Characteristic modes and fundamental singularities of partial differential equations[a)]

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Systems of linear partial differential equations with constant coefficients, like their ordinary differential equation counterparts, can be characterized by the properties of the matrices that form the coefficients of the differential operators. The question arises: Do the matrix operators that result from partial differential equations possess eigenvalues and eigensolutions in the same way that ordinary differential matrix operators do? The answer to this question is explored in some detail using as an example the linearized flow of a viscous fluid. It is shown that eigenfactors do exist for these equations, and that, of necessity, these involve hypercomplex algebra. This fact introduces significant new features to the problem. It is shown that eigenmodes exist and that each of these has its distinctive fundamental singularity. The fluid mechanical significance of these is examined in some detail. In addition, a representative group of other partial differential equations is examined and their eigenmodes and fundamental singularities are determined. It is shown that a number of basic differences exist between the eigenfunction theory for ordinary and for partial differential equations.

I. INTRODUCTION

The use of eigenfunctions associated with the eigenvalues of matrices is a well-established procedure for the solution of ordinary differential equations. However, no such procedure is customarily used for partial differential equations.

Any set of linear ordinary differential equations with constant coefficients can be presented in the form[1]

$$\frac{dz}{dt} + Az = Bf, \tag{1}$$

where z and f are column vectors whose components are the dependent variables and the independent input variables, and A and B are square matrices. The eigenvalues[2] for this system are the roots of the polynomial in T, $|\mathbb{I}T - A|$, where $\mathbb{I}$ is the unit matrix. If these roots are distinct, a similarity transformation exists which diagonalizes the matrix A. With A diagonal, the transformed equation (1) decouples completely, and the individual equations yield the eigenfunctions for the system. The most general behavior of the system becomes a simple linear combination of the individual behaviors of these eigenfunctions.

In Eq. (1), the derivative dz/dt, in reality, has the coefficient $\mathbb{I}$. Equally well we could have written

$$C\frac{dz}{dt} + Dz = Ef. \tag{2}$$

The characteristic polynomial for the eigenvalues becomes $|CT + D|$. An equivalency transformation then converts (2) to a decoupled set of equations.

Consider now a corresponding set of partial differential equations. It is not clear that these equations can be presented as a set of first-order equations with independent inputs which have the form

$$\sum_n A_n \frac{\partial z}{\partial x_n} + Bz = Cf,$$

and if this is possible, an attempt to find eigenrelations would involve factorizing $|\Sigma_n A_n \mathrm{X}_n + B|$, which generally cannot be accomplished within the realm of ordinary complex algebra. And finally the simultaneous diagonalization of the matrices A_n and B is required to obtain a decoupled set of equations whose solutions would be the eigenfunctions of the problem. No general procedure has been developed for the simultaneous diagonalization of multiple matrices.

Dirac[3] undertook a related problem when he sought to factorize the operator of the relativistic wave equation,

$$\frac{\partial^2\psi}{c^2\partial t^2} - \frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} + \frac{m^2c^2}{\hbar^2}\psi = 0. \tag{3}$$

Here the motivation was not to find the eigensolutions, but to find a wave equation which satisfied the general requirement of quantum mechanics that only the first-order term in the time derivative appear. Dirac factored the operator in (3) using hypercomplex algebra. The algebra he used contains 16 base elements. If α_1, α_2, α_3, and α_4 are four of these elements which obey the relation $\alpha_\mu\alpha_\nu + \alpha_\nu\alpha_\mu = 2\delta_{\mu\nu}$, then the factorized operator becomes

$$\left(\frac{\partial}{c\,\partial t} - \alpha_1\frac{\partial}{\partial x} - \alpha_2\frac{\partial}{\partial y} - \alpha_3\frac{\partial}{\partial z} - \alpha_4\frac{imc}{\hbar}\right) \times\left(\frac{\partial}{c\,\partial t} + \alpha_1\frac{\partial}{\partial x} + \alpha_2\frac{\partial}{\partial y} + \alpha_3\frac{\partial}{\partial z} + \alpha_4\frac{imc}{\hbar}\right).$$

In the decoupled equations, the first factor led to a proper relativistic theory for the electron including its intrinsic spin. The second factor led to Dirac's prediction of the existence of the positron.

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In the present article we shall explore the concepts of eigenfunction theory first for the case of the linear partial differential equations arising from the steady two-dimensional flow of a viscous fluid. Later we shall explore other examples.

II. VISCOUS FLOWS

In a two-dimensional viscous flow, the stresses (including the momentum fluxes) are

$$\sigma_{xx} = -\rho u'^2 - p' + \lambda'\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + 2\mu'\frac{\partial u'}{\partial x'},$$

$$\sigma_{xy} = -\rho u'v' + \mu'\left(\frac{\partial u'}{\partial y'} + \frac{\partial v'}{\partial x'}\right),$$

$$\sigma_{yy} = -\rho v'^2 - p' + \lambda'\left(\frac{\partial u'}{\partial x'} + \frac{\partial v'}{\partial y'}\right) + 2\mu'\frac{\partial v'}{\partial y'}.$$

Here u' and v' are the components of velocity along the x' and y' axes, p' is the pressure, ρ the density, and λ' and μ' are the two coefficients of viscosity.

The equations for steady flow are given by the three divergence equations:

$$\frac{\partial\sigma_{xx}}{\partial x'} + \frac{\partial x_{xy}}{\partial y'} = -F',$$

$$\frac{\partial\sigma_{xy}}{\partial x'} + \frac{\partial\sigma_{yy}}{\partial y'} = -G',$$

$$\frac{\partial\rho u'}{\partial x'} + \frac{\partial\rho v'}{\partial y'} = q'.$$

Here F', G', and q' are terms which allow us to influence the flow by external means, F' and G' being forces and q' a source strength, all per unit volume, In addition we shall make use of the two-dimensional vorticity:

$$\omega' = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'}.$$

If we linearize around a dominant free-stream velocity U, and if we introduce nondimensional variables

$$u = \frac{u'-U}{U},\ v = \frac{v'}{U},\ x = \frac{x'U\rho}{\mu'},\ y = \frac{y'U\rho}{\mu'},\ \omega = \frac{\omega'\mu'}{\rho U^2},$$

$$p = \frac{p'}{\rho U^2},\ \omega = \frac{\omega'\mu'}{\rho U^2},\ F = \frac{F'\mu'}{\rho^2 U^2},\ G = \frac{G'\mu'}{\rho^2 U^3},\ q = \frac{q'\mu'}{\rho^2 U^2},$$

then the equations for the flow of an incompressible fluid become

$$2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial p}{\partial x} - \left(\frac{\lambda'}{\mu'} + 2\right)\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = F,$$

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \left(\frac{\lambda'}{\mu'} + 2\right)\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = G,$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = q,$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + \omega = -\gamma.$$

Here we have introduced an external input $-\gamma$ into the vorticity equation.

We can rearrange these equations as follows:

$$\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} - \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = F - q + \left(\frac{\lambda'}{\mu'} + 1\right)\frac{\partial q}{\partial x},$$

$$\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 v}{\partial y^2} = G + \left(\frac{\lambda'}{\mu'} + 1\right)\frac{\partial q}{\partial y},$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = q,$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + \omega = -\gamma.$$

An alternative rearrangement which results in a set of first-order rather than second-order equations is

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} + \frac{\partial \omega}{\partial y} = f,\\ &\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} - \frac{\partial \omega}{\partial x} = g,\\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = q,\\ &\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} + \omega = -\gamma. \end{aligned} \tag{4}$$

In this latter set we have introduced new input functions f and g which are related to F and G as follows:

$$\begin{aligned} f &= F - q + \left(\frac{\lambda'}{\mu'} + 2\right)\frac{\partial q}{\partial x} - \frac{\partial\gamma}{\partial y},\\ g &= G + \left(\frac{\lambda'}{\mu'} + 2\right)\frac{\partial q}{\partial y} + \frac{\partial\gamma}{\partial x}. \end{aligned} \tag{5}$$

If we introduce the column vector W, whose components are u, v, p, ω, and the vector E whose components are f, g, q, and $-\gamma$, and the matrix operator

$$\mathbb{L} \equiv \begin{pmatrix} \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 & 0 \\ \frac{\partial}{\partial y} & -\frac{\partial}{\partial x} & 0 & 1 \end{pmatrix},$$

then Eqs. (4) can be written in the simple form

$$\mathbb{L}W = E. \tag{6}$$

III. EIGENFACTORS AND EIGENMODES FOR VISCOUS FLOWS

We now set out to explore the eigenmodes of this set of equations. The eigenvalues of the operator $\mathbb{L}$ are given by the factors of the determinant,

$$|\mathbb{L}| \equiv \begin{vmatrix} X & 0 & X & Y \\ 0 & X & Y & -X \\ X & Y & 0 & 0 \\ Y & -X & 0 & 1 \end{vmatrix} \equiv (X^2+Y^2)(X^2+Y^2-X).$$

For convenience, we have replaced $\partial/\partial x$ by X and $\partial/\partial y$ by Y. Since we are dealing with two entities, X and Y, rather than a single entity, it is clear that we are seeking eigenfactors involving the two variables rather than simple roots. We see immediately that two such factors are $X + iY$ and $X - iY$.

