DIFFRACTION OF A TRAPPED WAVE BY A SEMI-INFINITE METALLIC SHEET

by

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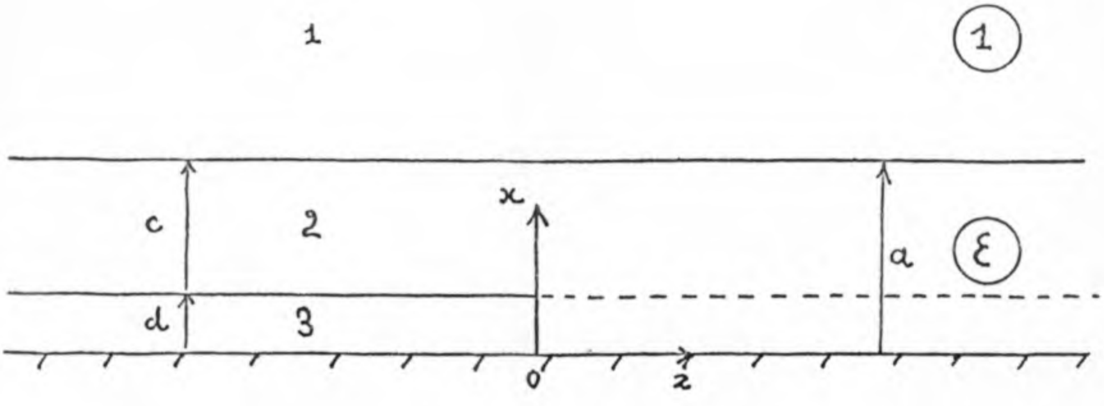

Figure 1.

I. Introduction

It is a well-known fact that dielectric coated infinite metallic structures such as planes and wires can propagate "surface modes". We are here chiefly concerned with a two-dimensional case. There is no theoretical difficulty in extending our solution to three-dimensional structures.

We are dealing here with a grounded dielectric slab of permittivity ε and thickness a. (The case in which the electric wall is replaced by a magnetic one involves only slight modifications.) The half-space over the slab is a dielectric of permittivity 1.

E modes and H modes can propagate in the slab. They are the so-called "trapped waves". The number of modes is connected with ε and a.

As an example of the treatment of the general case, we shall suppose that a is small enough to propagate only one E mode. Extension to the H case or to the multimode case is obvious.

We then suppose (see Fig. 1) that only one mode is propagating, coming from $z = +\infty$. This trapped wave will be diffracted by a semi infinite metallic sheet of zero thickness, which lies on $x = d$, $z < 0$. We are mainly interested in reflection and transmission coefficients for the trapped modes, the radiated power, and the far-field pattern.

II. Formulation of the Problem

The incoming mode is

$$H_y = e^{-i(\alpha x + \beta z)} + e^{-i(-\alpha x + \beta z)} \qquad \text{with } \alpha^2 + \beta^2 = k^2 \mathcal{E}$$

in the slab, and

$$H_y = Me^{-\gamma x - i\beta_z} \qquad \text{with } \beta^2 - \gamma^2 = k^2 \quad , \; k = \frac{2\pi}{\lambda}$$

in the upper half space.

Matching at the boundary $x = a$ gives:

$$M = 2e^{\gamma a} \cos \alpha a \quad ,$$

$$\cos \alpha a = \frac{\alpha}{k \sqrt{\mathcal{E} - 1}} \quad .$$

Since we have supposed that a is small enough, the last equation has only one solution for α.

In the following sections we put $H_y = \psi(x,z)$. We split the total field $\psi(x,z)$ into two parts: $\psi = \psi_i + \psi_s$, where ψ_i is the incoming field and ψ_s the scattered field. We apply now to the scattered field the Green's formula procedure in each of the three regions in Fig. 1.

Region 1. $x \geq a$.

Green function $G_1(x,x',z,z')$ with boundary condition $\frac{\partial G_1}{\partial x'}(x,a,z,z') = 0$

Region 2. $d \leq x \leq a$.

Green function $G_2(x,x',z,z')$ with boundary condition

$$\frac{\partial G_2}{\partial x'}(x,a,z,z') = \frac{\partial G_2}{\partial x'}(x,d,z,z') = 0$$

Region 3. $0 \leq x \leq d$.

Green function $G_3(x,x',z,z')$ with boundary condition

$$\frac{\partial G_3}{\partial x'}(x,d,z,z') = \frac{\partial G_3}{\partial x'}(x,0,z,z') = 0 .$$

We suppose that $k = k_r - ik_i$ where k_i is a small positive quantity which we can later let go to zero so the form of the radiation condition is $\psi(M) = O(e^{-ik \cdot OM})$ (or $O/e^{-ik\sqrt{\varepsilon} \cdot OM}$) when M goes to ∞.

Hence if we apply Green's formula

$$\psi_s = \int_{\Sigma} \left(G \frac{\partial \psi_s}{\partial n} - \psi_s \frac{\partial G}{\partial n}\right) d\sigma$$

to a region with some parts of the surface Σ at infinity, we get no contribution from those parts. We then have the following integral equations:

Region 1. $\psi_{s_1} = -\int_{-\infty}^{+\infty} G_1(x,a,z-z') \frac{\partial \psi_{s1}}{\partial x'}(a,z')\, dz'$

Region 2.

$$\psi_{s_2} = -\int_{-\infty}^{0} G_2(x,d,z-z') \frac{\partial \psi_{s2}}{\partial x'}(d,z')dz' - \int_{0}^{\infty} G_2(x,d,z-z') \frac{\partial \psi_{s2}}{\partial x'}(d,z')dz'$$

$$- \int_{-\infty}^{+\infty} G_2(x,a,z-z') \frac{\partial \psi_{s2}}{\partial x'}(a,z')dz' \qquad \text{(I)}$$

Region 3. $$\psi_{s_3} = \int_{-\infty}^{0} G_3(x,d,z-z')\,\frac{\partial\psi_{s_3}}{\partial x'}(d,z')\,dz'$$

$$+\int_{0}^{\infty} G_3(x,d,z-z')\,\frac{\partial\psi_{s_3}}{\partial x'}(d,z')\,dz' .$$

Integration along $x = 0$ does not contribute to ψ_{s_3} for along this surface $\frac{\partial\psi_{s_3}}{\partial x'} = 0$. We must of course use the fact that along $x = d$, $z \leq 0$

$$\frac{\partial\psi_s}{\partial x'} + \frac{\partial\psi_i}{\partial x'} = 0 \quad , \text{ i.e.}$$

$$\left(\frac{\partial\psi_s}{\partial x'}\right)_{\substack{d \\ z\leq 0}} = 2a \sin ad\, e^{-i\beta z} .$$

We substitute this in the integral equations, and suppress the subscript s since we are dealing only with the scattered field. We shall now define a function v by:

$$v(x,s) = \int_{-\infty}^{+\infty} \psi(x,z)\, e^{-sz}\, dz . \qquad s = \sigma + i\tau$$

From the assumption on the behavior of ψ as $z\to\infty$, it follows that $v(x,s)$ is analytic for $|\sigma| < k_i$.

