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ON THE APPLICATION OF A VARIATIONAL PRINCIPLE TO ANTENNA THEORY

by

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ABSTRACT:

The strong limitations on the applicability of exact methods of analysis to electromagnetic boundary-value problems have encouraged the development of certain approximation techniques. Among the most practical of these techniques is Schwinger's variational method. In this paper Schwinger's variational principle is derived from first principles and its application to antenna theory is critically examined.

La limitation de l'application des méthodes exactes d'analyses aux problèmes électromagnétiques comportant des conditions aux limites a encouragé le développement de certaines méthodes d'approximation. La plus pratique de ces méthodes est la méthode variationelle de Schwinger. Dans cette communication le principe variationnel de Schwinger sera dérivé de principes fondamentaux et on examinera de facon critique son application à la théorie des antennes.

INTRODUCTION:

The central problem of antenna analysis is the determination of the input impedance and the far field of radiators of arbitrary size and shape. At the present time three methods of handling such boundary-value problems are available, namely, the Fourier-Lamé method, the integral equation method, and the variational method.

The Fourier-Lamé method, that is, the method of particular solutions, which has been used in the past with some success, for example in computing the input impedance and radiation pattern of wide-angle conical antennas**, and the integral equation method, which has been applied to the problem of

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*An invited paper presented before the Congrès International des Circuits et Antennes Hyperfréquences at the Conservatoire National des Arts et Métiers, Paris, France, October 25, 1957.

**C.H.Papas and R.W.P.King, Proc.Inst.Radio Engrs., 37, 1269 (1949) and 39, 49 (1951); G.H.Brown and O.M.Woodward,Jr., R.C.A.Review 13, 425 (1952).

linear antennas*, are limited to a great degree by the special difficulties in solving the integral, integro-differential or matrix equations to which they ultimately lead. For this reason the variational method, which avoids these difficulties and opens up the possibility of solving the problem approximately and accurately, has received considerable attention.

In this paper will be discussed the variational method, originally devised by Schwinger to handle the boundary-value problems of waveguide discontinuities.** It is related to the integral equation of a boundary-value problem in the same manner as conventional variational principles are related to the Euler equations. Schwinger's variational principle provides the possibility of improving a trial solution until a satisfactory approximation is achieved, not unlike the Rayleigh-Ritz procedure.*** This is true for boundary-value problems of waveguide discontinuities where the variational principle leads to a stationary point that is an absolute maximum or minimum, but in antenna problems where the stationary point is a saddlepoint a moot question arises about how the method should be implemented. As a practical answer it is suggested that two complementary forms**** of the variational principle be used, one being a functional of surface electric fields and the other a functional of surface magnetic fields and that the smallness of the difference between the two results be taken as a measure of the quality of the approximation.

DERIVATION OF THE VARIATIONAL PRINCIPLE

The starting-point in the present derivation of Schwinger's variational principle is the integral-equation formulation of the boundary-value problem. It is assumed that the antenna problem in the steady state reduces to an integral equation such as

$$\psi(\underline{r}) = \int_S G(\underline{r},\underline{r}')\, K(\underline{r}')\, ds' \qquad (\underline{r} \text{ on } S) \tag{1}$$

* L.Brillouin, Quart.Appl.Math. 1, 201 (1943).

** N.Marcuvitz, Waveguide Handbook, Radiation Lab. Series, Vol.10, New York Toronto, London, (1951); F.Borgnis and C.H.Papas, Randwertprobleme der Mikrowellenphysik, Berlin, Gottingen, Heidelberg (1955).

***W.Ritz, J.fur reine und angew.Math. 135, 1 (1909); R.Courant and D. Hilbert, Methods of Mathematical Physics, New York, London 1953, 175,176.