We now grapple with the more formidable task of finding the factors of the simple looking expression, $X^2 + Y^2 - X$. If we attempt to find two factors such that

$$(X + \alpha Y + \beta)(X + aY + b) = X^2 + Y^2 - X,$$

then we must require that

$$a = -\alpha,\ \ \alpha a = 1,\ \ \alpha b = -\beta a,\ \ b + \beta = -1,\ \ \beta b = 0.$$

A little analysis shows that within the realm of complex algebra, no set of values for α, β, a, and b exists which will satisfy these equations.

To obtain a solution it is necessary to use hypercomplex algebra. (Hypercomplex algebras follow the same operational rules that square matrices do: Multiplication is associative but not necesssarily commutative and two nonzero quantities can have a zero product.)

The first two of the above equations can be satisfied by taking $\alpha = i$, $a = -i$. We introduce a new variable k and put $b = -(1 + k)/2$. The equation $b + \beta = -1$ shows that $\beta = -(1 - k)/2$. The equation $\beta b = 0$ then becomes $k^2 = 1$. It would be tempting to conclude that $k = \pm 1$, but neither of these permits satisfaction of the equation $\alpha b + \beta a = 0$ which becomes $ik + ki = 0$ or $ik = -ki$. Instead of being simply ± 1, we see that k must be a quantity which, in addition to having a square which is 1, also anticommutes with i. Let us put $ki = j$; then, $ik = -j$. Multiplying from the right and left by i and k, we find that $ij = -ji = k$ and $kj = -jk = i$ and $j^2 = 1$.

We now have a complete multiplication table for this hypercomplex algebra which has four base elements $1, i, j, k$:

	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	1	$-i$
k	k	j	i	1

Any number in this algebra can be represented in the form $a + bi + cj + dk$. With the above table we can carry out all the ordinary algebraic operations. However, the fact that we have to introduce hypercomplex algebra into the analysis for partial differential equations marks a distinct departure from the corresponding analysis for ordinary differential equations.

Returning now to the expression $X^2 + Y^2 - X$, we see that using the values we have just obtained, this can be written in factored form as $[X + iY - (1 - k)/2][X - iY - (1 + k)/2]$.

Our analysis has thus led to the following four eigenfactors for $|\mathbb{L}|$:

$$X + iY, X - iY, X + iY - (1 - k)/2, X - iY - (1 + k)/2. \tag{7}$$

Each factor should correspond to an eigenmode of behavior. It is worth noting that each of these factors should commute since from the physical problem we have no basis for distinguishing the order in which they are to be taken. At first glance, the first pair does not appear to commute with the second pair. In Sec. VIII we show that they do commute and explain why.

Let us suppose that in diagonal form the matrix operator $\mathbb{L}$ becomes Λ. Associated with this diagonal form is the eigensolution vector $Z = (a,b,c,d)$ and the input vector $K = (\kappa,\lambda,\mu,\nu)$. These vectors will be related to W and E by transformations

$$W = NZ, \quad E = MK, \tag{8}$$

where N and M are matrices whose elements are members of the hypercomplex algebra we have introduced.

In diagonal form (6) becomes

$$\Lambda Z = K. \tag{9}$$

Inserting (8) into (6) and comparing with (9) we find that

$$\mathbb{L}N \equiv M\Lambda.$$

These operators must be equal component by component as well as coefficient by coefficient of X and Y, and separately coefficient by coefficient of $1,i,j,k$. With the matrices written out we have

$$\begin{pmatrix} X & 0 & X & Y \\ 0 & X & Y & -X \\ X & Y & 0 & 0 \\ Y & -X & 0 & 1 \end{pmatrix} \begin{pmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{23} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{pmatrix}$$

$$\equiv \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix} \begin{pmatrix} X + iY & 0 & 0 & 0 \\ 0 & X - iY & 0 & 0 \\ 0 & 0 & X + iY - (1 - k)/2 & 0 \\ 0 & 0 & 0 & X - iY - (1 + k)/2 \end{pmatrix}. \tag{10}$$

The solution of this identity for the M's and N's is simplified by the fact that the equations decouple, each group containing N's from only one column and M's from the corresponding column. Details of the process of solving within the operational rules of hypercomplex algebra are given in the Appendix.

We find the following solutions

$$N = \begin{pmatrix} 1 & i & 1+k & i+j \\ i & 1 & i+j & 1+k \\ -1 & -i & 0 & 0 \\ 0 & 0 & i+j & 1+k \end{pmatrix},$$

$$M = \begin{pmatrix} 0 & 0 & 1+k & i+j \\ i & 1 & 0 & 0 \\ 1 & i & 1+k & i+j \\ -i & -1 & -i-j & -1-k \end{pmatrix}. \quad (11)$$

Since identity (10) is homogeneous in N and M the solution for these variables is not unique. Only ratios can be determined and these ratios go column by column.

The result of the above analysis is that we have reduced our problem of solving the original partial differential equations to one of solving the four independent equations:

$$\begin{aligned} &\frac{\partial a}{\partial x} + i\frac{\partial a}{\partial y} = \kappa, \\ &\frac{\partial b}{\partial x} - i\frac{\partial b}{\partial y} = \lambda, \\ &\frac{\partial c}{\partial x} + i\frac{\partial c}{\partial y} - \frac{1-k}{2}c = \mu, \\ &\frac{\partial d}{\partial x} - i\frac{\partial d}{\partial y} - \frac{1+k}{2}d = \nu. \end{aligned} \quad (12)$$

The solution of each of these equations will yield an individual mode of behavior. It is the combination of these individual modes that yields the overall behavior of the system. Once these individual equations have been solved the solutions for the original variables, u,v,p,ω are given by $W = NZ$; i.e., by

$$\begin{aligned} u &= a + ib + (1+k)c + (i+j)d, \\ v &= ia + b + (i+j)c + (1+k)d, \\ p &= -a - ib, \\ \omega &= (i+j)c + (1+k)d. \end{aligned} \quad (13)$$

IV. FUNDAMENTAL SINGULARITIES

Now that we have split the behavior of the flow field into four independent modes, our next step is to seek solutions of the individual equations of (12) and to show that each possesses a distinctive singularity.

For the first two of the above equations we are on familiar ground. If we formally "solve" for a and b, we have

$$a = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^{-1}\kappa, \quad b = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)^{-1}\lambda.$$

Here the question is one of giving meaning to the inverse operators. We can write these equations in the form

$$\begin{aligned} a &= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{-1}\kappa, \\ b &= \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{-1}\lambda. \end{aligned} \quad (14)$$

Classically we have the result that if $\partial^2\phi/\partial x^2 + \partial^2\phi/\partial y^2 = \kappa$, then for the infinite plane the general solution is

$$\phi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\kappa(x_0,y_0)\log R\, dx_0\, dy_0,$$

where $R = [(x-x_0)^2 + (y-y_0)^2]^{1/2}$. In words, this may be interpreted in two equivalent ways. One is that the inverse second-degree operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)^{-1}$ is to be interpreted as a double integration with the inversion kernel being $(1/2\pi)\log R$. The other is that any potential solution is the distributed sum of the contributions from all its sources (or charges, poles or masses as the case may be) where the potential for such a source is $(1/2\pi)\log R$.

If κ is zero everywhere except at the origin and there it has the singular behavior of an isolated point source, i.e., a Dirac delta function, then $\phi_s = (1/2\pi)\log r$, where $r = (x^2+y^2)^{1/2}$.

