\, x + c_2\, y \tag{33}$$

The second and fourth terms on the right hand side of Equation (32) represent the potential of two dipoles parallel to the x and y axes respectively. This is in accordance with the anisotropic behavior of ferrites by virtue of which magnetic fields are produced along both the x and y axes when the incident magnetic field is oriented along either x or y axes. It is also important to note that the assumption of uniform fields within the ferrite is valid only for elements of ellipsoidal shape. To evaluate the coefficients b_1, b_2 and c_1 and c_2, the boundary conditions (2) and (3) are imposed at different points on the sphere; namely $P_1(r,\theta,\psi) = (r_1,\pi/2,0)$ and $P_2(r,\theta,\psi) = (r_1,\pi/2,\pi/2)$, where r_1 represents the radius of the sphere. Imposing boundary condition (2) results in

$$b_1 = (c_1 + H_x)\, r_1^{\,3} \tag{34}$$

$$b_2 = (c_2 + H_y)\, r_1^{\,3} \tag{35}$$

while boundary condition (3), expanded in spherical coordinates, becomes

$$\left(\frac{\partial\varphi}{\partial r}\right)^+ = \left[\sin^2\theta\,(\mu - 1) + 1\right]\left(\frac{\partial\varphi}{\partial r}\right)^- +$$

$$+ (\mu - 1)\,\frac{\sin 2\theta}{2r}\left(\frac{\partial\varphi}{\partial\theta}\right)^- - j\,\frac{K}{r}\left(\frac{\partial\varphi}{\partial\psi}\right)^-$$

across the surface of the sphere (36)

Substituting the values of the derivatives of Equations (32) and (33) in the above at points P_1 and P_2 on the surface of the sphere leads to

$$- H_x - \frac{2b_1}{r_1^3} = \mu\, c_1 - j\, K\, c_2 \tag{37}$$

$$- H_y - \frac{2b_2}{r_1^3} = \mu\, c_2 + j\, K\, c_1 \tag{38}$$

The solution of simultaneous Equations (34), (35), (37) and (38) gives

$$b_1 = m_x = 4\pi\mu_o r_1^3\ \frac{\mu^2 + \mu - 2 - K^2}{(\mu + 2)^2 - K^2}\ H_x - 4\pi\mu_o r_1^3\ \frac{3\, j\, K}{(\mu + 2)^2 - K^2}\ H_y$$

$$b_2 = m_y = 4\pi\mu_o r_1^3\ \frac{3\, j\, K}{(\mu + 2)^2 - K^2}\ H_x + 4\pi\mu_o r_1^3\ \frac{\mu^2 + \mu - 2 - K^2}{(\mu + 2)^2 - K^2}\ H_y$$

(39)

These expressions agree with those obtained by Polder[6] for the magnetic moment of an ellipsoid when the demagnetizing factors are replaced by $\frac{4\pi}{3}$, which is the value of the demagnetizing factor for a spherically shaped dielectric object. By virtue of Equation (28), $b_z = \mu_o\, h_z$, and therefore the z-component of the magnetic moment vector, $m_z = 0$. It may be noted that each component of the incident magnetic field induces two perpendicular dipole fields in time quadrature.

From Equation (39), the magnetic polarizability tensor can be written as:

$$(a_m) = \begin{bmatrix} u & -j\, v & 0 \\ j\, v & u & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

where

$$u = 4\pi\mu_o r_1^{\,3}\,\frac{\mu^2 + \mu - 2 - K^2}{(\mu + 2)^2 - K^2}$$

$$v = 4\pi\mu_o r_1^{\,3}\,\frac{3\,j\,K}{(\mu + 2)^2 - K^2} \qquad (41)$$

The results obtained previously for the dielectric tensor of an array of anisotropic elements apply identically for the computation of the permeability tensor of such an array. Therefore the result of Equation (27) can be used in this application. For a cubical array the structural anisotropy tensor (T) reduces to a scalar to be denoted by A, [see Equation (14)] where $T_{xx} = T_{yy} = T_{zz} = A$. Substituting the values of (T) and (a) in Equation (27) gives