****G.V.Kisun'ko, Proc.Acad.Sci. USSR, 66, 863 (1949).

where $G(r,r')$ is a symmetrical Green's function of the observation and source points $\underline{r}$ and $\underline{r}'$ respectively, $\psi(\underline{r})$ is a wave function prescribed on the surface S, and $K(\underline{r})$ is an unknown surface current. It is also assumed that the quantity of physical interest X is a weighted average such as

$$X = \int_s \psi(\underline{r})\, K(\underline{r})\, ds \quad . \tag{2}$$

To construct a variational principle for the quantity of physical interest, X is written as a functional of $K(\underline{r})$ and $K(\underline{r}')$,

$$X = \int_s K(\underline{r})\, G(\underline{r},\underline{r}')\, K(\underline{r}')\, ds ds' \quad , \tag{3}$$

which follows from equation (1) and expression (2). The variation of X due to small variations of K is given by

$$\delta X = \int_s \delta K(\underline{r})\, G(\underline{r},\underline{r}')\, K(\underline{r}')\, ds ds' + \int K(\underline{r})\, G(\underline{r},\underline{r}')\, \delta K(\underline{r}') ds ds' . \tag{4}$$

With the aid of the assumed symmetry of the Green's function, i.e., $G(\underline{r},\underline{r}') = G(\underline{r}',\underline{r})$, this expression for δX may be written in the following form

$$\begin{aligned} \delta X = & \int_s \delta K(\underline{r}) ds \left\{ \int_s G(\underline{r},\underline{r}')\, K(\underline{r}')\, ds' \right\} \\ & + \int_s \delta K(\underline{r}') ds' \left\{ \int_s G(\underline{r}',\underline{r})\, K(\underline{r})\, ds \right\} \end{aligned} \tag{5}$$

placing in evidence the fact that δX does not disappear for arbitrary variations δK. It is clear that if the right side of this expression were

$$\begin{aligned} & \int_s \delta K(\underline{r})\, ds \left\{ \int_s G(\underline{r},\underline{r}')\, K(\underline{r}')\, ds' - \psi(\underline{r}) \right\} \\ & + \int_s \delta K(\underline{r}') ds' \left\{ \int_s G(\underline{r}',\underline{r})\, K(\underline{r})\, ds - \psi(\underline{r}') \right\} \quad , \end{aligned} \tag{6}$$

then by virtue of the integral equation (1) it would disappear. This clue leads to the consideration of the following functional

$$-X = \int_s K(\underline{r})\, G(\underline{r},\underline{r}')\, K(\underline{r}')\, ds ds' - \int_s K(r)\, \psi(\underline{r})\, ds - \int_s K(\underline{r}')\, \psi(\underline{r}')\, ds' \tag{7}$$

which is a consequence of subtracting expression (2) twice from expression (3). It is obvious that this functional is stationary, i.e., $\delta X = 0$ for arbitrary δK, because the variation of its right side leads to the terms (6).

Although expression (7) is stationary, it is not homogeneous in K. To modify it so that it will be homogeneous as well, K is written as the product of an amplitude constant α and a function of position $f(r)$, viz., $K(\underline{r}) = \alpha f(\underline{r})$. Under this circumstance the stationary expression (7) becomes

$$-X = \alpha^2 \int_s f(\underline{r})\; G(\underline{r},\underline{r}')\, f(\underline{r}')\, ds ds' - \alpha \int_s f(\underline{r})\, \psi(\underline{r})\, ds - \alpha \int_s f(\underline{r}')\, \psi(\underline{r}')\, ds' \; . \tag{8}$$

For this equation to be homogeneous it is necessary that X be independent of α, or alternatively it is necessary that $\partial x/\partial \alpha$ disappear. This condition is satisfied when

$$\alpha = \frac{\int_s f(r)\, \psi(r)\, ds}{\int_s f(\underline{r})\, G(\underline{r},\underline{r}')\, f(r')ds ds'} \; . \tag{9}$$

Substituting this value of α back into expression (8) yields

$$X = \frac{\left[\int_s f(\underline{r})\, \psi(r)\, ds\right]^2}{\int_s f(\underline{r})\, G(\underline{r},\underline{r}')\, f(\underline{r}')ds ds'} \; . \tag{10}$$

Since this form is homogeneous in f, it is permissible here to switch back from f to K. Thus,

$$X = \frac{\left[\int_s K(\underline{r})\,\psi(\underline{r})\,ds\right]^2}{\int_s K(\underline{r})\,G(\underline{r},\underline{r}')\,K(\underline{r}')\,ds\,ds'} \quad . \tag{11}$$

This is the desired expression for the quantity of physical interest X. It is clearly homogeneous in K; also it is stationary with respect to small variations of K about the true K determined by integral equation (1) of the boundary-value problem. The Euler equation of this variational principle is the integral equation (1).