For the a mode, from (14) we have

$$a_s = \frac{\partial\phi_s}{\partial x} - i\frac{\partial\phi_s}{\partial y} = \frac{x-iy}{2\pi r^2}.$$

Making use of (13), and retaining just the real parts for our physical flow we have

$$u_a = \frac{x}{2\pi r^2}, \quad v_a = \frac{y}{2\pi r^2}, \quad p_a = -\frac{x}{2\pi r^2}, \quad \omega_a = 0. \quad (15)$$

Correspondingly for the b mode we have

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = \lambda,$$

$$\psi(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\lambda(x_0,y_0)\log R\, dx_0\, dy_0,$$

$$\psi_s = \frac{1}{2\pi}\log r, \quad b_s = \frac{\partial\psi_s}{\partial x} + i\frac{\partial\psi_s}{\partial y} = \frac{x+iy}{2\pi r^2},$$

and again using (13), and retaining the real parts,

$$u_b = -\frac{y}{2\pi r}, \quad v_b = \frac{x}{2\pi r^2}, \quad p_b = \frac{y}{2\pi r^2}, \quad \omega_b = 0. \quad (16)$$

The a singularity (15) is a point source with fluid flowing out radially from the origin. Even though we are dealing with a linearized flow, the velocity field is the same as that for the exact field of a potential source. However, the pressure fields in the two cases appear to be quite different. For the exact potential source, the pressure follows the Bernoulli law and thus varies quadratically with the absolute velocity. In the linearized case, the pressure varies linearly with the x component of the velocity. When the exact potential source exists by itself, the pressure is circumferentially constant and there is no resultant force on the source. In the presence of a free stream, the pressure becomes unsymmetrical fore and aft and there exists a force, $\rho UQ'$ per unit length, in the upstream direction. The linearized source can be considered by itself, but in the linearization, account has already been taken of the existence of the free stream. As a result, the pressure field of the linearized source has a circumferential variation that is sinusoidal. If one computes the force acting on the linearized source by integrating the total stresses acting on a surface surrounding the source, one finds that here, too, there is a force per unit length of $\rho UQ'$, or in nondimensional terms, for a unit source the force is unity.

The real point of interest here is not the difference between the linearized and exact sources, but rather that in spite of the fact that we are dealing with a *viscous* fluid, one of the fundamental singularities is the same as that of a fluid without viscosity.

Turning to the b singularity (16) we see that it is a vortex with the same velocity distribution as that of an exact irrotational vortex. This linearized vortex has a pressure that, like the source, varies sinusoidally around the singularity, but shifted by 90° in phase. This together with the inertial and viscous stresses gives a force on a surface surrounding the vortex of $\rho U\Gamma'$ perpendicular to the free stream. The exact vortex, by its symmetry, experiences no force when it exists by itself, but when placed in a free stream experiences the same force as the linearized vortex.

Here again, the point of interest is that the circular vortex which is a fundamental singularity of an inviscid flow also turns out to be a fundamental singularity of a flow where viscous stresses are at work.

We now turn to the c and d modes which involve hypercomplex algebra. As before we "solve" for c and d;

$$c = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - \frac{1-k}{2}\right)^{-1}\mu,$$
$$d = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - \frac{1+k}{2}\right)^{-1}\nu.$$

For the i, j, k algebra, conjugate quantities are formed by reversing the signs of all the "imaginary," that is, of all the i, j, and k parts. As we did earlier, we multiply numerators and denominators of the above operators by their conjugates and write the results in the form

$$c = \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y} - \frac{1+k}{2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x}\right)^{-1}\mu,$$
$$d = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y} - \frac{1-k}{2}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{\partial}{\partial x}\right)^{-1}\nu.$$

To interpret the meaning of the operator as it occurs in $\xi = (\partial^2/\partial x^2 + \partial^2/\partial y^2 - \partial/\partial x)^{-1}\mu$, we use the classical result that for the infinite plane the general solution of $\partial^2\xi/\partial x^2 + \partial^2\xi/\partial y^2 - \partial\xi/\partial x = \mu$ is

$$\xi(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mu(x_0,y_0) \times \exp\left(\frac{x-x_0}{2}\right)K_0\left(\frac{R}{2}\right)dx_0\,dy_0,$$

where again $R = [(x-x_0)^2 + (y-y_0)^2]^{1/2}$. K_0 is the second solution of the zero-order Bessel equation of imaginary argument.

If we take μ to be the Dirac delta function with singularity at the origin, then

$$\xi_s = -(1/2\pi)e^{x/2}K_0(r/2).$$

For the c mode

$$c_s = \frac{\partial\xi_s}{\partial x} - i\frac{\partial\xi_s}{\partial y} - \frac{1+k}{2}\xi_s$$
$$= -\frac{e^{x/2}}{4\pi}\left[\frac{x+iy}{r}K_1\left(\frac{r}{2}\right) - kK_0\left(\frac{r}{2}\right)\right].$$

Making use of (13), and retaining only the real parts for our physical flow, we have

$$u_c = -\frac{e^{x/2}}{4\pi}\left[\frac{x}{r}K_1\left(\frac{r}{2}\right) - K_0\left(\frac{r}{2}\right)\right],$$
$$v_c = -\frac{e^{x/2}}{4\pi}\left[\frac{y}{r}K_1\left(\frac{r}{2}\right)\right],$$
$$p_c = 0,$$
$$\omega_c = -\frac{e^{x/2}}{4\pi}\left[\frac{y}{r}K_1\left(\frac{r}{2}\right)\right].$$

The streamlines for this fundamental singularity have been plotted in Fig. 1. Near the origin the flow superficially resembles that of the conventional source. However, the fluid emerges from the source with a strong variation in radial velocity. As a result the flow experiences high vorticity and large viscous stresses. Somewhat surprisingly the viscous and inertial stresses balance so that there is no variation of pressure even near the source where the velocities become infinite. As the flow emerges, it turns downstream and forms a rearward diffuse jet. Here we have a fundamental singularity which is characteristic of viscous flows, and which has no inviscid counterpart.

We can follow an analogous procedure for the d mode with the result that

$$\frac{\partial^2\chi}{\partial x^2} + \frac{\partial^2\chi}{\partial y^2} = \nu,$$
$$\chi(x,y) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\nu(x_0,y_0) \times \exp\left(\frac{x-x_0}{2}\right)K_0\left(\frac{R}{2}\right)dx_0\,dy_0,$$
$$\chi_s = -\frac{e^{x/2}}{2\pi}K_0\left(\frac{r}{2}\right),$$
$$d_s = \frac{\partial\chi_s}{\partial x} + i\frac{\partial\chi_s}{\partial y} - \frac{1-k}{2}\chi_s$$
$$= -\frac{e^{x/2}}{4\pi}\left[\frac{x-iy}{r}K_1\left(\frac{r}{2}\right) + kK_0\left(\frac{r}{2}\right)\right].$$

Making use of (13) and retaining only the real parts for our physical flow we have

$$u_d = -\frac{e^{x/2}}{4\pi}\left[-\frac{y}{r}K_1\left(\frac{r}{2}\right)\right],$$
$$v_d = -\frac{e^{x/2}}{4\pi}\left[\frac{x}{r}K_1\left(\frac{r}{2}\right) + K_0\left(\frac{r}{2}\right)\right],$$
$$p_d = 0,$$
$$\omega_d = -\frac{e^{x/2}}{4\pi}\left[\frac{x}{r}K_1\left(\frac{r}{2}\right) + K_0\left(\frac{r}{2}\right)\right].$$

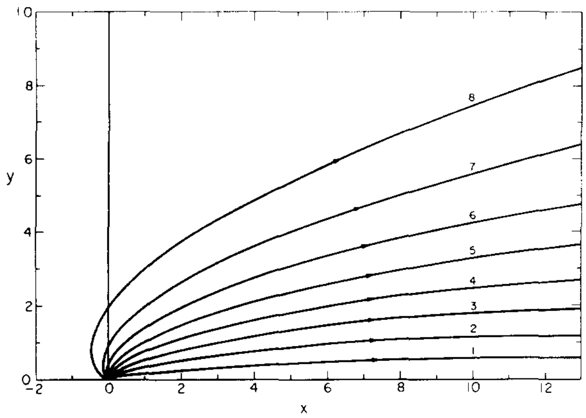

FIG. 1. Streamline pattern for the viscous source of the c mode. The free-stream velocity is from the left. The unit source is located at the origin. The flow is symmetrical about the x axis. The x and y coordinates are dimensionless Reynolds numbers. The stream function is dimensionless, and the numbers shown are multiples of 1/18.

In the c mode, if we had considered the "imaginary" parts, we would have found that the k parts gave the same results as the real parts. The i and j parts would have given the same results as we have obtained for the d mode. Analogous results hold for the "imaginary" parts of the d mode. In Sec. VII we examine this redundancy in greater detail.

We have plotted the streamlines for this fundamental singularity in Fig. 2.

Near the origin the flow superficially resembles that of the conventional irrotational vortex. However, superimposed upon the irrotational flow is one which has large variations in vorticity. The viscous stresses are large and so delicately in balance with the inertial stresses that there is no variation in pressure.

At greater distances, the fore and aft lack of symmetry increases with the streamlines being drawn out in great downstream loops. Here we have another fundamental singularity which is characteristic of viscous flows and which has no inviscid counterpart.

In summary, we see that the flow of a two-dimensional viscous fluid is characterized by the behavior of four different fundamental patterns. In the same way that in electrostatics the field of an electric charge is the fundamental pattern of the electric field, so do these four fields constitute the fundamental patterns for the flow of a two-dimensional viscous fluid. And just as any electrostatic field can be synthesized by a proper distribution of charges, so can any linearized viscous flow field be synthesized by a proper distribution of its four fundamental singularities.