We define also:

$$v_-(x,s) = \int_{-\infty}^{0} \psi(x,z)\, e^{-sz}\, dz \qquad\qquad v_+(x,s) = \int_{0}^{+\infty} \psi(x,z) e^{-sz}\, dz$$

where v_- is regular for $\sigma < k_i$ and v_+ is regular for $\sigma > -k_i$. And we note that

$$\int_{-\infty}^{0} e^{-i\beta z - sz}\, dz = \frac{1}{s + i\beta} .$$

III. Green's Functions for the Problem

We represent by $g_i(x,x',s)$ the two-sided Fourier transform of $G_i(x,x',z-z')$:

$$g_i(x,x',s) = \int_{-\infty}^{+\infty} G_i(x,x',z-z')\, e^{-s(z-z')}\, d(z-z')$$

with the following shorthand

$$g_i(p,q,s) = g_{i\,pq} .$$

The application of image method for determination of Green functions gives us easily:

$$g_1(x,x',s) = \frac{1}{\lambda}\, \mathrm{ch}\,\lambda(x'-a)e^{-\lambda(x-a)} \qquad x \geq x'$$

$$= \frac{1}{\lambda}\, \mathrm{ch}\,\lambda(x-a)e^{-\lambda(x'-a)} \qquad x \leq x'$$

$$g_2(x,x'xs) = \frac{1}{\mu} \cdot \frac{\mathrm{ch}\,\mu(x'-d)\ \mathrm{ch}\,\mu(x-a)}{\mathrm{sh}\,\mu\,(a-d)} \qquad x \geq x'$$

$$\frac{1}{\mu} \cdot \frac{\mathrm{ch}\,\mu(x-d)\ \mathrm{ch}\,\mu(x'-a)}{\mathrm{sh}\,\mu\,(a-d)} \qquad x \leq x'$$

$$g_3(x,x',s) = \frac{1}{\mu} \cdot \frac{\mathrm{ch}\,\mu x'\ \mathrm{ch}\,\mu(x-d)}{\mathrm{sh}\,\mu\, d} \qquad x \geq x'$$

$$\frac{1}{\mu} \cdot \frac{\mathrm{ch}\,\mu x\ \mathrm{ch}\,\mu(x'-d)}{\mathrm{sh}\,\mu\, d} \qquad x \leq x'$$

with $\lambda^2 = -(k^2+s^2)$

$\mu^2 = -(k^2\,\varepsilon + s^2)$.

We take the branch of λ which reduces to ik when $s \to 0$, and the branch of μ which reduces to $ik\sqrt{\varepsilon}$ (if ε real) when $s \to 0$.

IV. Fourier Transform for the Equations

Our system of integral equations becomes by Fourier transform:

$$v_1 = -g_{1\,\cdot a}\ \frac{\partial v_1(a)}{\partial x'}$$

$$v_2 = -g_{2\,\cdot d}\ \frac{\partial v_2-(d)}{\partial x'} - g_{2\,\cdot d}\ \frac{\partial v_2+(d)}{\partial x'} - g_{2\,\cdot a}\ \frac{\partial v_1(a)}{\partial x'}$$

$$v_3 = +g_{3\,\cdot d}\ \frac{\partial v_3-(d)}{\partial x'} + g_{3\,\cdot d}\ \frac{\partial v_3+(d)}{\partial x'}$$

and $\dfrac{\partial v_2-(d)}{\partial x'} = \dfrac{\partial v_3-(d)}{\partial x'} = -\dfrac{2\alpha \sin \alpha d}{s+i\beta}$.

Matching along $x = a$ gives:

$$(g_2-g_1)_{aa}\ \frac{\partial v_1(a)}{\partial x'} + g_{2\,da}\ \frac{\partial v_2+(d)}{\partial x'} - g_{2\,da}\ \frac{2\alpha \sin \alpha d}{s+i\beta} = 0$$

and along $x = d$, taking into account the continuity of $\dfrac{\partial v_2+(d)}{\partial x'}$ and $v_2+(d)$

$$v_2-(d) - v_3-(d) = -(g_2+g_3)_{dd}\ \frac{\partial v_2}{\partial x'}+(d) - g_{2\,ad}\ \frac{\partial v_2(a)}{\partial x'} + (g_2+g_3)\frac{2\alpha \sin \alpha d}{s+i\beta} .$$

Eliminating $\dfrac{\partial v_1}{\partial x'}(a) = \dfrac{\partial v_2}{\partial x'}(a)$ from the two last equations gives us:

$$(g_2-g_1)_{aa}\ (v_2-(d)-v_3-(d)) =$$

$$= \left[g^2_{2\,dd} - (g_2-g_1)_{dd}\ (g_2+g_3)_{dd} \right] \left(\frac{\partial v_2+(d)}{\partial x'} - \frac{2\alpha \sin \alpha d}{s+i\beta} \right)$$

or:

$$\left[v_2-(d) - v_3-(d)\right]\left[s+i\beta\right] = \frac{\left[g^2_{2\,dd} - (g_2-g_1)_{dd}\,(g_2+g_3)_{dd}\right]}{(g_2-g_1)_{aa}} \left[(s+i\beta)\frac{\partial v_2+(d)}{\partial x'} - 2\alpha \sin \alpha d\right]. \tag{A}$$

The left member is analytic for $\sigma < k_1$, the second bracket of the right member is analytic for $\sigma > -k_1$. We have now to factorize

$$K(s) = \frac{g_{2\,ad}^{2} - (g_2 - g_1)_{aa}\,(g_2 + g_3)_{dd}}{(g_2 - g_1)_{aa}} \quad .$$

Since

$$g_{1\,aa} = \frac{1}{\lambda} \qquad\qquad g_{3\,dd} = \frac{\coth \mu d}{\mu}$$

$$g_{2\,aa} = g_{2\,dd} = \frac{\coth \mu (a-d)}{\mu} \qquad\qquad g_{2\,ad} = \frac{1}{\mu \,\mathrm{sh}\, \mu (a-d)}$$

we can write $K(s)$ as

$$K(s) = -\frac{1}{\mu\,\mathrm{sh}\,\mu d} \cdot \frac{\lambda\,\mathrm{ch}\,\mu a - \mu\,\mathrm{sh}\,\mu a}{\lambda\,\mathrm{ch}\,\mu c - \mu\,\mathrm{sh}\,\mu c} \qquad c = a - d \quad .$$

We now have to factorize expressions of the type:

$$\frac{\cosh \mu a}{\mu} - \frac{\sinh \mu a}{\lambda}$$

with

$$\lambda = -i\sqrt{k^2 + s^2}$$

$$\mu = -i\sqrt{k^2 \varepsilon + s^2} \quad .$$

V. Factorization of $K(s)$

It is a well-known fact that the zeros $i\beta_n^h$ of $L(s) = \frac{\cosh \mu h}{\mu} - \frac{\sinh \mu h}{\lambda}$ are connected with the modes which can propagate along a grounded slab of permittivity ε and height h. We have supposed (h small enough) that only one mode can propagate. We call σ_0 the smaller of the two numbers

$$k_r \quad \text{and} \quad \left| \mathrm{Re}\, \beta_1^h \right| .$$

The singularities of $L(s)$ are branch points $s = \pm ik$, $s = \pm ik\sqrt{\varepsilon}$.