COMPLEMENTARY FORMS OF THE VARIATIONAL PRINCIPLE:

By reason of the fundamental uniqueness theorem it is possible to formulate the antenna problem in terms of surface electric currents (tangential magnetic field) or surface magnetic currents (tangential electric field.)

In the electric-current formulation the integral equation is

$$\psi_e(\underline{r}) = \int_S G_e(\underline{r},\underline{r}')\,K_e(\underline{r}')\,ds' \tag{12}$$

where K_e is the surface electric-current density, G_e is the appropriate Green's function, and ψ_e is the associated wave function, the quantity of physical interest X_e is given by

$$\frac{1}{X_e} = \int_S \psi_e(\underline{r})\,K_e(\underline{r})\,ds \quad , \tag{13}$$

and the homogeneous, stationary representation for X_e is

$$X_e = \frac{\int_S K_e(\underline{r})\,G_e(\underline{r},\underline{r}')\,K_e(\underline{r}')\,ds\,ds'}{\left[\int_S K_e(\underline{r})\,\psi_e(\underline{r})\,ds\right]^2} \quad . \tag{14}$$

In the magnetic-current formulation the integral equation is

$$\psi_m(\underline{r}) = \int_S G_m(\underline{r},\underline{r}') \, K_m(\underline{r}') \, ds' \tag{15}$$

where K_m is the surface magnetic-current density. The quantity of physical interest is

$$X_m = \int_S \psi_m(\underline{r}) \, K_m(\underline{r}) \, ds \quad , \tag{16}$$

and the homogeneous, stationary representation for X_m is

$$X_m = \frac{\left[\int_S K_m(\underline{r}) \, \psi_m(\underline{r}) \, ds\right]^2}{\int_S K_m(\underline{r}) \, G_m(\underline{r},\underline{r}') \, K_m(\underline{r}') ds ds'} \quad . \tag{17}$$

The two complementary forms (14) and (17) of the variational principle are particularly powerful in the problem of plane discontinuities in waveguide because one form yields an upper bound and the other a lower bound.* That is,

$$X_e \geq X_{exact} \geq X_m \tag{18}$$

when trial functions are substituted in representations (14) and (17) for K_e and K_m . If the trial functions happen to correspond with the true current distributions the equality signs hold and both representations give the exact result X_{exact} .

However, in the case of antennas where the quantity of physical interest is complex, these inequalities are not necessarily true and it becomes impossible to find upper and lower bounds by means of the two complementary forms of the variational principle. Nevertheless, the use of the two complementary forms is practically useful by virtue of the fact that as the trial functions approach the true currents, X_e and X_m approach X_{exact} . This fact suggests that the smallness of the real and

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*For a proof see F.Borgnis and C.H.Papas, "Randwertprobleme der Mikrowellen Physik", Springer-Verlag 1955, pp. 122-124.

imaginary parts of the difference $X_e - X_m$ be a measure of the quality of the approximation.

As an example of how the electric-current form of the variational principle comes into antenna theory, the case of the thin, linear antenna will now be considered. The antenna consists of a thin, straight, perfectly conducting wire of diameter 2a, driven at its center by a localized voltage. The wire extends from $z = -\ell$ to $z = \ell$ and the driving force $V(z)e^{-ikct}$ spans a narrow gap at $z = 0$. The well-known "linearized" integral equation for the electric current $I(z)$ in the wire is

$$V(z) = V_o \delta(z) = \int_{-\ell}^{\ell} G_e(z,z')\, I(z')\, dz' \tag{19}$$

where $\delta(z)$ is the Dirac delta function and V_o is a constant. The kernel $G_e(z,z')$ is the "linearized" version of the Green's function and is explicitly given by

$$G_e(z,z') = \frac{-i\omega\mu}{4\pi}\left(1 + \frac{1}{k^2}\frac{\partial^2}{\partial z^2}\right) \frac{e^{ik\sqrt{(z-z')^2 + a^2}}}{\sqrt{(z-z')^2 + a^2}} \; . \tag{20}$$