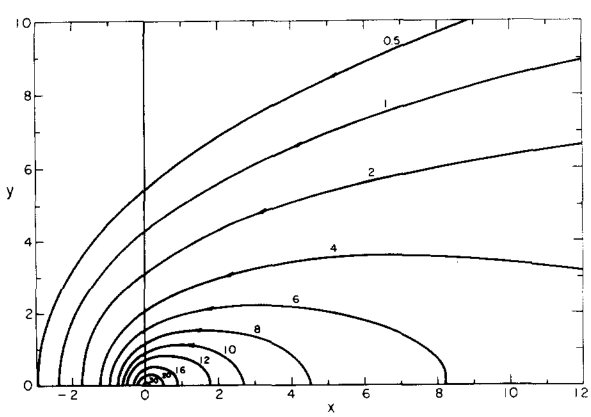

FIG. 2. Streamline pattern for the viscous vortex of the d mode. The unit vortex is located at the origin. The stream function numbers shown are multiples of $1/10\pi$. The viscous vortex is force-free.

V. FORCE LAWS FOR THE FUNDAMENTAL SINGULARITIES

In the last section, we were dealing with the modal input vector K. In (8) we saw that the input vector E of (6) was related to K by $E = MK$. We would like to go one step further and relate E to external force, source and vorticity densities per unit volume acting on or in the fluid.

Equations (5) give the forces F and G in terms of f and g. Combining these with (8) and (11), and retaining only real parts we can relate F, G, q, and γ to κ, λ, μ, and ν:

$$\begin{aligned} F &= \kappa + 2\mu - \left(\frac{\lambda'}{\mu'} + 2\right)\left(\frac{\partial \kappa}{\partial x} + \frac{\partial \mu}{\partial x}\right) + \frac{\partial \lambda}{\partial y} + \frac{\partial \nu}{\partial y}, \\ G &= \lambda - \left(\frac{\lambda'}{\mu'} + 2\right)\left(\frac{\partial \kappa}{\partial y} + \frac{\partial \mu}{\partial y}\right) - \frac{\partial \lambda}{\partial x} - \frac{\partial \nu}{\partial x}, \\ q &= \kappa + \mu, \\ \gamma &= \lambda + \nu. \end{aligned} \tag{17}$$

We apply these to the four fundamental singularities for which we would like to know the nondimensional total forces involved, the total fluid injected and the total circulation induced. We put

$$\begin{aligned} \mathscr{F} &= \iint F\,dx\,dy, \quad \text{total } x \text{ force}, \\ \mathscr{G} &= \iint G\,dx\,dy, \quad \text{total } y \text{ force}, \\ Q &= \iint q\,dx\,dy, \quad \text{total fluid injected}, \\ \Gamma &= \iint \gamma\,dx\,dy, \quad \text{total circulation}. \end{aligned} \tag{18}$$

Consider first the a mode where κ is a delta function. Inserting this into (17) and using (18) we find

$$\mathscr{F}_a = 1, \quad \mathscr{G}_a = 0, \quad Q_a = 1, \quad \Gamma_a = 0.$$

The results $\mathscr{F}_a = 1$ and $Q_a = 1$ are consistent with our earlier observation that the fundamental singularity of the a mode is a unit source and that the total force per unit length acting upon a source is $\rho U Q'$ in the streamwise direction.

For the b mode with λ as a delta function we find that

$$\mathscr{F}_b = 0, \quad \mathscr{G}_b = 1, \quad Q_b = 0, \quad \Gamma_b = 1.$$

These results, too, are consistent with our earlier observation that the fundamental singularity of the b mode is a unit vortex and that the total force per unit length acting upon such a vortex is $\rho U \Gamma'$ in a direction perpendicular to the free stream.

At this point we ask about Ω', the torque per unit length required to sustain this vortex. To compute the torque we must integrate the moment of the stress acting across a surface surrounding the singularity. If we do this and introduce a non-dimensional torque $\Omega = \Omega'/\rho v' U$, we find that $\Omega_b = 2$.

We now turn to the viscous modes where we find somewhat more unusual results. For the fundamental singularity of the c mode μ is a delta function. Inserting this in (17) and using (18) we find that $\mathscr{F}_c = 2$, $\mathscr{G}_c = 0$, $Q_c = 1$, $\Gamma_c = 0$. Since this singularity was earlier seen to be a source (Fig. 1), $Q_c = 1$ is an expected result. Also it is to be expected that an x-wise force should exist on this viscous source but the result $\mathscr{F}_c = 2$ showing that this force is twice as great as that on a conventional source is somewhat surprising. Later, we shall examine in some detail the structure of the flow that causes this result.

Turning to the singularity of the d mode where ν is a delta function, we find from (17) and (18) that $\mathscr{F}_d = 0$, $\mathscr{G}_d = 0$, $Q_c = 0$, $\Gamma_c = 1$. We saw from Fig. 2 that this is a vortex so that a circulation, $\Gamma_c = 1$ is to be expected. However, in contrast to the conventional vortex where $\mathscr{G}_b = 1$, and in contrast to the viscous source where $\mathscr{F}_c = 2$, we find here that there is no force acting on the viscous vortex. Later, we shall also examine the causes for this result. If we compute the torque required to maintain this vortex, we find it to be $\Omega_d = 2$.

By superposition it is possible to obtain other singularities that are both interesting and useful. The fact that the viscous singularities experience different forces than their conventional counterparts allows us to generate new singularities that are no longer sources nor vortices. Instead they are what might be called pure force singularities.

Let us combine a viscous source (mode c) with a conventional source (mode a) of equal and opposite strength. The sources completely cancel, leaving no net outflow from the origin, i.e., $Q_{c-a} = 0$. However, the forces do not cancel; leaving $\mathscr{F}_{c-a} = 1$. The flow pattern for this singularity is shown in Fig. 3. As the fluid passes through the origin a concentrated x-wise force acts continuously upon it, accelerating it to infinite velocity there. Viscous stresses impart high velocities to adjacent layers, all in the downstream direction.

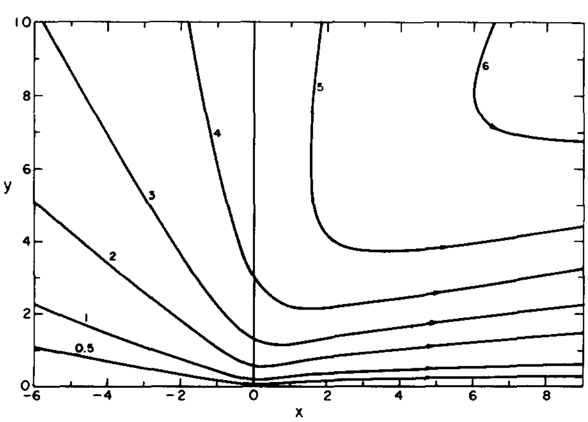

FIG. 3. Streamline pattern for a singularity which has a concentrated force acting in the x direction at the origin. It consists of the viscous source of mode c minus the potential source of mode a. The stream function numbers shown are multiples of 1/18.

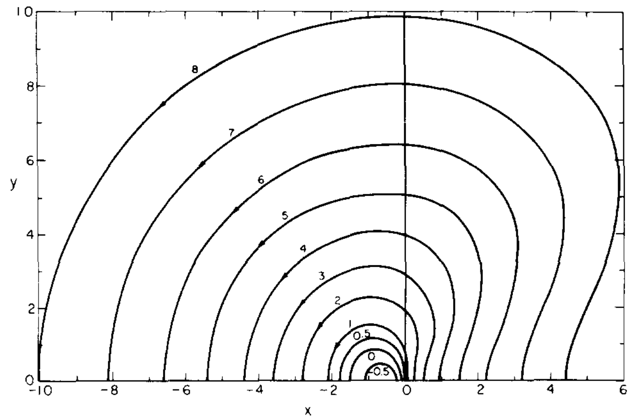

FIG. 4. Streamline pattern for a singularity which has a concentrated force acting in the y direction at the origin. It consists of a potential vortex of mode b minus a viscous vortex of mode d. The stream function numbers shown are multiples of $1/10\pi$.

As a result, fluid is drawn in from the upstream sector and from the sides and ejected as a diffuse jet downstream.

By canceling the viscous source against that of a conventional source, we see revealed in excellent detail why the nonsymmetrical structure of the flow of mode c (Fig. 2) leads to a force twice that of its conventional counterpart, mode a. The additional force is that associated with the flow of Fig. 3.

