A POSET CONNECTED TO ARTIN MONOIDS OF SIMPLY LACED TYPE

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Abstract. Let W be a Weyl group whose type is a simply laced Dynkin diagram. On several W-orbits of sets of mutually commuting reflections, a poset is described which plays a role in linear representations of the corresponding Artin group A. The poset generalizes many properties of the usual order on positive roots of W given by height. In this paper, a linear representation of the positive monoid of A is defined by use of the poset.

1. Introduction

The beautiful properties of the high root used in [5] to construct Lawrence-Krammer representations of the Artin group with non-commutative coefficients have analogues for certain sets of orthogonal roots. We study these properties and exploit them to construct a linear representation of the Artin monoid. In many instances, the monoid representation extends to an Artin group representation; this will be the subject of subsequent work.

Let M be a Coxeter diagram of simply laced type, i.e., its connected components are of type A, D or E. The Lawrence-Krammer representation ([1, 4, 7, 9]) has a basis consisting of positive roots of the root system of the Weyl group $W = W(M)$ of type M. Here we use instead, a W-orbit $\mathcal{B}$ of sets of mutually orthogonal positive roots. Not all W-orbits of this kind are allowed; we call those which are allowed, *admissible* (cf. Definition 1; a precise list for M connected is in Table 2). In the Lawrence-Krammer representation we used the partial ordering of the positive roots given by $\beta \leq \gamma$ iff $\gamma - \beta$ is a sum of positive roots with non-negative coefficients. In Proposition 3.1 we generalize this ordering to an ordering $(\mathcal{B}, <)$ on admissible W-orbits $\mathcal{B}$ of mutually orthogonal roots. In the action of $w \in W$ on a set $B \in \mathcal{B}$ the image wB is the set of positive roots in $\{\pm w\beta \mid \beta \in B\}$. In the single root case, there is a unique highest element, the well-known highest root. This property extends to $(\mathcal{B}, <)$: there is a unique maximal element B_0 in $\mathcal{B}$ (cf. Corollary 3.6).

In the Lawrence-Krammer representation, the coefficients were obtained from the Hecke algebra whose type is the subdiagram of M induced on the set of nodes i of M whose corresponding fundamental root α_i is orthogonal to the highest root. Here, the coefficients are obtained from the Hecke algebra Z whose type is the subdiagram of M induced on the nodes i whose corresponding fundamental root α_i is orthogonal to each element of B_0. Moreover, in the Lawrence-Krammer representation, to each pair of a positive root β and a node i with corresponding fundamental root α_i such that $(\alpha_i, \beta) = 0$, we assigned an element $h_{\beta,i}$ of the coefficient algebra. It occurs in the definition of the action of a fundamental generator of the Artin group A in the Lawrence-Krammer representation, on the basis element β. For the analogous purpose, we introduce elements $h_{B,i}$ (Definition 2) in the corresponding coefficient algebra Z. These elements are parameterized by pairs consisting of an element B of $\mathcal{B}$ and a node i of M such that the corresponding fundamental root α_i is orthogonal to all of B.

In analogy to the developments in [5] we define a right free Z-module with basis x_B ($B \in \mathcal{B}$) which is a left module for the positive monoid A^+ of the Artin group A of type M. For each node i of M, the i-th fundamental generator s_i of A^+ maps onto the linear transformation τ_i on V given by the following case division.

$$\tau_i x_B \quad = \quad \begin{cases} 0 & \text{if } \alpha_i \in B, \\ x_B h_{B,i} & \text{if } \alpha_i \in B^\perp, \\ x_{r_iB} & \text{if } r_iB < B, \\ x_{r_iB} - m x_B & \text{if } r_iB > B. \end{cases} \tag{1}$$

This leads to the following main result of this paper.

Theorem 1.1. *Let W be a Weyl group of simply laced type. For $\mathcal{B}$ an admissible W-orbit of sets of mutually orthogonal positive roots, there is a partial order $<$ on $\mathcal{B}$ such that the above defined map $s_i \mapsto \tau_i$ determines a homomorphism of monoids from A^+ to $\mathrm{End}(V)$.*

In the sections below, we deal with this construction in detail, the proof of the theorem is in Section 5.

When labeling the nodes of an irreducible diagram M, we will choose the labeling of [3]. If M is disconnected, the representations are easily seen to be a direct sum of representations corresponding to the components. Since the poset construction also behaves nicely, it suffices to prove the theorem only for M connected. Therefore, we will assume M to be connected for the greater part of this paper.

2. Admissible orbits

Let M be a spherical Coxeter diagram. Let (W, R) be the Coxeter system of type M with $R = \{r_1, \ldots, r_n\}$. Throughout this paper we shall assume that M is simply laced, which means that the order of each product r_ir_j is at most 3.