$$(k_m) = \begin{bmatrix} u' & -j\,v' & 0 \\ j\,v' & u' & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (42)$$

where

$$u' = 1 + \frac{N}{\mu_o}\,\frac{u + A(v^2 - u^2)}{(Au - 1)^2 - A^2v^2}$$

$$v' = \frac{N}{\mu_o}\,\frac{v}{(Au - 1)^2 - A^2v^2} \qquad (43)$$

(k_m) is the relative permeability tensor of the lattice medium; A is a constant equal to $\frac{N}{3\,\epsilon_o}$ [see Equation (23)]; and u and v are given by Equation (41). It is important to note that the permeability tensor $\mu_o(k_m)$ for the lattice medium has the same form as (Q), the permeability tensor of the infinite ferrite medium. Consequently, the quantitative description of the behavior of all phenomenon involving an infinite ferrite medium can by simple substitution be applied to the lattice medium. As an example, the phase velocities of the two counter-rotating circularly polarized plane waves, into which a plane wave traveling in the z-direction in the lattice medium can be resolved, are expressed as:

$$\beta_+ = \frac{\omega}{c'}\sqrt{k'_{ef}\,(\mu' - \nu')} \qquad \left(c' = \frac{1}{\sqrt{\mu_o \epsilon_o}} \right)$$

$$\beta_- = \frac{\omega}{c'}\sqrt{k'_{ef}\,(\mu' + \nu')} \tag{44}$$

The $\pm$ signs refer to positive and negative circularly polarized waves. A positive circularly polarized wave is one which rotates in the direction of the positive current producing the dc magnetic field. k'_{ef} represents the dielectric constant of the lattice medium, and is a scalar, because the lattice is cubical and the permittivity of the infinite ferrite medium is a scalar (see Section I). The value of k'_{ef} is given by

$$k'_{ef} = 1 + \frac{N\,a_e/\epsilon_o}{1 - \frac{N\,a_e}{3\epsilon_o}} \tag{45}$$

where a_e is the electric polarizability of a ferrite sphere and is given by

$$a_e = 4\pi\,\epsilon_o\,r_1^3\,\frac{k_{ef} - 1}{k_{ef} + 2}$$

k_{ef} is the dielectric constant of the infinite ferrite medium and is moderately high; values of k_{ef} in the range from 10 to 20 are common.

Similarly the Faraday rotation per unit length, $\frac{\theta}{\ell}$, of the infinite ferrite lattice can be computed from

$$\frac{\theta}{\ell} = \frac{\beta_- - \beta_+}{2} \tag{46}$$

Thus, the macroscopic behavior of the ferrite lattice medium is completely describable, the main restriction being that the ferrite elements and spacings be small compared to the wave length within and without the element.

Example 2 - A Tetragonal Array of Conducting Disks.

In Example 1, a cubical array of ferrite spheres was investigated and exemplified the case of an array possessing no structural anisotropy but only material anisotropy of the elements. This was, so to speak, lattice anisotropy on a truly molecular level. In this section, a tetragonal array of circular disks will be considered exemplifying structural anisotropy as well as element anisotropy due to the geometry of the latter.

The electric polarizability of a conducting disk is a tensor and is given by,

$$(\alpha_E) = \begin{bmatrix} \alpha_{exx} = \frac{16}{3}\epsilon_0 r_1^3 & 0 & 0 \\ 0 & \alpha_{eyy} = \frac{16}{3}\epsilon_0 r_1^3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where r_1 is the radius of the disk. The magnetic polarizability is given by

$$(\alpha_m) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_{mzz} = -\frac{8\mu_0 r_1^3}{3} \end{bmatrix}$$

For a tetragonal array the structural anisotropy tensor (T) reduces to

$$(T) = \begin{bmatrix} T_{xx} & 0 & 0 \\ 0 & T_{yy} & 0 \\ 0 & 0 & T_{zz} \end{bmatrix}$$