In a similar way, let us combine a conventional vortex (mode b) with a viscous vortex (mode d) of equal and opposite strength so that the vortices at the origin cancel, i.e., $\Gamma_{b-d} = 0$. Also the torques cancel, $\Omega_{b-d} = 0$. However, the forces do not cancel, but leave $\mathscr{G}_{b-d} = 1$. The flow pattern for this singularity is shown in Fig. 4. Here a cross-stream force acts continuously on the fluid as it passes through the origin, accelerating it to infinite velocity there. Viscous stresses impart high velocities to adjacent layers of fluid, all in the cross-stream direction. This motion induces an upstream whirlpool which rotates almost as a solid body. The center of this rotation is at approximately $x = -0.56$. At greater distances, the upstream flow is very nearly circular with the origin as center. Downstream, the streamlines are drawn in toward the origin and the flow is speeded up.

By canceling the viscous vortex with a conventional vortex, we see revealed the highly rotational and highly unsymmetrical character of the viscous flow which gives it such a different structure than that of its conventional counterpart. The viscous vortex, in addition to experiencing a force similar to that of the conventional vortex, also experiences an oppositely directed force (that of Fig. 4) which cancels the first.

For inviscid flows, the typical boundary condition imposed is that the flow not cross a solid boundary. This can be done by properly distributing conventional sources and vortices on or within these boundaries. For viscous flows, it is usual to impose, in addition, the "no slip" condition. In so doing, the boundary is subjected to shear stresses in addition to normal pressures. It is clear that the singularities of Figs. 3 and 4 play a central role in enabling these no slip conditions to be satisfied.

When a bouyant object such as an airship or a submarine travels at constant speed, there is no resultant force on

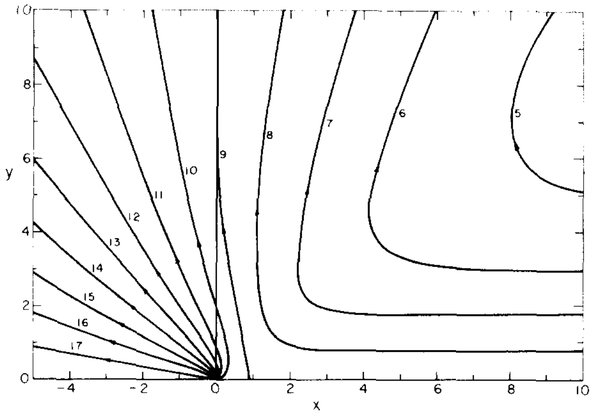

FIG. 5. Streamline pattern for a force-free source singularity. It consists of a double strength potential source of mode a minus a viscous source of mode c. The stream function numbers shown are multiples of 1/36.

the object. The propulsive forces just balance the drag or resistive forces. It is therefore of interest to ask about the existence of force-free singularities. One such, of course, is the viscous vortex of mode d which was shown in Fig. 3. We can obtain another by combining a viscious source with a conventional source of double strength and opposite sign. In this case, the forces cancel and leave a unit source with the same sign as that of the conventional source. Such a force-free source is shown in Fig. 5. All of the fluid from the source is expelled into the far field upsteam sector. Downstream, there is a stagnation point at approximately $x = 0.9$. Viscous stresses pull in fluid from downstream and expel it on the two downstream wings. Grossly, this will be the streamline pattern for any two-dimensional object that propels itself in a viscous environment by either ejecting fluid or taking it in. Similarly, the streamline pattern of Fig. 2, when combined with Fig. 5, will grossly represent the flow about any two-dimensional object that is freely moving except for an external torque that causes it to rotate.

VI. FLOWS WITH VANISHING VISCOSITY

If we examine our flow equations when the viscosity is zero we find that the determinantal operator (before nondimensionalization) becomes $(U/\rho)(\partial/\partial x')(\partial^2/\partial x'^2 + \partial^2/\partial y'^2)$. This leads to three modes, one corresponding to $\partial/\partial x' + i(\partial/\partial y')$, a second to $\partial/\partial x' - i(\partial/\partial y')$, and a third to $(\partial/\partial x')$ [cf. Eq. (7)]. The first and second correspond, of course, to modes a and b and thus have conventional sources and vortices as their fundamental singularities. But to which of the viscous modes does the third correspond? And what happened to the fourth viscous mode? The reader can readily verify that the third factor $\partial/\partial x'$ is the limiting remnant of mode c which has the viscous source as its fundamental singularity. If $v' \to 0$, with U, x', and y' being held constant, the nondimensional flow pattern of Fig. 1 shrinks until in the limit the fluid emanating from the origin goes straight downstream in a thin line. For the viscous vortex of Fig. 2, as $v' \to 0$ the flow pattern also shrinks and in the limit there is no flow except for an isolated spinning point at the origin. This mode thus vanishes in the inviscid limit.

In classical inviscid fluid mechanics the discontinuous flows of Helmholtz, Kirchoff, and Rayleigh were designed, in part, to incorporate resistance and drag into an otherwise lossless flow. In the linearized inviscid case, these discontinuous wakes are associated with distributions of the singularity that is the limit of mode c.

VII. EIGENMODES FOR OTHER PARTIAL DIFFERENTIAL EQUATIONS

Thus far our principal interest has been in the eigenmodes of viscous flows. At this point we examine some of the broader implications of the eigenmodes of partial differential equations.

Hypercomplex algebras not only follow the operational rules of matrices, they can be represented by matrices. The i,j,k algebra that we introduced in Sec. III has the following representation

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$j = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad k = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The reader can verify that these matrices obey the same multiplication table given earlier for the algebraic elements. It is readily apparent that any number $a + bi + cj + dk$ can be represented by a matrix

$$\begin{pmatrix} a+d & b+c \\ b-c & a-d \end{pmatrix},$$

and conversely any two-dimensional matrix

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

can be represented by $(A + D)/2 + [(B - C)/2]i + [(B + C)/2]j + [(A - D)/2]k$. Further, the modulus of any hypercomplex number, which is the product of the number and its conjugate, is the same as the determinant of the corresponding 2×2 matrix.

Consider now the product of a matrix M and a column vector V, i.e., MV. The question arises: If M is expressed algebraically using $1,i,j,k$, what is the proper treatment for the vector V? We introduce vector elements, m and n, and write

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

as $v_1 m + v_2 n$. The question then becomes: What are the multiplication rules for products of $1,i,j,k$ with m,n that make the product MV yield the same result for the matrix and algebraic representations. It is readily found that the appropriate multiplication table is the following:

	m	n
1	m	n
i	$-n$	m
j	n	m
k	m	$-n$

In Sec. V we used the real parts of the various solutions to obtain the fundamental singularities. We also noted a redundancy in which the imaginary parts of each of the solutions were the same as the real parts of other solutions. We now examine the cause for this redundancy and trace what simplifications are possible.

Consider a flow which is divergence-free and irrotational except for external inputs:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = f, \quad -\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = g. \tag{19}$$

The characteristic expression for the operator on the left is

$$\begin{vmatrix} X & Y \\ -Y & X \end{vmatrix} = X^2 + Y^2 = (X + iY)(X - iY).$$

These eigenfactors indicate that the diagonalized matrix should be

$$\begin{pmatrix} X + iY & 0 \\ 0 & X - iY \end{pmatrix}, \tag{20}$$

and one readily finds a similarity transformation which will produce this result.

But let us take a second look at (20). Here we are employing both a matrix representation *and* an algebraic representation. If we replace the algebraic representation by its equivalent matrix representation, (20) becomes

$$\begin{pmatrix} X & Y & 0 & 0 \\ -Y & X & 0 & 0 \\ 0 & 0 & X & -Y \\ 0 & 0 & Y & X \end{pmatrix}.$$

Here the redundancy is readily apparent. We not only have the original matrix

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix},$$

but its conjugate

$$\begin{pmatrix} X & -Y \\ Y & X \end{pmatrix}$$

as well. If we return to (19), we see that the characteristic matrix

$$\begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$$

for the operator of the original equation, when expressed in complex representation, is $X + iY$. If we now utilize the algebraic elements m and n, introduced earlier, for the vectors

$$\begin{pmatrix} u \\ v \end{pmatrix} \text{ and } \begin{pmatrix} f \\ q \end{pmatrix},$$

we see that (19) can be written directly as

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(mu + nv) = mf + ng.$$

In one short stroke this step accomplishes what the diagonalization procedure was attempting to accomplish, namely the conversion of (19) to an equation (or equations, each) in a single variable (here $mu + nv$).

The procedure for finding the fundamental singularities is straightforward. We write

$$mu + nv = \left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)^{-1}(mf + ng)$$
$$= \left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^{-1}(mf + ng). \tag{21}$$

The fundamental singularity associated with the operator $(\partial^2/\partial x^2 + \partial^2/\partial y^2)^{-1}$ is $(1/2\pi)\log/[(x - x_0)^2 + (y - y_0)^2]^{1/2}$. If f is a δ function at the origin we find from the m and n parts of (21) that $u = x/2\pi r^2$, $v = y/2\pi r^2$, which is a source at the origin. Similarly if g is a δ function $u = -y/2\pi r^2$, $v = x/2\pi r^2$, which is a vortex at the origin.