By Φ^+ we denote the positive root system of type M and by α_i the fundamental root corresponding to the node i of M. We are interested in sets B of mutually commuting reflections. Since each reflection is uniquely determined by a positive root, the set B corresponds bijectively to a set of mutually orthogonal roots of Φ^+. We will almost always identify B with this subset of Φ^+. The action of $w \in W$ on B is given by conjugation in case B is described by reflections and by $w\{\beta_1, \ldots, \beta_p\} = \Phi^+ \cap \{\pm w\beta_1, \ldots, \pm w\beta_p\}$ in case B is described by positive roots. The action of an element $w \in W$ on B should not be confused with the action of w on a root: in our case we have $w\{\alpha_i\} = \{\alpha_i\}$ whereas the usual action on roots implies $w\alpha_i = \alpha_i$. For example, if r_i is the reflection about α_i, $r_i\{\alpha_i\} = \{\alpha_i\}$ but $r_i\alpha_i = -\alpha_i$.

The W-orbit $\mathcal{B}$ of a set B of mutually orthogonal positive roots is the vertex set of a graph with edges labeled by the nodes of M, the edges with label j being the unordered pairs $\{B, r_jB\}$ (so $r_jB \neq B$) for $B \in \mathcal{B}$. The results of Section 3 show that if $\mathcal{B}$ is admissible, the edges of this graph can be directed so as to obtain a partially ordered set (poset) having a unique maximal element. This section deals with the notion of admissibility.

We let $\mathrm{ht}(\beta)$ be the usual height of a root $\beta \in \Phi^+$ which is $\sum_i a_i$ where $\beta = \sum a_i\alpha_i$.

Proposition 2.1. *Let M be of simply laced type. Every W-orbit of sets of mutually orthogonal positive roots satisfies the following properties for $B \in \mathcal{B}$, $j \in M$ and $\beta, \gamma \in B$.*

(i) There is no node i for which $(\alpha_i, \beta) = 1$, $(\alpha_i, \gamma) = -1$ and $\mathrm{ht}(\beta) = \mathrm{ht}(\gamma) + 1$.
(ii) Suppose $(\alpha_j, \beta) = -1$ and $(\alpha_j, \gamma) = 1$ with $\mathrm{ht}(\gamma) = \mathrm{ht}(\beta) + 2$. Then there is no node i for which $\alpha_i \in B^\perp$ and $i \sim j$.

Proof. Let B be a set of mutually orthogonal positive roots, and $\beta, \gamma \in B$.
(i). Suppose there is a node i for which $(\alpha_i, \beta) = 1$, $(\alpha_i, \gamma) = -1$ and $\mathrm{ht}(\beta) = \mathrm{ht}(\gamma) + 1$. As β and γ are orthogonal we have $(\beta, \gamma + \alpha_i) = 1$ so $\beta - \gamma - \alpha_i \in \Phi$. This is not possible as $\mathrm{ht}(\beta - \gamma - \alpha_i) = 0$.
(ii). Let β and γ be as in the hypothesis and assume there is an i for which $\alpha_i \in B^\perp$ and $i \sim j$. Then $(\alpha_i, \gamma - \alpha_j) = 1$, so $\gamma - \alpha_j - \alpha_i$ is a root. As $\mathrm{ht}(\gamma) = \mathrm{ht}(\beta) + 2$ we have $\mathrm{ht}(\gamma - \alpha_j - \alpha_i) = \mathrm{ht}(\beta)$. But $(\beta, \gamma - \alpha_j - \alpha_i) = 1$, so $\beta - \gamma + \alpha_j + \alpha_i$ is a root which contradicts $\mathrm{ht}(\beta - \gamma + \alpha_j + \alpha_i) = 0$. □

Definition 1. *Let $\mathcal{B}$ be a W-orbit of sets of mutually orthogonal positive roots. We say that $\mathcal{B}$ is* admissible *if for each $B \in \mathcal{B}$ and $i, j \in M$ with $i \not\sim j$ and $\gamma, \gamma - \alpha_i + \alpha_j \in B$, we have $r_i B = r_j B$.*

Not all W-orbits on sets of mutually orthogonal positive roots are admissible. The W-orbit of the triple $\{\alpha_1 + \alpha_2 + \alpha_3, \alpha_2 + \alpha_3 + \alpha_4, \alpha_1 + \cdots + \alpha_5\}$ of positive roots for $M = \mathrm{D}_5$ is a counterexample. Suppose that M is disconnected with components M_i. Then $\mathcal{B}$ is admissible if and only if each of the corresponding $W(M_i)$-orbits is admissible. So there is no harm in restricting our admissibility study to the case where M is connected. In that case, Proposition 2.3 below gives a full characterization of admissible orbits.

Lemma 2.2. *Let r be a reflection in W and let β, γ be two mutually orthogonal positive roots moved by r. Then there exists a reflection s which commutes with r such that $\{\beta\} = rs\{\gamma\}$.*

Proof. Let δ_r be the positive root corresponding to the reflection r. Now $r\{\beta\} = \{\pm\beta \pm \delta_r\} \cap \Phi^+$ and $(\gamma, \beta \pm \delta_r) = \pm(\gamma, \delta_r)$. Using this we can construct a new positive root δ depending on (β, δ_r), (γ, δ_r) as indicated in the table below.