The input impedance of the antenna at its driving point is defined by

$$\frac{Z_{input}}{V_o^2} = \frac{1}{\int_{-\ell}^{\ell} I(z)\, V(z)\, dz} \tag{21}$$

which reduces to $Z_{input} = V_o/I(0)$ when it is recalled that $V(z) = V_o\delta(z)$. By comparing equations (19) and (21) with equations (12) and (13) it is seen that $V(z)$, $I(z)$, and Z_{input}/V_o^2 of this problem correspond respectively to ψ_e, K_e, and X_e of the general electric-current formulation. Therefore, the homogeneous, stationary form (14) in the present case becomes

$$\frac{Z_{input}}{V_o^2} = \frac{\int_{-\ell}^{\ell}\int_{-\ell}^{\ell} I(z)\, G_e(z,z')\, I(z')\, dz\, dz'}{\left(\int_{-\ell}^{\ell} I(z)\, V(z)\, dz\right)^2} \; . \tag{22}$$

It is easy to show by direct operation on this representation that $\delta Z_{input} = 0$ provided $I(z)$ satisfies integral equation (19) and Z_{input}

is defined by relation (21).

This variational principle for linear antennas was used first by Storer* for the calculation of Z_{input}. As a trial function he chose the standing wave

$$I(z) = \sin k(\ell - |z|) \qquad (|z| \leq \ell) \quad . \tag{23}$$

An improved trial function in the form of a damped traveling wave,

$$I(z) = e^{i(k+i\alpha)(\ell - |z|)} - e^{-i(k+i\alpha)(\ell - |z|)} \qquad (|z| \leq \ell) \tag{24}$$

was used by Tai.** For the details of their results the reader is referred to the literature.*** The magnetic-current form of the variational principle has not as yet been applied to this problem. On the other hand, the magnetic-current form of the variational principle has been applied to the conical antenna whereas the electric-current form has not. Indeed, the only antenna which has been formulated in terms of both forms of the variational principle is the circular diffraction antenna, which will be discussed next.

THE CIRCULAR DIFFRACTION ANTENNA:

Now the two complementary forms of the variational principle will be applied for the calculation of the input impedance of the circular diffraction antenna. The antenna consists of a coaxial waveguide fitted with a limitless flange and open to free space, Figure 1. It is driven by an incident T-wave traveling from a remote source at $z = -\infty$ toward the aperture at $z = 0$. This type of excitation produces in the coaxial region $(z \leq 0,\ a \leq \rho \leq b)$ in addition to the incident T-wave a reflected T-wave and evanescent E-waves, and in the open half-space $(z \geq 0)$, advancing hemispherical waves. Since the incident wave and the structure are independent of the azimuthal angle ϕ, the non-vanishing field components, viz., $E_\rho(\rho,z), H_\phi(\rho,z)$ are independent of ϕ.

The field components of the T-waves are given by

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* J.E.Storer, Tech.Report No.101, Feb. 1950, Harvard U. Cruft Laboratory
**C.T.Tai, Tech.Report No.188, Aug. 1950, Harvard U. Cruft Laboratory
***F.Borgnis and C.H.Papas, loc.cit., pp.221-227

$$(H_\phi)_T = \alpha \frac{e^{ikz}}{\rho} + \beta \frac{e^{-ikz}}{\rho} \tag{25}$$

$$(E_\rho)_T = \frac{\mu}{\epsilon} \left(\alpha \frac{e^{ikz}}{\rho} - \beta \frac{e^{-ikz}}{\rho}\right) \tag{26}$$

where α and β are the amplitude constants of the incident and reflected waves respectively, k is the free-space wave number, and μ and ϵ are the permeability and dielectric constant of free-space respectively.