Let us now reexamine our treatment of the viscous flow problem. In (7) we found that the eigenfactors were $X + iY$, $X - iY$, $X + iY - (1 - k)/2$, and $X - iY - (1 + k)/2$. These led us to the diagonal matrix

$$\begin{pmatrix} X + iY & 0 & 0 & 0 \\ 0 & X - iY & 0 & 0 \\ 0 & 0 & X + iY - \dfrac{1-k}{2} & 0 \\ 0 & 0 & 0 & X - iY - \dfrac{1+k}{2} \end{pmatrix}.$$

If we convert this mixed representation (algebraic and matrix) to an all matrix representation we find the result is an 8×8 matrix.

If, instead, we select as our diagonal matrix

$$\begin{pmatrix} X + iY & 0 \\ 0 & X + iY - (1 - k)/2 \end{pmatrix}, \tag{22}$$

and convert this to an all matrix representation we have

$$\begin{pmatrix} X & Y & 0 & 0 \\ -Y & X & 0 & 0 \\ 0 & 0 & X & Y \\ 0 & 0 & -Y & X - 1 \end{pmatrix}, \tag{23}$$

which is of the fourth order as desired. We readily find a transformation which converts the characteristic matrix $\mathbb{L}$ to the "diagonal" form (22) for a revised version of Λ:

$$\begin{pmatrix}0&0&1&0\\0&1&0&0\\1&0&1&0\\0&-1&0&-1\end{pmatrix}^{-1}\begin{pmatrix}X&0&X&Y\\0&X&Y&-X\\X&Y&0&0\\Y&-X&0&1\end{pmatrix}\begin{pmatrix}1&0&1&0\\0&1&0&1\\-1&0&0&0\\0&0&0&1\end{pmatrix}=\begin{pmatrix}X&Y&0&0\\-Y&X&0&0\\0&0&X&Y\\0&0&-Y&X-1\end{pmatrix}.$$

We are now in a position to answer a question raised in Sec. III about the commutability of the factors $X+iY$ and $X+iY-(1-k)/2$. The physics of the problem dictates thay they should commute, but at first glance they appear not to commute. In the light of (22) and (23) we see that an all-matrix representation for $X+iY$ is

$$\begin{pmatrix}X&Y&0&0\\-Y&X&0&0\\0&0&1&0\\0&0&0&1\end{pmatrix}.$$

Correspondingly an all matrix representation for $X+iY-(1-k)/2$ is

$$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&X&Y\\0&0&-Y&X-1\end{pmatrix}.$$

These factors do in fact commute. An explanation for the original apparent lack of commutativity of the factors is that the $1,i$ used in one are not from the same algebra as the $1,i,j,k$ used in the other.

Let us now return to the fundamental singularities associated with the new "diagonal" form of the matrix given in (22). If the output and input vectors for this new form have components $a'b'c'd'$ and $k'\lambda'\mu'\nu'$, then the "decoupled" equations become

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right)(ma'+nb')=mk'+n\lambda',$$

$$\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}-\frac{1-k}{2}\right)(mc'+nd')=m\mu'+n\nu'.$$

From here on the procedure follows very similar lines to the example we have given earlier. Now however, we are solving two equations instead of four, and we must use the imaginary parts as well as the real ones to obtain the complete set of solutions.

The examples we have considered so far have been elliptic equations. Now let us look at a simple parabolic equation, the heat transfer equation,

$$k\frac{\partial^2\theta}{\partial x'^2}-c\frac{\partial\theta}{\partial t'}=0.$$

The conductivity k and the heat capacity c can be absorbed into the variables x' and t' so that this equation becomes $\partial^2\theta/\partial x^2-\partial\theta/\partial t=0$.

To factor the characteristic expression X^2-T we must resort to hypercomplex algebra. The factors are $\{X+[(i+j)/2]T+(i-j)/2\}\{X-[(i+j)/2]T-(i-j)/2\}$. The experience we have had with the preceding examples indicates that instead of using the diagonal matrix

$$\begin{pmatrix}X+\frac{i+j}{2}T+\frac{i-j}{2}&0\\0&X-\frac{i+j}{2}T-\frac{i-j}{2}\end{pmatrix},$$

we should use simply $X+(i+j)/2T+(i-j)/2$. Thus the "diagonalized" equation with external inputs should be

$$\left(\frac{\partial}{\partial x}+\frac{i+j}{2}\frac{\partial}{\partial t}+\frac{i-j}{2}\right)(m\phi+n\theta)=mf+ng. \qquad (24)$$

If we convert this to matrix notation and interpret it as a pair of equations we have

$$\frac{\partial\phi}{\partial x}+\frac{\partial\theta}{\partial t}=f,\quad \phi+\frac{\partial\theta}{\partial x}=g. \qquad (25)$$

This set is readily seen to be the classic pair of heat transfer equations[4] with θ as the temperature and ϕ as the heat flux.

To obtain the fundamental singularities for these equations, we write (24) in the form

$$\begin{aligned}m\phi+n\theta&=\left[\frac{\partial}{\partial x}+\left(\frac{i+j}{2}\right)\frac{\partial}{\partial t}+\frac{i-j}{2}\right]^{-1}(mf+ng)\\&=\left[\frac{\partial}{\partial x}-\left(\frac{i+j}{2}\right)\frac{\partial}{\partial t}-\frac{i-j}{2}\right]\\&\quad\times\left(\frac{\partial^2}{\partial x^2}-\frac{\partial}{\partial t}\right)^{-1}(mf+ng).\end{aligned} \qquad (26)$$

The fundamental singularity associated with the operator $(\partial^2/\partial x^2-\partial/\partial t)^{-1}$ is

$$[4\pi(t-t_0)]^{-1/2}\exp[-(x-x_0)^2/4(t-t_0)]. \qquad (27)$$

If f is a δ function at the origin, we can insert (27) in (26) to yield

$$\phi=-(x/4\pi^{1/2}t^{3/2})\exp(-x^2/4t),$$
$$\theta=(4\pi t)^{-1/2}\exp(-x^2/4t).$$

Similarly if g is a δ function at the origin, we find that

$$\phi=(4\pi^{1/2}t^{3/2})^{-1}(1-x^2/2t)\exp(-x^2/2t),$$
$$\theta=-(x/4\pi^{1/2}t^{3/2})\exp(-x^2/2t).$$

These, then, are the fundamental singularities for Eqs. (25).

Let us now examine the hyperbolic wave equation. The equation for a taut string is

$$\rho\frac{\partial^2\xi}{\partial t'^2}-\tau\frac{\partial^2\xi}{\partial x'^2}=0.$$

If ρ and τ, the mass per unit length and the tension, are absorbed into the coordinates x' and t', this equation becomes

$$\frac{\partial^2\xi}{\partial t^2}-\frac{\partial^2\xi}{\partial x^2}=0. \qquad (28)$$

Ordinarily the characteristic expression T^2-X^2 is factored with the real factors $(T-X)(T+X)$. However if we introduce

$$v = \frac{\partial \xi}{\partial t}, \quad \theta = \frac{\partial \xi}{\partial x},$$

then (28) can be written as a pair of equations:

$$\frac{\partial v}{\partial t} - \frac{\partial \theta}{\partial x} = f, \quad -\frac{\partial v}{\partial x} + \frac{\partial \theta}{\partial t} = g. \tag{29}$$

Here we have inserted input functions f and g.

Now we see that when the characteristic matrix for (29),

$$\begin{pmatrix} T & -X \\ -X & T \end{pmatrix},$$

is expressed in complex form it becomes $T - jX$, where j is an element of the i, j, k algebra. Thus we see that an alternate to the usual factorization of $T^2 - X^2$ as $(T-X)(T+X)$ is $(T-jX)(T+jX)$. If we use this latter factorization, the procedure for solving the hyperbolic equations (29) follows a directly parallel course to that used for the elliptic and parabolic equation. We write (29) as

$$\left(\frac{\partial}{\partial t} - j\frac{\partial}{\partial x}\right)(mv + n\theta) = mf + ng,$$

or

$$mv + n\theta = \left(\frac{\partial}{\partial t} - j\frac{\partial}{\partial x}\right)^{-1}(mf + ng)$$

$$= \left(\frac{\partial}{\partial t} + j\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)^{-1}(mf + ng). \tag{30}$$

The fundamental singularity at the origin for the operator $(\partial^2/\partial t^2 - \partial^2/\partial x^2)^{-1}$ is a function which has unit value everywhere in the wedge extending in the positive time direction between the lines $x + t = 0$ and $x - t = 0$. Elsewhere it is zero.