Substituting these values in Equation (27) yields,

$$\left(k_e\right) = \begin{bmatrix} 1 + \dfrac{N\alpha_{exx}/\epsilon_o}{1 - \alpha_{exx}T_{xx}} & 0 & 0 \\ 0 & 1 + \dfrac{N\alpha_{eyy}/\epsilon_o}{1 - \alpha_{eyy}T_{yy}} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{47}$$

and

$$\left(k_m\right) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 + \dfrac{N\alpha_{mzz}/\epsilon_o}{1 - \alpha_{mzz}T_{zz}} \end{bmatrix} \tag{48}$$

Estrin[8] in a paper on disk arrays realized that the conventional Clausius-Mosotti relations, to account for interactions between elements, were not valid for such arrays, and did not include interaction consideration in his treatment of the effects of array anisotropy on the propagation of incident plane waves. He did, however, indicate that some valid interaction analysis was necessary to strengthen the theory. The above expressions, (47) and (48), supply this correction for general tetragonal arrays of disks, provided the packing is not so close as to invalidate the representation of elements as a set of dipoles. The results could be extended to any uniform lattice as discussed earlier.

To illustrate the method suggested in Section I for the computation of the components of the structural anisotropy tensor, the dielectric constant of a tetragonal array will be numerically evaluated and its dependence on the lattice spacing will be discussed. Consider a tetragonal array with spacings $a = b$ and c, as shown in Figure (1). From Section I,

$$\begin{aligned} T_{xx} = T_{yy} &= T'_{xx} + T''_{xx} \\ &= \frac{N}{3\epsilon_o} + \frac{L_{xx}}{4\pi\epsilon_o} \end{aligned} \tag{49}$$

where $L''_{xx} = \sum_{\mu'} \frac{3x^2 - r^2}{r^5}$ and where the summation index μ' extends over all elements within a sphere or cube of finite dimensions centered around the origin of coordinates, except the single element at the origin. Insertion of this value of T_{xx} into Equation (47) gives

$$k_{exx} = 1 + \frac{N\alpha_{exx}/\epsilon_o}{1 - \frac{N\alpha_{exx}}{\epsilon_o}\left(\frac{1}{3} + \frac{L_{xx}}{4\pi N}\right)} \tag{50}$$