The current $I(z)$ and the voltage $V(z)$ of the coaxial waveguide are defined by

$$I(z) = \int_0^{2\pi} (H_\phi)_T \, \rho \, d\phi \tag{27}$$

$$V(z) = \int_a^b (E_\rho)_T \, d\rho \quad . \tag{28}$$

The eigenfunctions $R_n(\rho)$ and the eigenvalues λ_n of the E-waves are determined by the boundary-value problem

$$\left(\frac{\partial^2}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial}{\partial\rho} - \frac{1}{\rho^2} + \frac{1}{\lambda_n^2}\right) R_n(\rho) = 0 \qquad (a \leq \rho \leq b) \tag{29}$$

$$\left(\frac{\partial}{\partial\rho} + \frac{1}{\rho}\right) R_n(\rho) = 0 \qquad (\rho = a, \ \rho = b) \quad . \tag{30}$$

It follows from equations (29) and (30) that the eigenfunctions are orthogonal with one another and with $1/\rho$. Hence, with appropriate normalization

$$\int_a^b R_n(\rho) \, \rho \, d\rho = \delta_{nm} \tag{31}$$

$$\int_a^b R_n(\rho) \, d\rho = 0 \quad . \tag{32}$$

The integral equation for the determination of the aperture electric field $\xi(\rho) = E_\rho(\rho,0)$ is

$$\frac{\alpha+\beta}{\rho}\int_a^b \xi(\rho')\,\rho' d\rho' d\phi' \left\{-\frac{i\omega\epsilon}{2\pi}\sum_{n=1}^{\infty}\frac{R_n(\rho)\,R_n(\rho')}{\sqrt{\lambda_n^2-k^2}} - \frac{i\omega\epsilon}{2\pi}\,C(\rho,\rho',\phi',0)\right\} \tag{33}$$

where

$$C(\rho,\rho',\phi',z) = \cos\phi' \; \frac{\exp\left(ik\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi'+z^2}\right)}{\sqrt{\rho^2+\rho'^2-2\rho\rho'\cos\phi'+z^2}} \;.$$

By comparing integral equations (15) and (33) it is seen that

$$\psi_m(\rho) = \frac{\alpha+\beta}{\rho} \quad , \qquad K_m(\rho) = \xi(\rho) \quad ,$$

$$G_m(\rho,\rho') = -\frac{i\omega\epsilon}{2\pi}\sum_{n=1}^{\infty}\frac{R_n(\rho)\,R_n(\rho')}{\sqrt{\lambda_n^2-k^2}} - \frac{i\omega\epsilon}{2\pi}\,C(\rho,\rho',\phi,0) \;. \tag{34}$$

Accordingly the corresponding quantity of physical interest (16) is

$$X_m = \int_0^{2\pi}\int_a^b \psi_m(\rho)\,K_m(\rho)\,\rho\,d\rho\,d\phi = \int_0^{2\pi}\int_a^b \frac{\alpha+\beta}{\rho}\,\xi(\rho)\,d\rho\,d\phi \;,$$

which, with the aid of definitions (27) and (28) becomes

$$X_m = I(0)\,V(0) = Z_{input}\,I^2(0) \tag{35}$$

where Z_{input}, the input impedance of the antenna, is equal to $V(0)\,I(0)$. The numerator of expression (17) becomes

$$\int_0^{2\pi}\int_a^b \psi_m(\rho)\,K_m(\rho)2\,d\rho\,d\phi = \int_0^{2\pi}\int_a^b \frac{\alpha+\beta}{\rho}\,\xi(\rho)\rho\,d\rho\,d\phi = I(0)\int_a^b \xi(\rho)\,d\rho \tag{36}$$

and the denominator becomes

$$\int_0^{2\pi}\int_a^b\int_0^{2\pi}\int_a^b K_m(\rho)\,G_m(\rho,\rho')\,K_m(\rho')\,\rho\,d\rho\,d\phi\,\rho'\,d\rho'\,d\phi' =$$

$$-2\pi\,i\omega\epsilon\sum_{n=1}^{n=\infty}\frac{1}{\sqrt{\lambda_n^2-k^2}}\left[\int_a^b \xi(\rho)\,R_n(\rho)\rho\,d\rho\right]^2 \tag{37}$$

$$-i\omega\epsilon\int_a^b \rho\,d\rho\int_a^b \rho'\,d\rho'\int_0^{2\pi} d\phi'\,C(\rho,\rho',\phi,0)\,\xi(\rho)\,\xi(\rho').$$