We have then:

$$L_h(s) = \frac{\sin(h\sqrt{k^2 \varepsilon + s^2})}{\sqrt{k^2 + s^2}} + \frac{i\cos(h\sqrt{k^2\varepsilon + s^2})}{\sqrt{k^2 \varepsilon + s^2}}$$

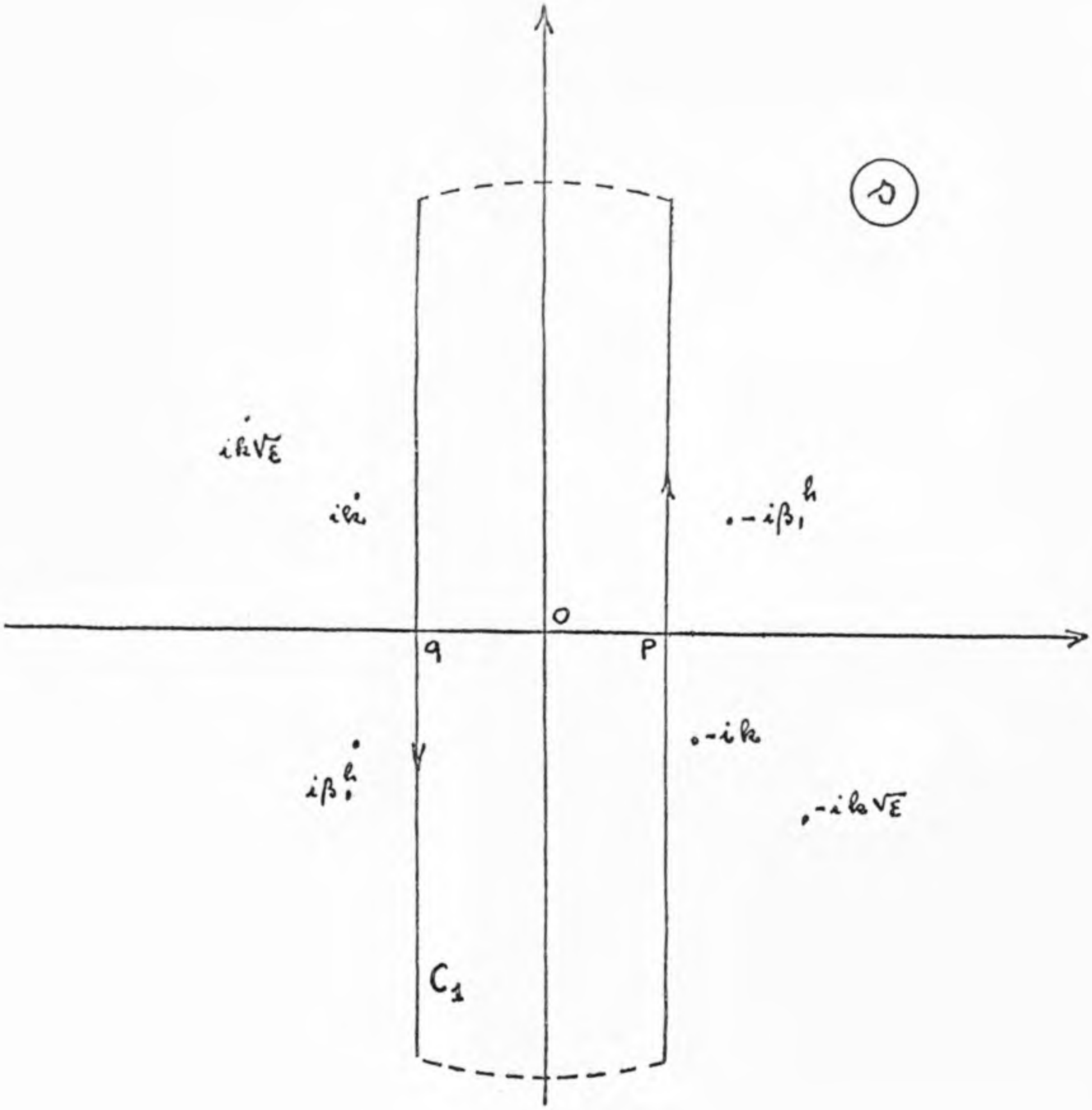

Figure 2.

and we know there exist only two zeros $\pm i\beta_1^h$ of $L_h(s)$. Putting

$$r_1 = \sqrt{k^2 + s^2} \qquad C = \cos(hr_2)$$

$$r_2 = \sqrt{k^2 \varepsilon + s^2} \qquad S = \sin(hr_2)$$

we get:

$$\frac{L'_h(s)}{L_h(s)} = \frac{s}{r_1^2 r_2^2}\left[-\frac{r_2^3 S + i r_1^3 C}{r_2 S + i r_1 C}\right] + \frac{s \cdot h}{r_2}\left[\frac{Cr_2 - i\ Sr_1}{Sr_2 + i\ Cr_1}\right] = s\left[f_1(s) + f_2(s)\right] .$$

When $|s| \to \infty$ the first bracket is $O(\frac{1}{s})$. We have to study the asymptotic behavior of the second bracket. We shall write it as:

$$\frac{h \cdot s}{ir_2}\left[\frac{(r_2 - r_1)\, e^{ir_2} + (r_2 + r_1)e^{-ir_2}}{(r_1 - r_2)\, e^{ir_2} + (r_1 + r_2)e^{-ir_2}}\right]$$

so when $\mathcal{J}(r_2) < 0$ its asymptotic value is:

$$-\frac{h\,s}{ir_2} \quad \text{with remainder} \quad \frac{2e^{-ir_2}\,(r_1 + r_2)}{(r_1 - r_2)e^{ir_2} + (r_1 + r_2)e^{-ir_2}} ,$$

and when $\mathcal{J}(r_2) > 0$ its asymptotic value is:

$$+\frac{s\,h}{ir_2} \quad \text{with remainder} \quad \frac{2e^{ir_2}\,(r_2 - r_1)}{(r_1 - r_2)\, e^{ir_2} + (r_1 + r_2)e^{-ir_2}} .$$