(β, δ_r)	(γ, δ_r)	$\pm\delta$
1	1	$\beta + \gamma - \delta_r$
1	-1	$\beta - \gamma - \delta_r$
-1	1	$\beta - \gamma + \delta_r$
-1	-1	$\beta + \gamma + \delta_r$

It is easy to check that the reflection s with root δ commutes with r and indeed $\{\beta\} = rs\{\gamma\}$. □

Proposition 2.3. *Let M be connected. The following statements concerning a W-orbit $\mathcal{B}$ of sets of mutually orthogonal positive roots are equivalent.*

(i) $\mathcal{B}$ is admissible.
(ii) For each pair r, s of commuting reflections of W and each $B \in \mathcal{B}$ such that γ and $rs\gamma$ both belong to B, we have $rB = sB$.
(iii) For each reflection r of W and each $B \in \mathcal{B}$ the size of $rB \setminus B$ is one of $0, 1, 2, 4$.

(iv) $\mathcal{B}$ is one of the orbits listed in Table 2.

Below in the proof we show that four is the maximum possible roots in $rB \setminus B$ which can be moved and so only three is ruled out in part (iii).

Proof. (i) $\Longrightarrow$ (ii). By (i), assertion (ii) holds when r and s are fundamental reflections. The other cases follow by conjugation since each pair of commuting reflections is conjugate to a pair of fundamental conjugating reflections. (As each reflection is conjugate to a fundamental reflection, the reflections orthogonal to it can be determined and the system of roots orthogonal to a reflection has the type obtained by removing nodes connected to the extending node of the affine diagram.)

(ii) $\Longrightarrow$ (iii). When all $r, s \in W$ move at most two mutually orthogonal roots, the implication holds trivially. If r would move five mutually orthogonal roots then the 6×6 Gram matrix for these roots together with the root of r is not positive semi-definite as its determinant is -16, a contradiction. Hence r moves at most 4 roots.

Assume we have a $B \in \mathcal{B}$ such that r moves precisely three roots of B, say $\beta_1, \beta_2, \beta_3$. By Lemma 2.2 we know there exists a reflection s such that $\beta_1 = rs\beta_2$. Now $\beta_2 = \beta_1 \pm \delta_r \pm \delta_s$ with δ_r, δ_s the positive roots corresponding to r and s, respectively. As β_3 is orthogonal to β_1 and β_2, we find $(\beta_3, \delta_s) = \pm(\beta_3, \delta_r)$, so s moves β_3 as well. But obviously $r\beta_3 \neq s\beta_3$, so $rB \neq sB$, which contradicts (ii).

(iii) $\Longrightarrow$ (i). Let $B \in \mathcal{B}$ and $i, j \in M$ with $i \not\sim j$ and $\gamma, \gamma - \alpha_i + \alpha_j \in B$. When both r_i, r_j do not move any other root then $r_iB = r_jB$. Without loss of generality we can assume r_i moves four roots of B. Let β be a third root in B moved by r_i. As β has to be orthogonal to $\gamma, \gamma - \alpha_i + \alpha_j$ we find $(\alpha_i, \beta) = (\alpha_j, \beta)$, so $\beta - \alpha_i - \alpha_j$ or $\beta + \alpha_i + \alpha_j$ is a positive root as well. This root is also moved by r_i and mutually orthogonal to $\gamma, \gamma - \alpha_i + \alpha_j$ and β.

So now $\{\gamma, \gamma - \alpha_i + \alpha_j, \beta, \beta - \alpha_i - \alpha_j\} \subseteq B$. But these 4 roots are also the roots moved by r_j. We know from above that 4 is the maximal number of mutually orthogonal roots moved by r_i (or by r_j for that matter). We find $r_iB = r_jB$ which proves $\mathcal{B}$ is admissible.

At this point we have achieved equivalence of (i), (ii), and (iii), a fact we will use throughout the remainder of the proof.

(iii) $\Longrightarrow$ (iv). In Table 1, we have listed all W-orbits of sets of mutually orthogonal positive roots. It is straightforward to check this (for instance by induction on the size t of such a set), so we omit the details. For all orbits in Table 1 but not in Table 2 we find, for some set B in the orbit $\mathcal{B}$, a reflection r which moves precisely three roots.

We will use the observation that if B belongs to a non-admissible orbit for W of type M, then it also does not belong to an admissible orbit for W of any larger type.

For $M = \mathrm{D}_n$ the sets not in Table 2 contain at least one pair of roots $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ but also at least one root $\varepsilon_p \pm \varepsilon_q$ without the corresponding other positive root containing $\varepsilon_p, \varepsilon_q$. (Here the ε_i are the usual orthogonal basis such that $\Phi^+ = \{\varepsilon_i \pm \varepsilon_j \mid i < j\}$.) For these sets the reflection corresponding to a positive root $\varepsilon_j \pm \varepsilon_q$ moves precisely three roots.