If we insert this result in (30) for the case where f is a δ function at the origin we find that v is a distributed δ function along the line $x + t = 0$ extending from the origin outward in the positive time direction. We also find that $\theta = 0$.

Correspondingly we find that when g is a δ function at the origin, $v = 0$ and θ is a distributed δ function along the line $x - t = 0$ from the origin outward in the positive time direction.

Both of these are typical wave solutions and are similar in character to the solution we found in Sec. VI for the case of the viscous source with vanishing viscosity. Except that here the δ functions, being inclined to the axes, are called waves, whereas when such a function is aligned with the flow axis, it is called a wake.

In the examples we have considered thus far, the most significant difference that has emerged between the eigenfunction theory for ordinary and partial differential equations has been the need to use hypercomplex algebra in treating the latter. We now consider examples where other differences are present.

Consider the linearized equations for the three-dimensional steady inviscid flow of an incompressible fluid. The free stream coming from infinity is assumed to be uniform; when nondimensional variables are used, the equations become

$$\begin{aligned} &\frac{\partial u}{\partial x} + \frac{\partial p}{\partial x} = 0, \\ &\frac{\partial v}{\partial x} + \frac{\partial p}{\partial y} = 0, \\ &\frac{\partial w}{\partial x} + \frac{\partial p}{\partial z} = 0, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \end{aligned} \tag{31}$$

The first of these equations may be integrated to show that $p + u$ can be an arbitrary function of y and z. However, since the flow is assumed to be uniform upstream, this indicates that $p = -u$. Using this result, Eqs. (31) become

$$\begin{aligned} &-\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = f, \\ &-\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = g, \\ &\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x} = q. \end{aligned} \tag{32}$$

Here we have inserted input functions f, g, and q. The characteristic expression for the operator of this set is

$$X(X^2 + Y^2 + Z^2).$$

Factorization of $(X^2 + Y^2 + Z^2)$ cannot be accomplished with the i, j, k algebra we have used previously. The simplest suitable algebra has three complex elements, which we shall call α, β, γ. These have the following multiplication table:

	α	β	γ
α	-1	γ	$-\beta$
β	$-\gamma$	-1	α
γ	β	$-\alpha$	-1

Using these, we have $X^2 + Y^2 + Z^2 = (X + \alpha Y + \beta Z)(X - \alpha Y - \beta Z)$.

Our earlier experience would lead us to believe that the characteristic matrix for (32), i.e.,

$$\begin{pmatrix} -Y & X & 0 \\ -Z & 0 & X \\ X & Y & Z \end{pmatrix}, \tag{33}$$

should be transformable to a "diagonal" matrix

$$\begin{pmatrix} X + \alpha Y + \beta Z & 0 \\ 0 & X \end{pmatrix}. \tag{34}$$

Now it is possible to find matrix representations for the α, β, γ algebra. The simplest mixed representation is

$$\alpha = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \gamma = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

These matrices are related to the Pauli spin matrices as follows: $\alpha = i\sigma_x$, $\beta = i\sigma_z$, $\gamma = i\sigma_y$. The simplest pure matrix representation is

$$\alpha = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$

$$\beta = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$\gamma = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}.$$

If we insert either of these in (34), we find that no transformation exists which will transform (33) to (34).

In spite of this situation we can still find fundamental singularities for Eqs. (32). We write (32) as follows

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} -\dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} & 0 \\ -\dfrac{\partial}{\partial z} & 0 & \dfrac{\partial}{\partial x} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \end{pmatrix}^{-1} \begin{pmatrix} f \\ g \\ q \end{pmatrix}$$

$$= \begin{pmatrix} -\dfrac{\partial^2}{\partial x\,\partial y} & -\dfrac{\partial^2}{\partial x\,\partial z} & \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial z^2} & -\dfrac{\partial^2}{\partial y\,\partial z} & \dfrac{\partial^2}{\partial x\,\partial y} \\ -\dfrac{\partial^2}{\partial y\,\partial z} & \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} & \dfrac{\partial^2}{\partial x\,\partial z} \end{pmatrix}$$

$$\times \left(\frac{\partial}{\partial x}\right)^{-1} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{-1} \begin{pmatrix} f \\ g \\ q \end{pmatrix}. \qquad (35)$$

In the case of ordinary differential equations, the operator on the right can always be separated into partial fractions, leading to an operator of the form

$$A\left(\frac{\partial}{\partial x}\right)^{-1} + B\left(\frac{\partial}{\partial x} + \alpha\frac{\partial}{\partial y} + \beta\frac{\partial}{\partial y}\right)^{-1} + C\left(\frac{\partial}{\partial x} - \alpha\frac{\partial}{\partial y} - \beta\frac{\partial}{\partial y}\right)^{-1},$$

where A, B, and C are numerical matrices. In this way the modes corresponding to the operator factors would be decoupled and expressed separately.

However, in (35) no such partial fraction decomposition is possible. It is not even possible to separate the two real factors $\partial/\partial x$ and $\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ in the form

$$A\left(\frac{\partial}{\partial x}\right)^{-1} + \left(B\frac{\partial}{\partial x} + C\frac{\partial}{\partial y} + E\frac{\partial}{\partial z} + F\right) \times \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{-1}.$$

However, we can find three independent singular solutions. We can rewrite (35)

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \left(\frac{\partial}{\partial x}\right)^{-1} \begin{pmatrix} -f \\ -g \\ +q \end{pmatrix}$$

$$+ \begin{pmatrix} \dfrac{\partial^2}{\partial x\,\partial y} & \dfrac{\partial^2}{\partial x\,\partial z} & \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} & \dfrac{\partial^2}{\partial y\,\partial z} & \dfrac{\partial^2}{\partial x\,\partial y} \\ \dfrac{\partial^2}{\partial y\,\partial z} & \dfrac{\partial^2}{\partial z^2} & \dfrac{\partial^2}{\partial x\,\partial z} \end{pmatrix} \left(\frac{\partial}{\partial x}\right)^{-1}$$

$$\times \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)^{-1} \begin{pmatrix} -f \\ -g \\ +q \end{pmatrix}. \qquad (36)$$

First let us suppose that f is a $-\delta$ function at the origin. The expression $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial x^2)^{-1} f$ is then the potential function for a unit source at the origin. Further $(\partial/\partial x)^{-1}(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)f$ is the potential for a line source stretching from the origin to downstream infinity along the x axis. And $(\partial/\partial y)(\partial/\partial x)^{-1}(\partial^2/\partial x^2 + \partial/\partial y^2 + \partial^2/\partial z^2)^{-1} f$ is the potential for doublets, distributed along the x axis from the origin to downstream infinity. The axis of the doublets points in the y direction. If we call this potential ϕ_f then (36) yields the result,

$$u = \frac{\partial\phi_f}{\partial x}, \quad v = -\left(\frac{\partial}{\partial x}\right)^{-1} f + \frac{\partial\phi_f}{\partial y}, \quad w = \frac{\partial\phi_f}{\partial z}.$$

This is the velocity field of distributed doublets with a v correction concentrated along the downstream x axis. When this correction is made, it converts the distributed doublet system to a vortex pair with the vortices displaced from one another by an infinitesimal amount in the z direction. This, of course, is the well-known Prandtl "horseshoe" vortex system. In this case, it is associated with a lifting element located at the origin and for which the lift is in the y direction.

If g is taken to be a $-\delta$ function, the flow field is rotated so that the trailing vortices are displaced relative to one another in the y direction and the lifting element has its lift in the z direction.

Finally if we take q to be a δ function, each of the components, u,v,w has a factor $\partial/\partial x$ in the numerator which cancels the $(\partial/\partial x)^{-1}$ of the denominator, leaving the gradient of the potential for a simple source at the origin as the third fundamental singularity.

It should be noted that the trailing vortices of the "horseshoe" singularities give a physical insight into the mathematical nonseparability of the operator $(\partial/\partial x)^{-1}(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)^{-1}$ into partial fractions. The "trailing" part arises from the factor $(\partial/\partial x)^{-1}$, whereas the vortex is associated with the factor $(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2)^{-1}$. If separation were possible there would exist linear combinations such that one singularity would be purely trailing and one would be purely potential. However, since lifting vortices cannot end in the fluid, then (as Prandtl reasoned) lifting elements must have trailing vortices, which here means nonseparability.