For the first bracket we see similarly that if $\mathcal{J}(r_2) < 0$,

$$\frac{s}{r_1^2\, r_2^2}\left[-(r_1^2 + r_1\, r_2 + r_2^2) + \frac{r_1\, r_2(r_1 + r_2)(C - iS)}{r_1\, C - ir_2\, S}\right] \qquad \text{and if}$$

$\mathcal{J}(r_2) > 0$,

$$\frac{s}{r_1^2\, r_2^2}\left[-(r_1^2 - r_1\, r_2 + r_2^2) + \frac{r_1\, r_2(r_2 - r_1)(C + iS)}{r_1 C - ir_2\, S}\right] .$$

(We are only interested in terms like $\dfrac{L'_a}{L_a} - \dfrac{L'_a - d}{L_a - d}$ so we can drop the first term in the bracket.)

Let us now apply the standard Cauchy integral factorization method:

$$\frac{L'_h(s)}{L_h(s)} = \frac{1}{2\pi i}\int_{C_1} \frac{tf_1(t)}{t - s}\, dt + \frac{1}{2\pi i}\int_{C_1} \frac{tf_2(t)}{t - s}\, dt$$

$$= \frac{1}{2\pi i}\int_{C_1} \left[f_1(t) + f_2(t)\right] dt + \frac{s}{2\pi i}\int_{C_1} \frac{f_1 + f_2}{t - s}\, dt ,$$

where path C_1 is shown in Fig. 2.

The first integral is zero by Cauchy theorem. For the second integral, the contribution from the dotted lines is zero. So we have:

$$\frac{L'_h(s)}{L_h(s)} = \left[\frac{s}{2\pi i}\int_{p-i\infty}^{p+i\infty}\frac{f_1+f_2}{t-s}\,dt\right] + \left[\frac{s}{2\pi i}\int_{q+i\infty}^{q-i\infty}\frac{f_1+f_2}{t-s}\,dt\right] = I^h_p - I^h_q .$$

We know that the first bracket I^h_p is regular in the half plane $R(s) < p$, the second bracket in the right half plane $R(s) > q$, so that:

$$I^h_p = \frac{L'_h-(s)}{L_h-(s)} \qquad\qquad I^h_q = \frac{L'_h+(s)}{L_h+(s)} .$$

We are now interested in the asymptotic values of

$$\frac{L'_h-(s)}{L_h-(s)} = \frac{s}{2\pi i}\int_{p-i\infty}^{p+i\infty}\frac{f_1(t)+f_2(t)}{t-s}\,dt .$$

Writing

$$\frac{1}{t-s} = i\int_0^{\infty} e^{i(s-t)u}\,du ,$$

we have to evaluate integrals like:

$$I^h_p = \frac{is}{2\pi i}\int_{p-i\infty}^{p+i\infty}(f_1+f_2)dt\int_0^{\infty} e^{i(s-t)u}\,du$$

$$= \frac{si}{2\pi i}\int_0^{\infty} e^{ius}\,du\int_{p-i\infty}^{p+i\infty}\left[f_1(t)+f_2(t)\right]e^{-iut}\,dt$$

where the last equality results from the assumption that the order of integration may be interchanged.

For large values of t, when $\mathcal{J}(r_2) < 0$

$$(f_1+f_2) = -\frac{h}{ir_2} + \frac{4t^2}{r_2} e^{-2ir_2} + O(e^{-2ir_2})$$

$$-\left(\frac{1}{r_2^2} + \frac{1}{r_1 r_2} + \frac{1}{r_1^2}\right) + \frac{t^2}{r_1 r_2} e^{-2ir_2} + O(e^{-2ir_2})$$

and when $\mathcal{J}(r_2) > 0$

$$(f_1+f_2) = \frac{h}{ir_2} - \frac{4}{r_2 t^2} e^{2ir_2} + O(e^{+2ir_2})$$

$$-\left(\frac{1}{r_2^2} - \frac{1}{r_1 r_2} + \frac{1}{r_1^2}\right) - \frac{4t^2}{r_1 r_2} e^{2ir_2} + O(e^{2ir_2}) \ .$$

The terms in the bracket can be dropped because we are only interested in expressions like $I_p^a - I_p^{a-d}$. So from well-known theorems in asymptotic expansion of Laplace transform,

$$I_p^a - I_p^{a-d} \sim -\frac{d}{\pi} \log s + \alpha \frac{\log s}{s^2} + O\left(\frac{s}{s^2}\right)$$

and

$$L_a(s) - L(a-d)(s) \sim -\frac{d}{\pi}(s \log s - s) + C + O\left(\frac{\log s}{s}\right)$$

the last term going to zero as $s \to -\infty$. We can, in addition to this, study directly the integral:

$$M_h^{(s)} = \frac{hs}{2\pi} \int_{p-i\infty}^{p+i\infty} \frac{dt}{(t-s)\sqrt{k^2\mathcal{E} + t^2}} \ .$$

Changing the path of integration from the solid to the dotted line we remain with (Fig. 3)

$$\frac{sh}{\pi} \int_{-ik\sqrt{\mathcal{E}}-\infty}^{-ik\sqrt{\mathcal{E}}} \frac{dt}{(t-s)\sqrt{k^2\mathcal{E}+t^2}} = \frac{sh}{\pi} \frac{1}{\sqrt{k^2\mathcal{E}+s^2}} \log \frac{ik\sqrt{\mathcal{E}}}{s-\sqrt{k^2\mathcal{E}+s^2}}$$

$$\sim h\left(A \log s + B\frac{\log s}{s^2} + \frac{C}{s^2} + O\left(\frac{1}{s^2}\right)\right)$$