M	$\|B\|$	B
A_n	t	$\{\alpha_1, \alpha_3, \ldots, \alpha_{2t-1}\}$
D_n	t	$\{\alpha_i, \beta_i \mid i = 1, 3, \ldots, .., 2k-1\} \quad \cup \quad \{\alpha_i \mid i = 2k+1, 2k+3, \ldots, .., 2t-1\}$
D_n	$n/2$	$\{\alpha_1, \alpha_3, \ldots, \alpha_{n-3}, \alpha_n\}$
E_6	1	$\{\alpha_2\}$
E_6	2	$\{\alpha_2, \alpha_5\}$
E_6	3	$\{\alpha_2, \alpha_3, \alpha_5\}$
E_6	4	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$
E_7	1	$\{\alpha_2\}$
E_7	2	$\{\alpha_2, \alpha_5\}$
E_7	3	$\{\alpha_2, \alpha_5, \alpha_7\}$
E_7	3	$\{\alpha_2, \alpha_3, \alpha_5\}$
E_7	4	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7\}$
E_7	4	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$
E_7	5	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_0\}$
E_7	6	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_0\}$
E_7	7	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_0\}$
E_8	1	$\{\alpha_2\}$
E_8	2	$\{\alpha_2, \alpha_5\}$
E_8	3	$\{\alpha_2, \alpha_3, \alpha_5\}$
E_8	4	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7\}$
E_8	4	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5\}$
E_8	5	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_0\}$
E_8	6	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_0, \bar{\alpha}_0\}$
E_8	7	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_0, \bar{\alpha}_0\}$
E_8	8	$\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7, \alpha_0, \bar{\alpha}_0\}$

TABLE 1. For $M = \mathrm{A}_n$ we have $t \leq \frac{n+1}{2}$ and for $M = \mathrm{D}_n$ we have $t \leq \frac{n}{2}$. For $M = \mathrm{D}_n$ we write $\beta_{n-1} = \alpha_n$ and $\beta_{2t+1} = \alpha_n + \alpha_{n-1} + 2\alpha_{n-2} + \ldots + 2\alpha_{2t+2} + \alpha_{2t+1}$. In E_7 and E_8 we use $\alpha_0 = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$ and $\bar{\alpha}_0 = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, the respective highest roots.

Suppose $M = \mathrm{E}_n$. The orbit of sets of three mutually orthogonal roots which is not in Table 2 is the orbit of $\{\alpha_2, \alpha_3, \alpha_5\}$, which is not admissible in the subsystem of type D_4 corresponding to these three roots and α_4, as r_4 moves all three roots.

The orbit of four mutually orthogonal positive roots not in Table 2 contains the set $\{\alpha_2, \alpha_3, \alpha_5, \alpha_7\}$ and r_4 moves exactly three of these.

The orbit of five mutually orthogonal positive roots not in Table 2 contains the set $\{\alpha_2, \alpha_3, \alpha_5, \alpha_7, \alpha_0\}$ and r_4 moves again exactly three of these.

If $M = \mathrm{E}_7$, the orbit of sets of six mutually orthogonal positive roots containing $\{\alpha_2, \alpha_5, \alpha_7, \alpha_3, \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5, \alpha_0\}$ remains. Clearly the reflection r_1 moves only the last three roots. If $M = \mathrm{E}_8$, the orbit of sets of six mutually orthogonal positive roots is not admissible as it contains the orbit of E_7 we just discussed.

M	$\lvert B\rvert$	$\lvert\mathcal{B}\rvert$	Y	C	$N_W(B)$
A_n	t	$\frac{(n+1)!}{2^t t!(n-2t+1)!}$	A_{n-2t}	A_{n-2t}	$2^t\mathrm{S}_t\mathrm{S}_{n+1-2t}$
D_n	t	$\frac{n!}{t!(n-2t)!}$	$\mathrm{A}_1^t\mathrm{D}_{n-2t}$	$\mathrm{A}_1\mathrm{D}_{n-2t}$	$2^{2t}\mathrm{S}_t W(\mathrm{D}_{n-2t})$
D_n	$2t$	$\frac{n!}{2^t t!(n-2t)!}$	D_{n-2t}	A_{n-2t-1}	$2^{2t}W(\mathrm{B}_t)W(\mathrm{D}_{n-2t})$
E_6	1	36	A_5	A_5	$2\mathrm{S}_6$
E_6	2	270	A_3	A_2	$2^{2+1}\mathrm{S}_4$
E_6	4	135	$\emptyset$	$\emptyset$	$2^4\mathrm{S}_4$
E_7	1	63	D_6	D_6	$2W(\mathrm{D}_6)$
E_7	2	945	$\mathrm{A}_1\mathrm{D}_4$	$\mathrm{A}_1\mathrm{D}_4$	$2^{2+1+1}W(\mathrm{D}_4)$
E_7	3	315	D_4	A_2	$2^3\mathrm{S}_3W(\mathrm{D}_4)$
E_7	4	945	A_1^3	A_1	$2^{4+3}\mathrm{S}_4$
E_7	7	135	$\emptyset$	$\emptyset$	$2^7\mathrm{L}(3,2)$
E_8	1	120	E_7	E_7	$2W(\mathrm{E}_7)$
E_8	2	3780	D_6	A_5	$2^{2+1}W(\mathrm{D}_6)$
E_8	4	9450	D_4	A_2	$2^4\mathrm{S}_3W(\mathrm{D}_4)$
E_8	8	2025	$\emptyset$	$\emptyset$	$2^{8+3}\mathrm{L}(3,2)$

TABLE 2. Each row contains the type M, the size of $B \in \mathcal{B}$, the size of the W-orbit $\mathcal{B}$ containing B, the Coxeter type Y of the roots orthogonal to B, the type of the Hecke Algebra C defined in Corollary 4.4, and the structure of the normalizer in W of B, respectively. In the first line for D_n, we define D_{n-2t} as being empty if $n-2t \leq 1$. Only one of the roots $\varepsilon_i \pm \varepsilon_j$ occur for roots in the first line of D_n. For roots in the second line, both occur.