Let us now consider a second example which illustrates other differences from the theory for ordinary differential equations. We again consider the taut string, but this time we assume that it is elastically restrained in space. In this case we have the three equations

$$\frac{\partial v}{\partial t} - \frac{\partial \theta}{\partial x} + \xi = f, \quad \frac{\partial \xi}{\partial t} - v = g, \quad \frac{\partial \xi}{\partial x} - \theta = h. \tag{37}$$

The characteristic expression for the operator of these equations is

$$\begin{vmatrix} 1 & T & -X \\ T & -1 & 0 \\ X & 0 & -1 \end{vmatrix} = T^2 - X^2 + 1,$$

which is seen to be the two-variable equivalent of the characteristic expression for the Dirac equation (3). This expression can be factorized using the i, j, k algebra:

$$T^2 - X^2 + 1 = (T + jX - i)(T - jX + i).$$

We are thus led to believe that the equation for an elastically restrained string should be expressible in the form

$$\left(\frac{\partial}{\partial t} + j\frac{\partial}{\partial x} - i\right)(m\phi + n\psi) = mF + nG. \tag{38}$$

Converting this to matrix form and interpreting it as a set of equations it becomes

$$\frac{\partial \phi}{\partial t} + \frac{\partial \psi}{\partial x} - \psi = F, \quad \frac{\partial \phi}{\partial x} + \phi + \frac{\partial \psi}{\partial t} = G. \tag{39}$$

If we compare (39) with (37) we find significant differences between the two sets of equations which supposedly represent the same system. One consists of three equations involving three variables with three inputs, the other with two each of these quantities. Within the framework of the Lorentz transformation, the v and θ of (37) constitutes a vector with ξ being a scalar. In contrast, the ϕ and ψ of (39) constitute a spinor.

A transformation exists which linearly relates the scalar and vector to the spinor:

$$\begin{aligned} \phi &= (\xi - \theta)\cosh \alpha/2 + v \sinh \alpha/2, \\ \psi &= -(\xi + \theta)\sinh \alpha/2 + v \cosh \alpha/2. \end{aligned} \tag{40}$$

Here α is a hyperbolic phase angle which determines the orientation of the spinor to the vector-scalar pair. If now the t and x axes are "rotated' by a hyperbolic angle β, α will correspondingly increase by β.

It is also possible to find a relationship between the F and G of (39) and the f, g, and h of (37). When $\alpha = 0$,

$$F = g - \frac{\partial g}{\partial x} + \frac{\partial h}{\partial t}, \quad G = f + h.$$

Using the format of (38) it is a straightforward procedure to find the fundamental singularities of (39). We write

$$\begin{aligned} m\phi + n\psi &= \left(\frac{\partial}{\partial t} + j\frac{\partial}{\partial x} - i\right)^{-1}(mF + nG) \\ &= \left(\frac{\partial}{\partial t} - j\frac{\partial}{\partial x} + i\right)\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 1\right)^{-1} \\ &\quad \times (mF + nG). \end{aligned}$$

The fundamental singularity associated with the operator $(\partial^2/\partial t^2 - \partial^2/\partial x^2 + 1)^{-1}$, when placed at the origin, is[5] $J_0(\rho)$ within the wedge along the positive t axis bounded by $t - x = 0$ and $t + x = 0$, and zero elsewhere. Here J_0 is the Bessel function of zero order and $\rho = (t^2 - x^2)^{1/2}$. When F is a δ function at the origin,

$$\phi = (t/\rho)J_0'(\rho), \quad \psi = (x/\rho)J_0'(\rho) - J_0(\rho).$$

When G is a δ function at the origin

$$\phi = (x/\rho)J_0'(\rho) + J_0(\rho), \quad \psi = (t/\rho)J_0'.$$

Now let us return to (36) and seek its fundamental singularities. We write it in the form

$$\begin{pmatrix} \xi \\ v \\ \theta \end{pmatrix} = \begin{pmatrix} 1 & \frac{\partial}{\partial t} & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} & -1 & 0 \\ \frac{\partial}{\partial x} & 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} f \\ g \\ h \end{pmatrix}$$

$$= \left[\begin{pmatrix} 1 & \frac{\partial}{\partial t} & -\frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} & \frac{\partial^2}{\partial t^2} & -\frac{\partial^2}{\partial x \partial t} \\ \frac{\partial}{\partial x} & \frac{\partial^2}{\partial x \partial t} & -\frac{\partial^2}{\partial x^2} \end{pmatrix}\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} + 1\right)^{-1} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\right]\begin{pmatrix} f \\ g \\ h \end{pmatrix}.$$

Recalling the fundamental singularity associated with $(\partial^2/\partial t^2 - \partial^2/\partial x^2 + 1)^{-1}$, we find that when f is a δ function at the origin,

$$\xi = J_0(\rho), \quad v = (t/\rho)J_0'(\rho), \quad \theta = -xJ_0'(\rho)/\rho.$$

When g is a δ function at the origin,

$$\begin{aligned} \xi &= (t/\rho)J_0'(\rho), \\ v &= -\frac{(t^2 + x^2)}{\rho}J_0'(\rho) - \frac{t^2 + 1}{\rho}J_0(\rho), \\ \theta &= (xt/\rho^3)[2\rho J_0'(\rho) + J_0(\rho)]. \end{aligned}$$

Similarly when h is a δ function at the origin

$$\begin{aligned} \xi &= (x/\rho)J_0'(\rho), \\ v &= -(xt/\rho^3)[2\rho J_0'(\rho) + J_0(\rho)], \\ \theta &= \frac{t^2 + x^2}{\rho^3}J_0'(\rho) - \frac{(t^2 + 1)}{\rho^2}J_0(\rho). \end{aligned}$$

At first glance we appear to have three fundamental singularities for the Eqs. (37) [in contrast to the two for Eq. (39)]. However, we find that the fundamental singularities associated with f, g, and h are not independent since the last two are the t and x derivatives of the f singularity. This latter corresponds to the G singularity we found for (39).

VIII. CONCLUSIONS

The theory of eigenmodes promises to provide a deep insight into the fundamental patterns of behavior of partial differential equations. As the examples we have explored show, this theory for partial differential equations is not just a simple extension of the corresponding theory for ordinary differential equations. Rather, it is considerably more complex and the ramifications are much broader.

APPENDIX: OBTAINING THE TRANSFORMATION MATRICES FOR DIAGONALIZING THE OPERATOR MATRIX

In identity (10) we have

$$\begin{pmatrix} X & 0 & X & Y \\ 0 & X & Y & -X \\ X & Y & 0 & 0 \\ Y & -X & 0 & 1 \end{pmatrix}\begin{pmatrix} N_{11} & N_{12} & N_{13} & N_{14} \\ N_{21} & N_{22} & N_{23} & N_{24} \\ N_{31} & N_{32} & N_{33} & N_{34} \\ N_{41} & N_{42} & N_{43} & N_{44} \end{pmatrix}$$

$$\equiv \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{pmatrix}\begin{pmatrix} X+iY & 0 & 0 & 0 \\ 0 & X-iY & 0 & 0 \\ 0 & 0 & X+iY-(1-k)/2 & 0 \\ 0 & 0 & 0 & X-iY-(1+k)/2 \end{pmatrix}$$

and the task is to determine the M's and N's that make this transformation possible. As mentioned earlier, the relations between the M's and N's decouple, each group containing N's from only one column and M's from the corresponding column.

The first two columns can be treated with ordinary complex algebra and present no problem. An arbitrary multiplier is involved and in each case this has been chosen so that the fundamental singularity for the a mode is a pure source and that for the b mode a pure vortex.

When we consider columns three and four, we encounter significant new problems because here we are dealing with a hypercomplex algebra. The principal difficulty is the following. Consider the equation $XA = 0$, where A is a known hypercomplex quantity and we are seeking a solution for X. The correct answer is not $X = 0$ but rather arbitrary multiples of those quantities which give a zero product with A. These are called the divisors of zero of A. As an example suppose that A is $1 - k$. Now $1 + k$ is a divisor of zero of $1 - k$. Also $i(1 + k) = i - j$, and any other left-hand multiples of $1 + k$ are left-hand divisors of zero of $1 - k$. It is easily shown that the most general solution to $X(1 - k) = 0$ is $X = \alpha(1 + k) + \beta(i - j)$, where α and β are arbitrary real numbers.

Now let us turn to the problem of finding N's and M's from the third column of their respective matrices. Using the above results we find the general solution to be

$$N_{13} = \alpha(1+k) + \beta(i-j), \quad M_{13} = \alpha(1+k) + \beta(i-j),$$
$$N_{23} = \alpha(i+j) - \beta(1-k), \quad M_{23} = 0,$$
$$N_{33} = 0, \quad M_{33} = \alpha(1+k) + \beta(i-j),$$
$$N_{43} = +\alpha(i+j) - \beta(1-k),$$
$$M_{43} = -\alpha(i+j) + \beta(1-k).$$

In the matrices of (11) we have elected to take $\alpha = 1$ and $\beta = 0$ so that the fundamental singularity for the c mode is a pure source and not a mixture of viscous source and vortex. The calculations for the fourth column follow a similar procedure to that for the third column.

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