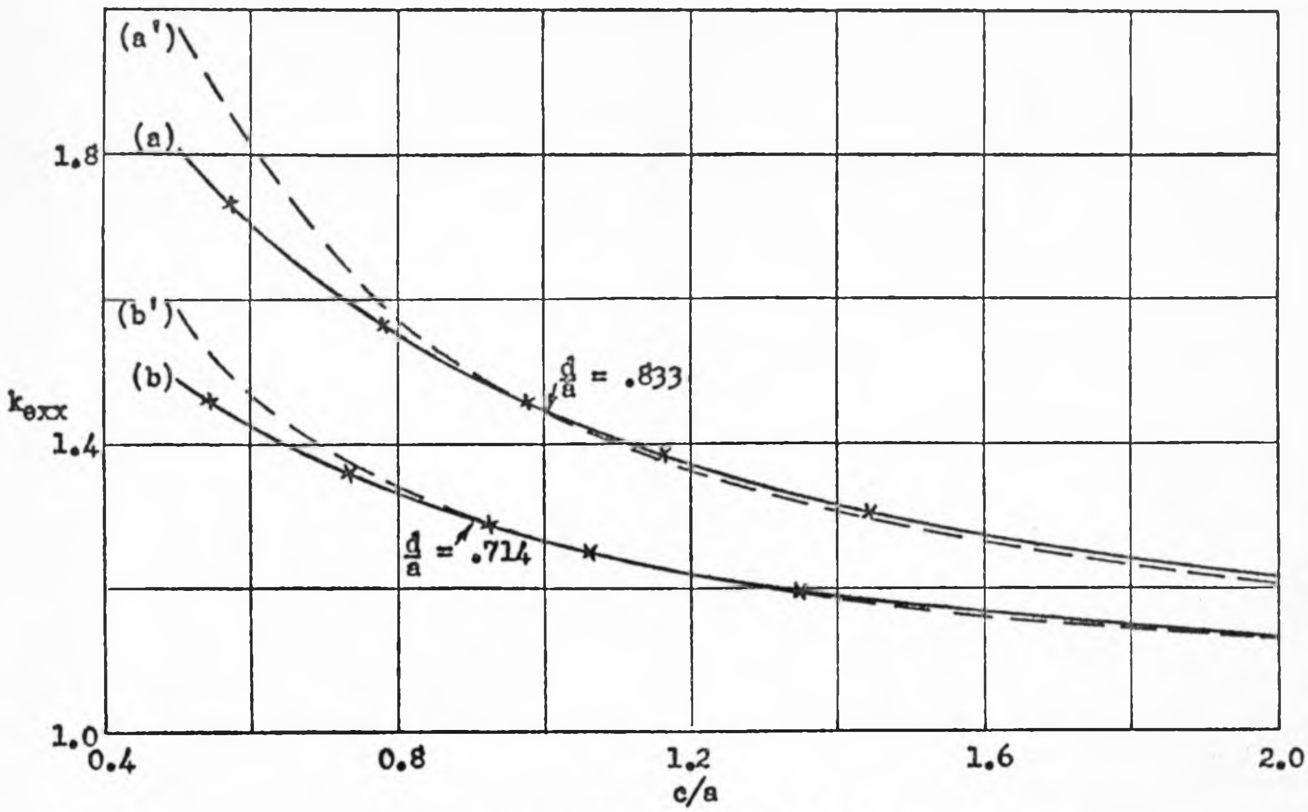
For a circular metallic disk

$$\alpha_{exx} = \alpha_{eyy} = \frac{2}{3} d^3 \epsilon_o$$

where d is the diameter of the disk. Substituting this value of α_{exx} in Equation (50) yields

$$k_{exx} = k_{eyy} = 1 + \frac{\frac{2}{3}\left(\frac{d}{a}\right)^3 \frac{a}{c}}{1 - \frac{2}{3}\left(\frac{d}{a}\right)^3 \frac{a}{c}\left(\frac{1}{3} + \frac{L_{xx}a^2c}{4\pi}\right)} \tag{51}$$

Figure (5) is a plot of $\frac{L_{xx}}{4\pi N}$ as a function of $\frac{c}{a}$ within a cube 30 units on a side. Curves (a) and (b) in Figure (4) are plots of Equation (51) against $\frac{c}{a}$, for values of $\frac{d}{a} = 0.833$ and $\frac{d}{a} = 0.714$ respectively. Curves (a') and (b') represent the same curves where L_{xx} is taken to be equal to zero. The experimental results obtained by El-Kharadly and Jackson[9] are also shown in Figure 4 and are in good agreement with the theoretical curves (a) and (b). The Clausius-Mosotti curves (a') and (b') fall below the experimental curves for values of $\frac{c}{a} < 0.9$. This inadequacy of the classical Clausius-Mosotti relation for the disk array was first pointed out by Süsskind[10]. However, as demonstrated by the curves of Figure (4), the Clausius-Mosotti relation is a good approximation for values of $\frac{c}{a} > 0.9$ but this region is not within



× × × × × Experimental Values

Figure 4 - Dielectric Constant of the Tetragonal Array of Disks.