Therefore, by substituting expressions (35), (36), and (37) into expression (17), the following stationary expression for Z_{input} in terms of the aperture electric field (ρ) is obtained

$$\left(\frac{Z_{input}}{Z_o}\right)_m = \frac{-\dfrac{1}{ik \log (b/a)}\left(\displaystyle\int_a^b \mathcal{E}(\rho)\, d\rho\right)^2}{\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{\lambda_n^2 - k^2}}\left[\int_a^b \mathcal{E}(\rho) R_n(\rho)\rho\, d\rho\right]^2 + \frac{1}{2\pi}\int_a^b \rho\, d\rho \int_a^b \rho' d\rho' \int_0^{2\pi} d\phi\; C(\rho,\rho',\phi,0)\mathcal{E}(\rho)\mathcal{E}(\rho')} \tag{38}$$

where $Z_o = \frac{1}{2\pi}\sqrt{\frac{\mu}{\epsilon}} \log\left(\frac{b}{a}\right)$ is the characteristic impedance of the coaxial waveguide. In a similar manner the complementary variational principle in terms of the magnetic field $\mathcal{H}(\rho) = H_\phi(\rho,0)$ can be set up. For the sake of brevity the result is given without derivation.

$$\left(\frac{Z_o}{Z_{input}}\right)_e = \frac{-\dfrac{ik}{\log \frac{b}{a}}\left(\displaystyle\int_a^b \mathcal{H}(\rho)\, d\rho\right)^2}{\displaystyle\sum_{n=1}^{\infty} \sqrt{\lambda_n^2 - k^2}\left[\int_a^b \mathcal{H}(\rho) R_n(\rho)\rho\, d\rho\right]^2 - \int_0^{\infty} \rho\, d\rho\, \mathcal{H}(\rho)\int_0^{\infty} \rho'\, d\rho' \mathcal{H}(\rho')\frac{\partial}{\partial z}\frac{\partial}{\partial z}\Gamma(\rho,\rho',0,0)} \tag{39}$$

where $\Gamma(\rho,\rho',z,z') = \frac{1}{4\pi}\int_0^{2\pi} d\phi \left\{C(\rho,\rho',\phi,z-z') - C(\rho,\rho',\phi,z+z')\right\}$.

Expressions (38) and (39) are the two complementary forms of the variational principle for the input impedance of the antenna.*

On account of its mathematical simplicity and its physical suitability $1/\rho$ is chosen as a trial function for $\mathcal{E}(\rho)$ in expression (38). From this assumption for the aperture electric field the corresponding trial function for $\mathcal{H}(\rho)$ is calculated and then inserted into expression (39). Thus

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*For complete details on the derivation of these two forms of the variational principle the reader is referred to H. Levine and C. H. Papas, J.Appl.Phys.22, (1951) 29.

$$\left(\frac{Z_o}{Z_{input}}\right)_m = \frac{-ik}{\log(\frac{b}{a})} \int_0^{\infty} \frac{d\zeta}{\zeta\sqrt{\zeta^2 - k^2}} \left[J_o(\zeta a) - J_o(\zeta b)\right]^2 = A \qquad (40)$$

where the path of integration is along the positive real axis of the complex ζ-plane with a downward indentation at $\zeta = k$, and

$$\left(\frac{Z_o}{Z_{input}}\right)_e = \frac{A^2}{A - \frac{ik}{\log\frac{b}{a}} \sum_{n=1}^{n=\infty} \sqrt{\lambda_n^2 - k^2} \left\{\int_0^{\infty} \frac{d\zeta}{\sqrt{\zeta^2 - k^2}} \left[J_o(\zeta a - J_o(\zeta b)\right] f_n(\zeta)\right\}^2} \qquad (41)$$

where

$$f_n(\zeta) = \frac{2}{\pi} \frac{N_n}{\lambda_n} \frac{1}{J_o(\lambda_n b)} \frac{\zeta}{\lambda_n^2 - \zeta^2} \left[J_o(\zeta b)\, J_o(\lambda_n a) - J_o(\zeta a)\, J_o(\lambda_n b)\right]$$

$$\frac{1}{N_n} = \frac{\sqrt{2}}{\pi\lambda_n} \left[\frac{J_o^2(\lambda_n a)}{J_o^2(\lambda_n b)} - 1\right]^{1/2} .$$