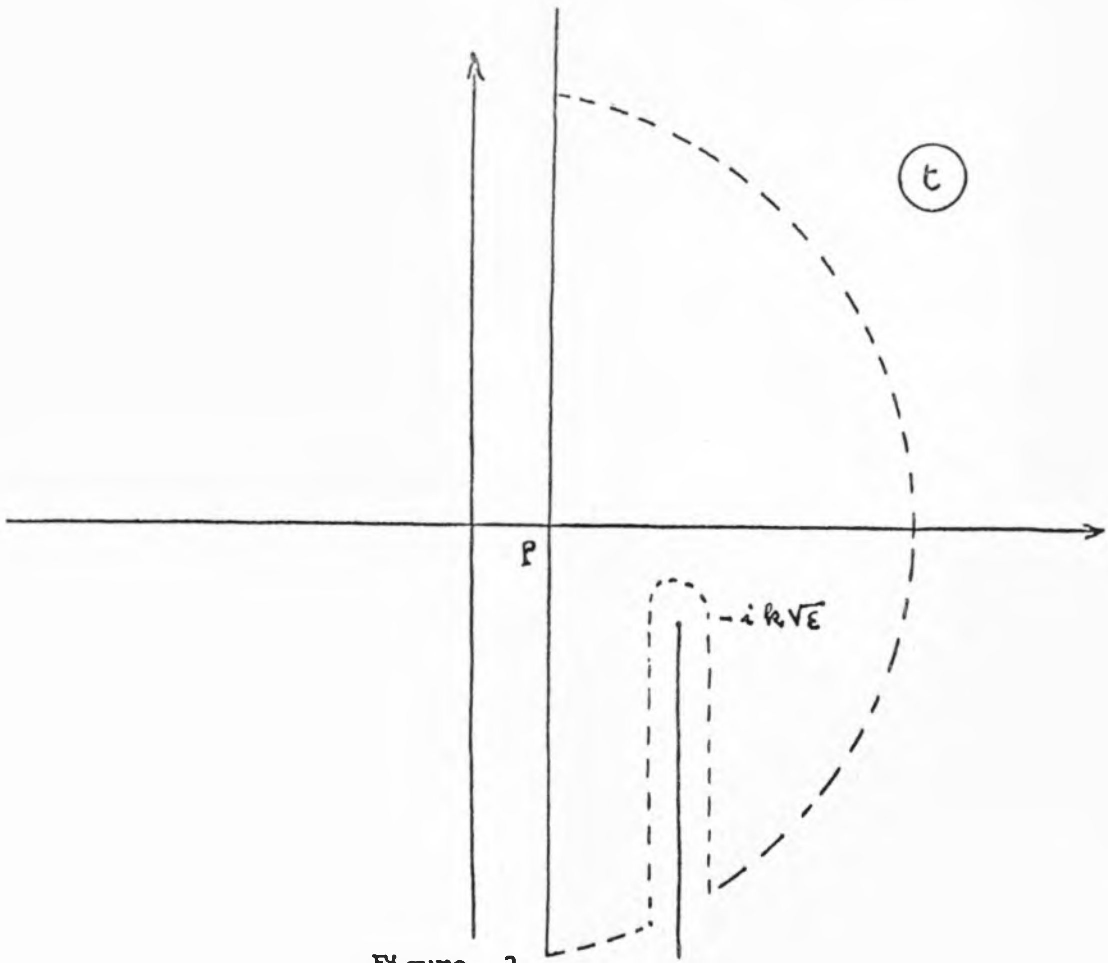

Figure 3.

and also

$$hf(s) = \int_s M_h(s)\, ds = \frac{h}{\pi}\left[\sqrt{k^2\varepsilon + s^2}\ \log \frac{ik\sqrt{\varepsilon}}{s - \sqrt{k^2\varepsilon + s^2}} + s + C^*\right]$$

which is equivalent to

$$-\frac{h}{\pi}\left[s \log s - s\right] + C' + \frac{h}{\pi} s \log ik\sqrt{\varepsilon} + R(s)\ ,\ R(s) \to 0 \text{ if } s \to \infty.$$

For practical computation we shall write the expression $\frac{L'_h(s)}{L_h(s)}$ in the following form:

$$\frac{L'_h-(s)}{L_h-(s)} = \frac{s}{2\pi i}\int_{p-i\infty}^{p+i\infty}\left\{\frac{1}{r_1^2\ r_2^2}\left[-\frac{(r_2^3 S+ir_1^3 C)}{r_2\ S+ir_1\ C}\right]+\frac{h}{r_2}\cdot\frac{Cr_2-iSr_1}{r_2 S+ir_1 C}\right\}\frac{1}{t-s}\ dt.$$

We have to modify the path of integration from the dotted to the solid line; the only branch cut corresponds to $t = -ik$. So we have (contribution of the large circle is zero)

$$\frac{L'_h-(s)}{L_h-(s)} = \frac{s}{\pi}\int_{-ik+\infty}^{-ik}\left[\frac{(r_2^2-r_1^2)SC}{r_1 r_2(r_2^2 S^2+r_1^2 C^2)} - \frac{hr_1}{r_2^2 S^2+r_1^2 C^2}\right]\frac{1}{t-s}\ dt = \frac{dT_-^h(s)}{dS}$$