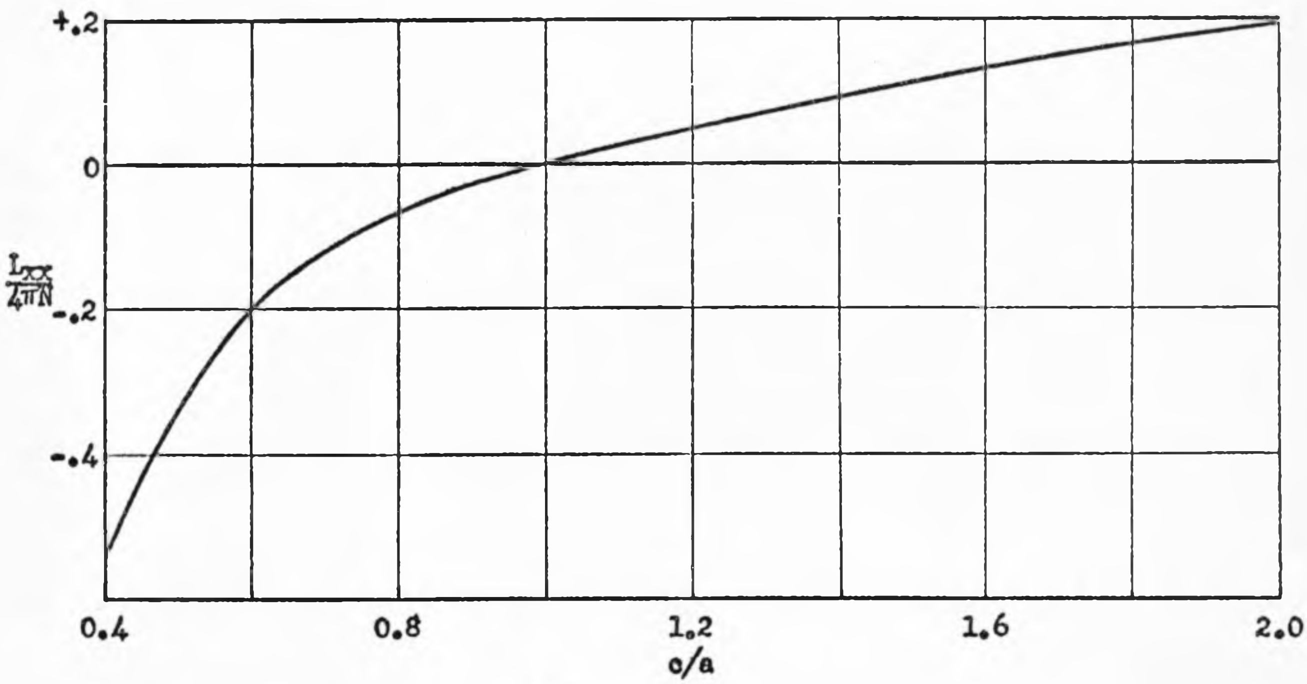


Figure 5 - $\frac{L_{xx}}{4\pi N}$ For the Tetragonal Disk Array.

the range of values of interest in microwave applications. Brown and Jackson[4] have derived approximate relations for the dielectric constant of a tetragonal array of metallic disks by approximating the lattice by one sheet of disks when $\frac{c}{a} > 0.6$, and by a two dimensional array of cylinders when $\frac{c}{a} < 0.6$. Their formulas are in good agreement with experiments within these ranges. However, their results are applicable only for a tetragonal array of disks and their use thereby is very limited. On the other hand, the method developed in this note is most general in that it applies for any element shape or material as well as any array geometry. The only restrictions on the method are with regard to the element size and spacing with respect to wave length. In particular, the range of validity is determined by requirement that the element packing should not be so close as to require the consideration of higher order multipoles in the calculation of interaction effects. In the example considered, the results are valid only for values of $\frac{c}{a} > 0.4$.

Conclusion

It has been possible to derive the constitutive dielectric parameters for uniform space arrays of generalized structural geometry composed of similarly oriented elements of completely generalized material and shape. The usual restrictions that element size and spacings be small compared to wave length are imposed. Interactions between elements have been completely accounted for on the basis of the single additional assumption that the packing is sufficiently loose that the induced field in each element can be described by a dipolar field only. One of the results has been the evaluation of certain conditionally convergent series representing the components of the structural anisotropy tensor (T).

The results have been applied to two examples: A cubical array of ferrite spheres and a tetragonal array of disks. Published experimental curves for the latter show good agreement with results obtained.

The results have direct application in the design of material for the control and direction of microwaves. Work is continuing on this topic and will include a treatment for arrays of high element concentration which will require a consideration of higher order multipoles.

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