Finally the orbit of seven mutually orthogonal positive roots in E_8 contains $\{\alpha_2, \alpha_3, \alpha_5, \alpha_2+\alpha_3+2\alpha_4+\alpha_5, \alpha_7, \alpha_0, \bar{\alpha}_0\}$. Here the reflection r_8 moves only the last three roots.

(iv) $\Longrightarrow$ (iii). All orbits for type A_n are admissible as here every reflection moves at most two mutually orthogonal roots.

All sets in the first collection of orbits in D_n contain from every pair of roots $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ at most one element. So again as for A_n, every reflection moves at most two mutually orthogonal roots.

All sets in the second collection of orbits in D_n contain from every pair of roots $\varepsilon_i - \varepsilon_j, \varepsilon_i + \varepsilon_j$ both roots or none of them. So every reflection here will always move an even number of roots, so the size of $rB \setminus B$ is equal to $0, 2$ or 4.

The orbits in E_n of sets containing fewer than three roots are admissible as every reflection will never move more than two roots. For the remaining six orbits it is an easy exercise to verify for one chosen set that every reflection moves indeed $0, 1, 2$ or 4 roots. □

We finish this section with some further comments on Table 2. If B is a set of mutually orthogonal positive roots as indicated in Table 2, then the type Y of the system of all roots orthogonal to B is listed in the table. In the final column of the table we list the structure of the stabilizer in W acting on $\mathcal{B}$. If $\mathcal{B}$ has an element B all of whose members are fundamental roots, this stabilizer can be found in [8].

Two distinct lines represent different classes; sometimes even more than two, in which case they fuse under an outer automorphism (so they behave identically). This happens for instance for $M = \mathrm{D}_n$ (first line) with $n = 2t$. In the second line for D_n, the permutation action is not faithful.

3. Posets

In this section we show that admissible orbits carry a nice poset structure. An arbitrary W-orbit of sets of mutually orthogonal positive roots satisfies all of the properties of the proposition below except for (iii).

Proposition 3.1. *Let M be a spherical simply laced diagram and $\mathcal{B}$ an admissible W-orbit of sets of mutually orthogonal positive roots. Then there is an ordering $<$ on $\mathcal{B}$ with the following properties.*

- *(i) For each node i of M and each $B \in \mathcal{B}$, the sets B and r_iB are comparable. Furthermore, if $(\alpha_i, \beta) = \pm 1$ for some $\beta \in B$, $r_iB \neq B$.*
- *(ii) Suppose $i \sim j$ and $\alpha_i \in B^\perp$. If $r_jB < B$, then $r_ir_jB < r_jB$. Also, $r_jB > B$ implies $r_ir_jB > r_jB$.*
- *(iii) If $i \not\sim j$, $r_iB < B$, $r_jB < B$, and $r_iB \neq r_jB$, then $r_ir_jB < r_jB$ and $r_ir_jB < r_iB$.*
- *(iv) If $i \sim j$, $r_iB < B$, and $r_jB < B$, then either $r_ir_jB = r_jB$ or $r_ir_jB < r_jB$, $r_jr_iB < r_iB$, $r_ir_jr_iB < r_ir_jB$, and $r_ir_jr_iB < r_jr_iB$.*

It readily follows from the existence result that there is a unique minimal ordering $<$ satisfying the requirements of the proposition (it is the transitive closure of the pairs (B, r_jB) for $B \in \mathcal{B}$ and j a node of M such that $r_jB > B$). This poset $(\mathcal{B}, <)$ with this minimal ordering is called the *monoidal poset* (with respect to W) on $\mathcal{B}$ (so $\mathcal{B}$ should be admissible for the poset to be monoidal).

Proof. We define the relation $<$ on $\mathcal{B}$ as follows: for $B, C \in \mathcal{B}$ we have $B < C$ iff there are $\beta \in B \setminus C$ and $\gamma \in C \setminus B$, of minimal height in $B \setminus C$, respectively $C \setminus B$, such that $\mathrm{ht}(\beta) < \mathrm{ht}(\gamma)$. It is readily verified that $<$ is an ordering. We show that it also satisfies properties (i),...,(iv).

We will need various properties involving the actions of the r_i on $\mathcal{B}$. Clearly, if $\alpha_i \in B^\perp$, then $r_iB = B$. As described earlier, if α_i is in B we replace $r_i\alpha_i = -\alpha_i$ with α_i and say α_i is fixed. Note that then also $r_iB = B$. If $(\alpha_i, \beta) = \pm 1$, we say r_i moves β. In this case with Proposition 3.1(i) in mind, we see $r_iB \neq B$ and we say r_i *lowers* B if $B > r_iB$ and we say r_i *raises* B if $B < r_iB$. We also use this for a single root β: if $\beta + \alpha_i$ is a root, we say r_i raises β and if $\beta - \alpha_i$ is a root we say r_i lowers β.