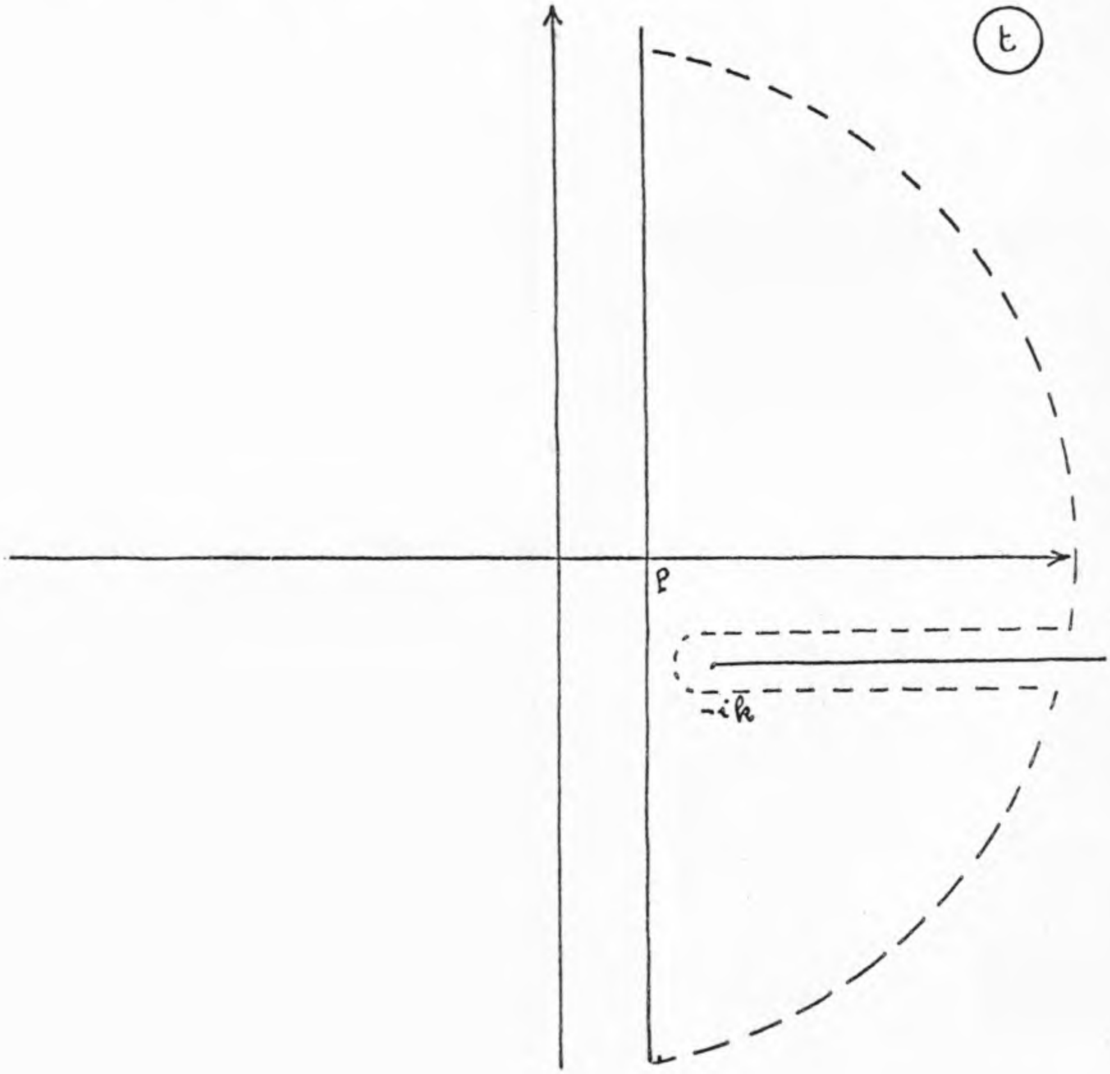

Figure 4.

The remaining part of our factorization is

$$\frac{\sin d\sqrt{k^2\mathcal{E}+s^2}}{d\sqrt{k^2\mathcal{E}+s^2}} = \prod_1^\infty \left(1 - \frac{k^2\mathcal{E}\,d^2 + d^2 s^2}{n^2\pi^2}\right)$$

$$= \prod_1^\infty \left(\sqrt{1-(\tfrac{kd}{\pi n})^2} - \frac{sd}{\pi n}\right) e^{\frac{sd}{\pi n}} \cdot \prod_1^\infty \left(\sqrt{1-(\tfrac{kd}{\pi n})^2} + \frac{sd}{\pi n}\right) e^{-\frac{sd}{\pi n}}$$

where the exponential terms are added to ensure convergence. The inverse of the first product $\prod_-$ is regular in the left half plane. The inverse of the second product $\prod_+$ is regular in the right half plane. We will now study the asymptotic values of

$$\prod_+(s) = \prod_1^\infty \left(\sqrt{1-(\tfrac{kd}{\pi n})^2} + \frac{sd}{\pi n}\right) e^{-\frac{sd}{\pi n}} \qquad \mathrm{Re}(s) > 0 \ .$$

Let us study the asymptotic value of

$$P(s) = \frac{\prod_1^\infty \left(\sqrt{1-(\tfrac{kd}{\pi n})^2} + \frac{sd}{\pi n}\right)}{\prod_1^\infty \left(1 + \frac{sd}{\pi n}\right)} \ .$$

We have:

$$\log P(s) = \sum_1^\infty \log\left[\frac{\frac{sd}{\pi} + n\sqrt{1-(\frac{kd}{\pi n})^2}}{\frac{sd}{\pi} + n}\right] .$$

We can write:

$$\log P(s) = \sum_1^N \log\left[\frac{\frac{sd}{\pi} + n\sqrt{1-(\frac{kd}{\pi n})^2}}{\frac{sd}{\pi} + n}\right] + \sum_N^\infty \log\left[\frac{\frac{sd}{\pi} + n\sqrt{1-(\frac{kd}{\pi n})^2}}{\frac{sd}{\pi} + n}\right] .$$

We suppose $\left|\frac{kd}{\pi}\right| < 1$ so that $\sqrt{1-(\frac{kd}{\pi n})^2} = 1 - \frac{1}{2}(\frac{kd}{\pi n})^2 + \frac{\varepsilon_1}{n^3}$. Let us consider first the finite sum:

$$S_N = \sum_1^N \log \frac{\left[\frac{sd}{\pi} + n - \frac{1}{2}\left(\frac{kd}{\pi}\right)^2 \cdot \frac{1}{n} + \frac{\varepsilon_1}{n^2}\right]}{\frac{sd}{\pi} + n} = \sum_1^N \log \left[1 - \frac{\frac{1}{2}\left(\frac{kd}{\pi}\right)^2 - \frac{\varepsilon_1}{n}}{n\left(n + \frac{sd}{\pi}\right)} \right]$$

$$\left|S_N\right| \leq \sum_1^N \left|\log\right| \leq \sum_1^N \log \left| 1 - \frac{\frac{1}{2}\left(\frac{kd}{\pi}\right)^2}{n\left(n + \frac{sd}{\pi}\right)} + \frac{\varepsilon_1}{n^2\left(n + \frac{sd}{\pi}\right)} \right|$$

$$\leq \sum_1^N \log \left| 1 + \frac{\frac{1}{2}\left|\frac{kd}{\pi}\right|^2}{n\left|n + \frac{sd}{\pi}\right|} + \frac{\left|\varepsilon_1\right|}{n^2\left|n + \frac{sd}{\pi}\right|} \right| .$$

Let us suppose s in the positive half-plane (Fig. 5) so that $\left|\frac{sd}{\pi} + n\right| > \left|\frac{sd}{\pi}\right|$.

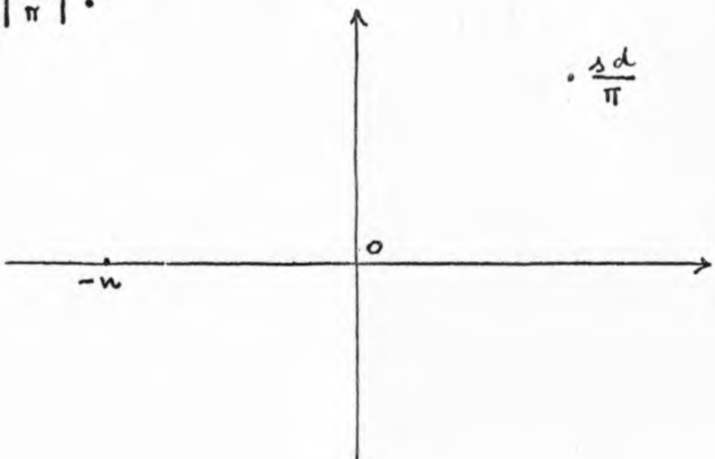

Figure 5.