It is seen that expressions (40) and (41) differ by the presence of the quantity

$$D = -\frac{ik}{\log\frac{b}{a}} \sum_{n=1}^{\infty} \sqrt{\lambda_n^2 - k^2} \left\{\int_0^{\infty} \frac{d\zeta}{\sqrt{\zeta^2 - k^2}} \left[J_o(\zeta a) - J_o(\zeta b)\right] f_n(\zeta)\right\}^2 \qquad (42)$$

It follows from expressions (40), (41), and (42) that

$$\frac{(Z_o/Z_{input})_m}{(Z_o/Z_{input})_e} = 1 + \frac{D}{A} . \qquad (43)$$

By actual calculation it can be shown that $\left|\frac{D}{A}\right|$ is small compared with 1 for ka ranging from 0 to 2.0 when $b/a = 1.57$ and $b/a = 2.36$. Hence, within these limits it is a good approximation to write

$$(Z_o/Z_{input})_m \approx (Z_o/Z_{input})_e . \qquad (44)$$

In view of this approximate equality it should be expected that $(Z_o/Z_{input})_m$ yield results in close agreement with measured values. Indeed, as shown in Figure 2, the agreement between computed and measured values of Z_o/Z_{input} is good.

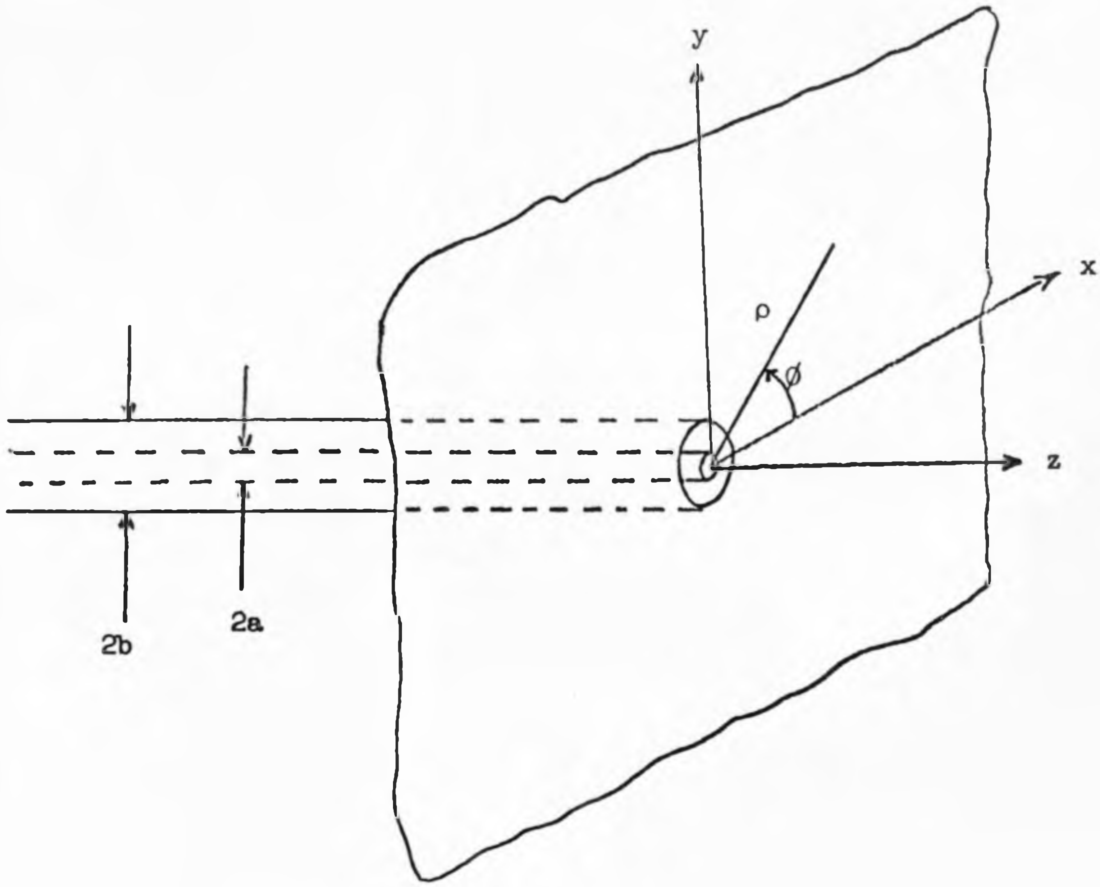

Fig. 1. Circular diffraction antenna consisting of open end of coaxial waveguide with limitless baffle.