(i). If α_i is orthogonal to each member of B or $\alpha_i \in B$, then $r_iB = B$, so B and r_iB are comparable. So we may assume that $(\alpha_i, \beta) = \pm 1$ for at least one β. Notice if $(\beta, \alpha_i) = \pm 1$, that $r_i\beta = \beta \pm \alpha_i$ is in Φ but not in B as $(\beta, \beta \pm \alpha_i) = 2 \pm 1 \neq 0$ whereas different elements of B are orthogonal. In particular, $r_iB \neq B$. If $(\beta, \alpha_i) = \pm 1$ holds for exactly one member of B, then clearly r_iB and B are comparable. Suppose now B and r_iB are not comparable. Then there exist at least one $\beta \in B \setminus r_iB$ and one $\gamma \in B$ such that $\mathrm{ht}(\beta) = \mathrm{ht}(r_i\gamma)$ and $r_i\gamma$ is an element of minimal height in $r_iB \setminus B$. Clearly $\beta \neq r_i\gamma$ as they are in different sets by their definition. As $\gamma \in B \setminus r_iB$ we have $\mathrm{ht}(\gamma) \geq \mathrm{ht}(\beta)$ by our assumption that β is of minimal height in $B \setminus r_iB$. As $\mathrm{ht}\, r_i\gamma = \mathrm{ht}\, \beta$ we have $(\alpha_i, \gamma) = 1$. Also $r_i\beta \in r_iB \setminus B$ and as $r_i\gamma$

with $\mathrm{ht}(r_i\gamma) = \mathrm{ht}(\beta)$ is of minimal height in $r_iB \setminus B$ we see $\mathrm{ht}(r_i\beta) \geq \mathrm{ht}(r_i\gamma)$. In particular we must have $(\alpha_i, \beta) = -1$. But according to Condition (i) Proposition 2.1, this never occurs.

(ii). By the assumption $r_jB < B$, there is a root $\beta \in B$ of minimal height among those moved by r_j such that $\beta - \alpha_j \in r_jB$. Then $\beta - \alpha_j$ is minimal among those moved by r_i in r_jB and $\beta - \alpha_i - \alpha_j \in r_ir_jB$, so $r_ir_jB < r_jB$.

The proof for the second assertion is a bit more complicated. By the assumption $r_jB > B$, there is no root $\beta \in B$ of minimal height among those moved by r_j such that $\beta - \alpha_j \in r_jB$. Indeed all that are moved go to $\beta + \alpha_j$ in r_jB. Suppose that δ has minimal height among the roots moved by r_i in r_jB. This implies $\delta = \gamma \pm \alpha_j$ for some $\gamma \in B$. If $\delta = \gamma + \alpha_j$ for all choices of δ, then $r_ir_jB > r_jB$, as $r_i(\gamma + \alpha_j) = \gamma + \alpha_j + \alpha_i$. So assume that $\delta = \gamma - \alpha_j$ has minimal height among the roots moved by r_i in r_jB for some $\gamma \in B$. Let h be the minimal height of all elements of B moved by r_j. We know each of these roots is raised in height by r_j and so γ is not one of them. In particular $\mathrm{ht}(\gamma) > h$. Also $\mathrm{ht}(\gamma) - 1 = \mathrm{ht}(\delta) \leq h + 1$. It follows that $\mathrm{ht}(\gamma) \leq h + 2$. The two cases are γ has height $h + 1$ or $h + 2$. By Condition (i) for Proposition 2.1 for γ and β the case $h + 1$ is ruled out. But then Condition (ii) of Proposition 2.1 with α_i, γ, and β rules out the case $h + 2$.

(iii). Suppose $r_jB < B$ and $r_iB < B$ with $r_iB \neq r_jB$. Choose β an element of smallest height in B moved by r_j for which $\beta - \alpha_j$ is a root. Choose γ an element of smallest height in B moved by r_i with $\gamma - \alpha_i$ a root. We are assuming $\beta - \alpha_j$ is a root. This means as $(\alpha_i, \alpha_j) = 0$ that $\beta \pm \alpha_i$ is a root if and only if $\beta - \alpha_j \pm \alpha_i$ is a root.

To prove the result we will get a contradiction if we assume r_i raises r_jB. Suppose then r_i raises r_jB. In this case all elements ζ of smallest height in r_jB which are moved by r_i have $\zeta + \alpha_i$ as roots. We will show first that $\gamma - \alpha_j$ is not a root. If it were, r_i lowers it as $\gamma - \alpha_i$ is a root. This means it is a root of smallest height moved by r_i as γ is a root of smallest height moved by r_i in B and in r_jB this has height one smaller. But it is lowered, not raised. This means $\gamma - \alpha_j$ is not a root.

Depending on (γ, α_j), either γ or $\gamma + \alpha_j$ is a root of r_jB. Suppose $(\gamma, \alpha_j) = 0$ and so γ is a root of r_jB. As r_i raises r_jB, all elements of smallest height moved by r_i must be raised. As γ is lowered, there must be an $r_j\delta \in r_jB$ with $\delta - \alpha_j$ a root of r_jB and $\mathrm{ht}(\delta - \alpha_j)$ less than $\mathrm{ht}(\gamma)$. Its height must be one less than $\mathrm{ht}(\gamma)$ as heights are lowered at most one by r_j. Now in r_jB, the elements $\delta - \alpha_j$ and γ contradict condition (i) for Proposition 2.1. Suppose then $\gamma + \alpha_j$ is root. The smallest height of elements for which r_i moves roots in r_jB is now either $\mathrm{ht}(\gamma)$ or $\mathrm{ht}(\gamma) - 1$. (It cannot be $\mathrm{ht}(\gamma) + 1$ as $\gamma + \alpha_j$ is lowered.) If it is height $\mathrm{ht}(\gamma)$ there is an element δ of height $\mathrm{ht}(\gamma)$ which is raised by r_i. Now δ and $\gamma + \alpha_j$ contradict Condition (i) of Proposition 2.1.