We have then

$$\left|S_N\right| \leq \sum_1^N \left[\frac{\frac{1}{2}\left|\frac{kd}{\pi}\right|^2}{n\left|\frac{sd}{\pi}\right|} + \frac{\left|\varepsilon_2\right|}{n\left|\frac{sd}{\pi}\right|} \right] \leq N \left[\frac{\frac{1}{2}\left|\frac{kd}{\pi}\right|^2}{\left|\frac{sd}{\pi}\right|} + \frac{\left|\varepsilon_2\right|}{\left|\frac{sd}{\pi}\right|} \right] .$$

Let us now consider the second sum, R_N

$$R_N = \sum_N^\infty \log \frac{\frac{sd}{\pi} + n - \frac{1}{2}\left(\frac{kd}{\pi}\right)^2 \frac{1}{n} + \frac{\varepsilon_1}{n^2}}{\frac{sd}{\pi} + n} = \sum_n^\infty \log \left[1 - \frac{\frac{1}{2}\left(\frac{kd}{\pi}\right)^2 \cdot \frac{1}{n} + \frac{\varepsilon_1}{n^2}}{\frac{sd}{\pi} + n} \right]$$

$$\left|R_N\right| \leq \sum_N^{\infty} \log \left| 1 + \frac{\frac{1}{2}\left(\frac{kd}{\pi}\right)^2}{n\left(n + \frac{sd}{\pi}\right)} + \frac{1}{n^2\left(n + \frac{sd}{\pi}\right)} \right| .$$

If s in the positive half-plane,

$$\left| s + \frac{sd}{\pi} \right| > n \quad , \text{ so}$$

$$\left|R_N\right| \leq \sum_N^{\infty} \left(\frac{\frac{1}{2}\left|\frac{kd}{\pi}\right|^2}{n^2} + \frac{\left|\varepsilon_3\right|}{n^2} \right) < \varepsilon_4 \quad \text{for } N > N_1 .$$

N_1 being fixed in that way,

$$\left|S_N\right| \leq A \cdot \frac{N_1}{|s|} .$$

So if $|s|$ is greater than $\frac{AN_1}{\varepsilon_5}$, $\left|S_N\right| \leq \varepsilon_5$.

So for $|s|$ great enough, $\left|\log P(s)\right| < \varepsilon$, and $\left|P(s)\right| < 1 + \eta$.
The asymptotic value of $\prod_+(s)$ for $|s|$ great enough in the right half plane is independent of (kd) .

VI. Solution of the Equation A in the x,s Space

We write then:

$$K_+(s) = \frac{C_+(s - i\beta^a) \exp\left[T_+^{a}(s) - T_+^{a-d}(s)\right]}{\sqrt{d}\ \prod_+(s)\ (s - i\beta^{a-d})(s - ik\sqrt{\varepsilon})}$$

$$K_-(s) = \frac{C_-\sqrt{d}\ \prod_-(s)\ (s + i\beta^{a-d})(s + ik\sqrt{\varepsilon}) \exp\left[T_-^{a}(s) - T_-^{a-d}(s)\right]}{(s + i\beta^a)} .$$

C_+ and C_- may be determined from the asymptotic expansions.
Our equation (A) may then be written as:

$$(s + i\beta^a)\left[v_{2-}(d) - v_{3-}(d)\right]K_-(s) = K_+(s)(s + i\beta^a)\left[\frac{\partial v_{2+}(d)}{\partial x'} - 2\alpha \sin \alpha d\right] .$$

The first member is analytic in the left half plane, the second one in the right half plane. So both represent the same integral function. Let us study now the asymptotic behavior of $K_+(s)$. For large value of $|s|$ in the right half plane,

$$T_+^a(s) - T_+^{a-d}(s) \sim -(s \log s - s)\frac{d}{\pi} - \frac{sd}{\pi}\log ik\sqrt{\mathcal{E}} + \rho(s)$$

$$\rho(s) \to 0 \text{ if } |s| \to \infty .$$

$$\frac{1}{\Pi_+(s)} \sim \Gamma\left(\frac{sd}{\pi}\right) e^{+\gamma\frac{sd}{\pi}} \quad \sqrt{2\pi}\sqrt{\frac{sd}{\pi}} \cdot \exp\left[\frac{sd}{\pi}\log\frac{sd}{\pi} - \frac{sd}{\pi} + \gamma\frac{sd}{\pi}\right]$$

$\gamma =$ Euler constant,

so the growth of the exponential term is

$$\exp -\left[\frac{sd}{\pi}\left(\gamma - \log ik\sqrt{\mathcal{E}} + \log\frac{d}{\pi}\right)\right] = \exp(\chi s) .$$

Then we have to multiply both members of the equation by $\exp(-\chi s)$ to get an algebraic growth.

From the Meixner edge condition $\frac{\partial\psi_2}{\partial x}(d,z) \sim \frac{1}{\sqrt{z}}$ for $z \to +0$, $\frac{\partial v_2}{\partial x} \sim \sqrt{s}$ for $|s| \to \infty$ in the right half plane. So that the asymptotic value of $\exp(-\chi s)K+(s)(s+i\beta)\frac{\partial v_2}{\partial x'}+(d)$ is $O(s)$.

By an extension of Liouville theorem, the said integral function is a first degree polynomial, $a + b(s+i\beta)$, and we have:

$$\begin{cases} \dfrac{\partial v_{2+}(d)}{\partial x'} = \dfrac{2\alpha \sin \alpha d}{s + i\beta} + \left[\dfrac{a}{K_+(s)(s+i\beta^a)} + \dfrac{b}{K_+(s)}\right] \exp\chi s , \\ v_{2-}(d) - v_{3-}(d) = \left[\dfrac{a}{K_-(s)(s+i\beta^a)} + \dfrac{b}{K_-(s)}\right] \exp\chi s . \end{cases}$$

We have now to evaluate a and b.

VII. Solution of the Equations I in the x,z Space

First, $\dfrac{\partial v_{2+}(d)}{dx'}$ cannot have a pole at $s = -i\beta^a$ so

$$2\alpha \sin \alpha d + \frac{a}{K_+(-i\beta)} e^{-\chi i\beta} = 0$$

$$a = -2\alpha \sin \alpha d \, K_+(-i\beta) e^{\chi i\beta} \quad .$$

So we have:

$$(g_2 - g_1)_{aa} \frac{\partial v_1(a)}{\partial x'} + g_{2\,da} \left[- \frac{K_+(-i\beta) e^{\chi i\beta}}{K_+(s)\, e^{-\chi s}} \cdot \frac{2\alpha \sin \alpha d}{s + i\beta^a} + \frac{b}{K_+(s)\, e^{-\chi s}} \right]$$

$$\frac{\partial v_1(a)}{\partial x'} = \frac{g_{2\,da}}{(g_2 - g_1)_{aa}} \left[- \frac{K_+(-i\beta) e^{\chi i\beta}}{K_+(s) e^{-\chi s}} \cdot \frac{2\alpha \sin \alpha d}{s + i\beta^a} + \frac{b}{K_+(s) e^{-\chi s}} \right] .$$