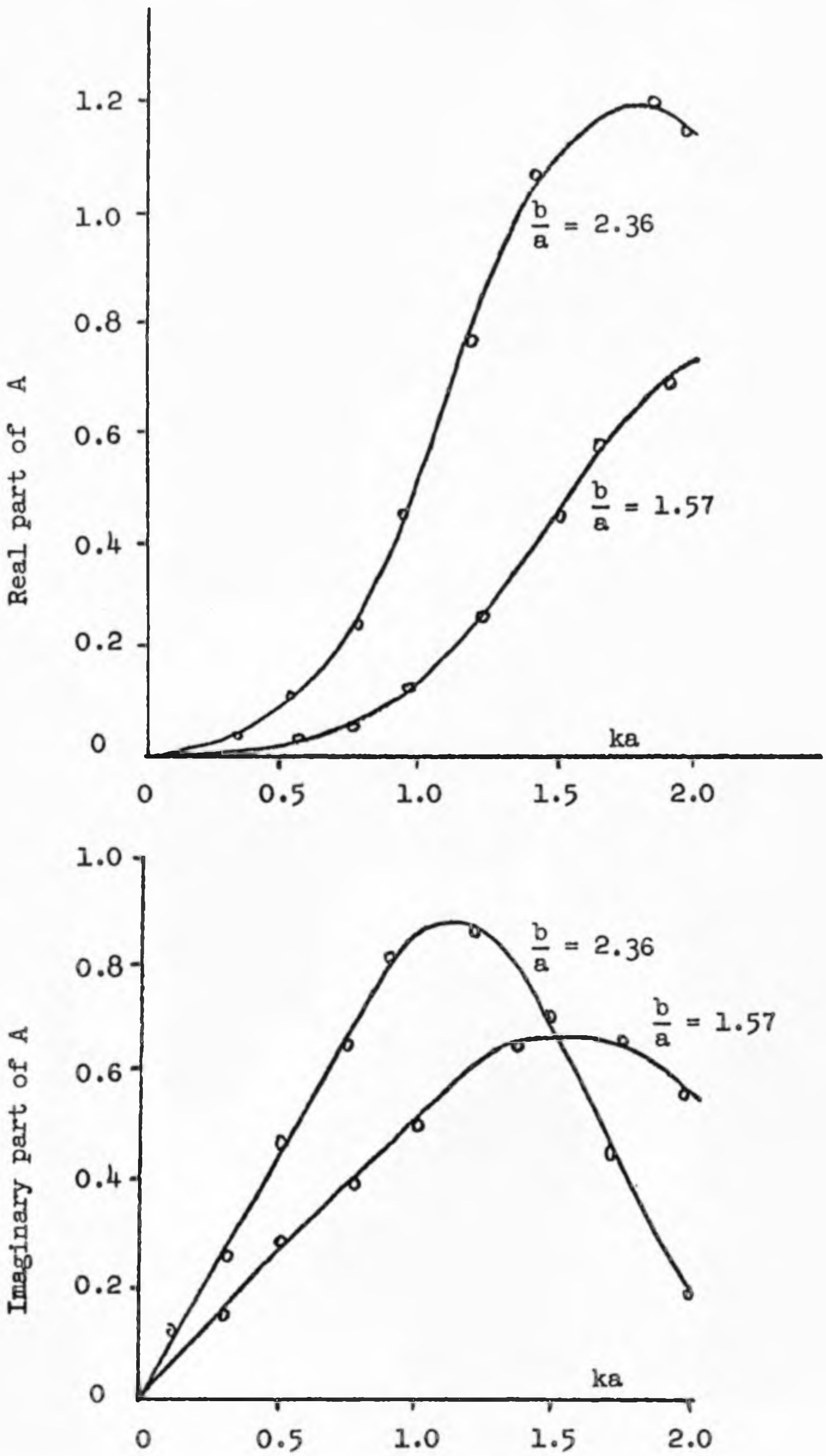

Fig. 2. Real and imaginary parts of A versus ka for b/a = 1.57 and b/a = 2.36 . Smooth curve is theoretical result. Dots are measured values.