We are left with one case in which an element of height $\mathrm{ht}(\gamma) - 1$ in r_jB is raised by r_i. This means there is δ in B of height $\mathrm{ht}(\gamma)$ which is lowered by r_j and raised by r_i. Recall γ is lowered by r_i and raised by r_j. As $i \not\sim j$ we have $\delta + \alpha_i - \alpha_j, \gamma + \alpha_j - \alpha_i$ in r_ir_jB. Now $(\delta, \gamma + \alpha_j - \alpha_i) = 2$ so $\delta = \gamma + \alpha_i - \alpha_j$. By admissibility of $\mathcal{B}$ we have $r_iB = r_jB$ contradicting the starting assumptions.

(iv). We shall use the results of the following computations throughout the proof. Let $\epsilon \in B$ and write $\rho = (\alpha_i, \epsilon)$ and $\sigma = (\alpha_j, \epsilon)$. Then $\epsilon - \rho\alpha_i = r_i(\varepsilon)$, $\epsilon - \sigma\alpha_j = r_j(\varepsilon)$, $\epsilon - (\rho+\sigma)\alpha_i - \sigma\alpha_j = r_ir_j(\varepsilon)$, $\epsilon - \rho\alpha_i - (\rho+\sigma)\alpha_j = r_jr_i(\varepsilon)$, $\epsilon - (\rho+\sigma)(\alpha_i+\alpha_j) = r_jr_ir_j(\varepsilon) = r_ir_jr_i(\varepsilon)$. Note that $\rho = \sigma = 1$ would imply $(\alpha_j, \epsilon - \alpha_i) = 2$, whence

$\epsilon = \alpha_i + \alpha_j$. This means $r_i r_j B = r_j B$. To see this, suppose $\alpha_i + \alpha_j \in B$. Then $\alpha_i \in r_j B$ as it is $r_j(\alpha_i + \alpha_j)$. Now all other elements of $r_j B$ are orthogonal to α_i as α_i is one of the elements. Now $r_i r_j B = r_j B$. So we can assume this does not occur.

By assumption, there are $\beta, \gamma \in B$ such that $\beta - \alpha_i \in r_i B$ and $\gamma - \alpha_j \in r_j B$ and such that $\mathrm{ht}(\beta), \mathrm{ht}(\gamma)$ are minimal with respect to being moved by α_i, α_j respectively. By symmetry, we may also assume that $\mathrm{ht}(\beta) \leq \mathrm{ht}(\gamma)$.

If $\beta = \gamma$, then $(\alpha_i, \beta) = (\alpha_j, \gamma) = 1$, a case that has been excluded. Therefore, we may assume that β and γ are distinct. In particular, (α_j, β) is 0 or -1.

Suppose first that a β can be chosen so that $(\alpha_j, \beta) = 0$. This is certainly the case if $\mathrm{ht}(\beta) < \mathrm{ht}(\gamma)$. Now $\beta - \alpha_i - \alpha_j \in r_i r_j B$, so $\beta - \alpha_i \in r_i B$ is a root of smallest height moved by r_j and so $r_j r_i B < r_i B$. Recall from our choice no root of height smaller than $\mathrm{ht}(\beta)$ is moved by r_i.

Since $\beta \in r_j B$ and $\beta - \alpha_i \in r_i r_j B$, we have $r_i r_j B < r_j B$ unless there is $\delta \in B$ with $\delta - \alpha_j \in r_j B$, $\mathrm{ht}(\delta - \alpha_j) = \mathrm{ht}(\beta) - 1$, and $(\alpha_i, \delta - \alpha_j) = -1$. But then the inner products show $\delta = -\alpha_i$ is not a positive root. Notice this shows $\delta - \alpha_j + a_i$ cannot be a root. Hence, indeed, $r_i r_j B < r_j B$.

Since $\beta - \alpha_i - \alpha_j \in r_j r_i r_j B$ and $\beta - \alpha_i \in r_i r_j B$, a similar argument to the previous paragraph shows that $r_j r_i r_j B < r_i r_j B$.

It remains to show $r_j r_i r_j B < r_j r_i B$. Both sides contain $\beta - \alpha_i - \alpha_j$ and r_i does not lower or raise $r_j r_i \beta$. We need to look at the δ in B of height up to $\mathrm{ht}(\gamma)$. We know that for δ with $\mathrm{ht}(\delta) < \mathrm{ht}(\gamma)$ that $(\delta, \alpha_j) = 0$. Looking at the equations above with $\sigma = 0$ we see r_i does not change $r_j r_i \delta = \delta - \rho(\alpha_i + \alpha_j)$. We also know that $(\gamma, \alpha_j) = 1$. This means $\sigma = 1$. Using the equations again with $\sigma = 1$ and ρ we must compare $\gamma - \rho(\alpha_i + \alpha_j) - \alpha_j$ with $\gamma - \rho(\alpha_i + \alpha_j) - \alpha_j - \alpha_i$ which is lower. In particular, $r_i r_j r_i B < r_j r_i B$.