To determine b let us write that $\dfrac{\partial v_1}{\partial x'}(a)$ has no mode corresponding to $s = i\beta^{a-d} = i\beta'$, i.e.,

$$\frac{b\, e^{\chi i\beta'}}{K_+\,(i\beta')} - \frac{K_+(-i\beta)}{K_+(i\beta')} e^{\chi(i\beta + i\beta')} \cdot \frac{2\alpha \sin \alpha d}{i(\beta + \beta')}$$

$$b = \frac{K_+(-i\beta)\, e^{\chi i\beta} \cdot 2\alpha \sin \alpha d}{i(\beta + \beta')}$$

so that finally:

$$\psi s_1(x,z) = -\frac{1}{2\pi i} \int_{c - i\infty}^{c + i\infty} \frac{g_2(x,a,s) g_2(d,a,s)}{(g_2 - g_1)_{aa}}$$

$$\left\{ \frac{K_+(-i\beta)}{K_+(s)} e^{(i\beta + s)\chi} 2\alpha \sin \alpha d \left[\frac{1}{i(\beta + \beta')} - \frac{1}{(s + i\beta)} \right] \right\} e^{sz}\, ds \qquad \text{(II)}$$

$$\Psi_{s_2}(x,y) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_2(x,d,s) + \frac{g_1(x,a,s)g_2(d,a,s)}{(g_2-g_1)_{aa}}$$

$$\left\{\frac{K_+(-i\beta)}{K_+(s)}\, e^{(i\beta+s)\chi}\, 2\alpha \sin \alpha d \left[\frac{1}{i(\beta+\beta')} - \frac{1}{(s+i\beta)}\right]\right\} e^{sz}\, ds$$

$$\Psi_{s_3}(x,z) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} g_3(x,d,s)\left\{\frac{K_+(-i\beta)}{K_+(s)}\, e^{(i\beta+s)\chi}\, 2\alpha \sin \alpha d\right.$$

$$\left.\left[\frac{1}{i(\beta+\beta')} - \frac{1}{(s+i\beta)}\right]\right\} e^{sz}\, ds \,.$$

where c real is in the strip of analyticity. We are now able to determine the reflection and transmission coefficients, and the radiation pattern.

It is obvious to show that the term containing $\frac{1}{s+i\beta}$ produces a term that cancels the incoming mode for $z < 0$. The transmitted mode in the upper open waveguide corresponds for $x = a$ to a residue for the pole $s = -i\beta'$. The corresponding amplitude is:

$$-2\alpha \sin \alpha d\; \frac{K_+(-i\beta)}{K_+(-i\beta')}\, e^{i\chi(\beta-\beta')}\, \frac{g_1(x,a,-i\beta')\cdot g_2(d,a,-i\beta')}{\left[\frac{d}{ds}(g_2-g_1)_{aa}\right]_{-i\beta'}}$$

$$e^{i\beta' z}\left[\frac{1}{i(\beta+\beta')} - \frac{1}{i(\beta-\beta')}\right] \,.$$

The incoming wave has an amplitude $2\cos\alpha \cdot e^{i\beta z}$, so that

$$t_c = -\frac{\alpha \sin \alpha d}{\cos \alpha a} \frac{K_+(-i\beta)}{K_+(-i\beta')} e^{i\chi(\beta-\beta')} \frac{g_1(a,a,-i\beta')\, g_2(d,a,-i\beta')}{\left[\frac{d}{ds}(g_2-g_1)_{aa}\right]_{-i\beta'}}$$

$$\left[\frac{1}{i(\beta+\beta')} - \frac{1}{i(\beta-\beta')}\right] .$$

In the same way the reflection coefficient is

$$r_a = -\frac{\alpha \sin \alpha d}{\cos \alpha a} \frac{K_+(-i\beta)}{K_+(+i\beta)} e^{i\chi 2\beta} \frac{g_2(a,a,i\beta)\, g_2(d,a,i\beta)}{(g_2-g_1)_{a,a,i\beta}} \left[\frac{1}{i(\beta+\beta')} - \frac{1}{2i\beta}\right] .$$

It is quite easy to compute the transmission coefficient in the parallel plate waveguide: (residue for $s = -ik$) from formulas (II)

$$t_d = \alpha \tan \alpha d \, \frac{K_+(-i\beta)}{K_-(-ik\sqrt{\varepsilon})} e^{i\chi(\beta - k\sqrt{\varepsilon})} \left[\frac{1}{i(\beta+\beta')} - \frac{1}{i(\beta - k\sqrt{\varepsilon})}\right] .$$

We now must find the radiation pattern. Consider:

$$\psi_{s_1}(x,z) = -\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{e^{-\lambda(x-a)+sz}}{\lambda} \frac{g_2(d,a,\partial)}{(g_2-g_1)_{aa}} \left\{\frac{K_+(-i\beta)}{K_+(s)} e^{i\chi(i\beta+s)}\right\} ds$$

with $\lambda = -i\sqrt{k^2+s^2}$.

We have to find the value of the integral for $R = \sqrt{(x-a)^2+z^2}$ large. We shall write:

$$s = k \operatorname{sh} \tau$$

$$x - a = R\cos\theta$$

$$z = R \sin\theta$$

$$\lambda = -ik \operatorname{ch} \tau$$

so

$$e^{-\lambda(x-a)+sz} = e^{Rk(\operatorname{sh}\tau \sin\theta + i\,\operatorname{ch}\tau \cos\theta)} = e^{ikR\cos(\theta - i\tau)}$$

A straightforward application of the saddle point method gives us the radiation pattern. (Saddle point $\theta = i\tau$) . The first term is (for $\theta \neq \frac{\pi}{2}$)

$$-\frac{2\alpha \sin \alpha d}{ik\cos\theta}\ \frac{g_2(d,a,-ik\sin\theta)}{(g_2-g_1)a,a,-ik\sin\theta}\ \frac{K_+(-i\beta)}{K_+(-ik\sin\theta)}\ e^{i\chi[\beta - k\sin\theta]}$$

$$\left[\frac{1}{i(\beta+\beta')} - \frac{1}{i(\beta - k\sin\theta)}\right] .$$

VIII. Conclusion

Through Green functions and Wiener-Hopf techniques we have been able to determine the characteristic parameters of a simple obstacle embedded in a dielectric slab. Transmission and reflection coefficients are in general complex, showing the characteristic mode phase shift we described in our thesis. Application of such radiating obstacles can be made to dielectric line antennas, acting in this case like arrays.

Another application of the preceding study would be the launching efficiency of surface modes by a parallel plate waveguide. (The only modification necessary is to put $d > a$, and to have the primary wave coming from $z = +\infty$.)

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