Suppose then that $(\alpha_j, \beta) = -1$. This means in particular that $\mathrm{ht}(\beta) = \mathrm{ht}(\gamma)$. If $(\alpha_i, \gamma) = 0$ we can use the argument above. We are left then with the case in which $(\alpha_i, \gamma) = 1$, $(\alpha_j, \beta) = 1$, $(\alpha_i, \beta) = -1$, $(\alpha_j, \gamma) = -1$, and of course $(\alpha_i, \alpha_j) = -1$.

This means $\gamma - \alpha_i$ and $\beta - \alpha_j$ are positive roots of height $\mathrm{ht}(\beta) - 1$. But $(\gamma - \alpha_i, \beta - \alpha_j) = 0 + 1 + 1 - 1 = 1$ so by subtracting one root from the other, we should get another positive root. As both roots are of the same height, this would give a root of height 0 which is not possible, proving this case never arises. □

We showed during the proof that if $\alpha_i + \alpha_j \in B$ and $i \sim j$, then $r_i r_j B = r_j B$. This is case (iv) of Proposition 3.1. The following lemma shows this is if and only if.

Lemma 3.2. *Suppose that $(\mathcal{B}, <)$ is a monoidal poset for (W, R) for which $r_i r_j B = r_j B$ with $i \sim j$. If $r_i B < B$ and $r_j B < B$, then $\alpha_i + \alpha_j \in B$.*

Proof. Suppose $\alpha_i + \alpha_j$ is not in B. Let β be an element of smallest height moved in B by r_j for which $\beta - \alpha_j$ is a root. Such a root exists because $r_j B < B$. As $\alpha_i + \alpha_j$ is not in B, we know $\beta - \alpha_j \neq \alpha_i$, and even α_i is not in $r_j B$. It follows as $r_i r_j B = r_j B$ that $\alpha_i \in (r_j B)^{\perp}$. In particular $r_i(\beta - \alpha_j)$ is in $r_j B$. As $\beta - \alpha_j \pm \alpha_i$ is not orthogonal to $\beta - \alpha_j$ we must have $\beta + \alpha_i$ a root. Now r_j lowers β and r_i raises β.

As $r_i B < B$ there exists γ, an element of smallest height in B moved by r_i for which $\gamma - \alpha_i$ is a root. We know $\mathrm{ht}(\gamma) \leq \mathrm{ht}(\beta)$ as r_i moves β. Suppose $(\gamma, \alpha_j) = 0$.

Then $\gamma \in r_j B$ and $r_i\gamma = \gamma - \alpha_i$ is also in $r_j B$. This contradicts the hypothesis that elements of $r_i B$ are all orthogonal. This implies $(\gamma, \alpha_j) = \pm 1$. This in turn means $\mathrm{ht}(\beta) \leq \mathrm{ht}(\gamma)$ as $\mathrm{ht}(\beta)$ is the height of the smallest element moved by r_j. Now we have $\mathrm{ht}(\beta) = \mathrm{ht}(\gamma)$. If $\gamma - \alpha_i$ and $\gamma - \alpha_j$ were both roots, an inner product computation would show $(\gamma, \alpha_i + \alpha_j) = 2$ so $\gamma = \alpha_i + \alpha_j$. This means $\gamma + \alpha_j$ is a positive root, so in $r_j B$ we have $\beta - \alpha_j$ and $\gamma + \alpha_j$ contradicting (i) of Proposition 2.1. □

In order to address the monoid action later we will need some more properties of this action in terms of lowering and raising. We begin with the case in which two different fundamental reflections act the same on a member B of $\mathcal{B}$.

Before we begin we need to examine the case in which some B has two indexes which raise it to the same B'. In particular we have

Lemma 3.3. *Suppose $B \in \mathcal{B}$ and $r_i B = r_k B > B$ with $k \neq i$. If β is the element of B of smallest height moved by either r_i or r_k, then $\beta + \alpha_i + \alpha_k$ is also in B. Furthermore, $i \not\sim k$.*

Proof. Let β be an element of smallest height B moved by either r_i or r_k. We know that all elements of smaller height are not moved by r_i and r_k. Elements of the same height could be moved by r_i or r_k, but then the root would have to be added. Suppose $(\alpha_i, \beta) = -1$, so $r_i\beta = \beta + \alpha_i$. If $(\alpha_k, \beta) = 0$, then $\beta \in r_k B = r_i B$ as is $\beta + \alpha_i$ and so $(\beta, \beta + \alpha_i) = 2 - 1 \neq 0$, which contradicts that elements of $r_i B$ are mutually orthogonal. In particular $(\alpha_k, \beta) = -1$ (for otherwise, $(\alpha_k, \beta) = 1$ and so $r_k B < B$).

If $i \sim k$, then $(\alpha_i, \alpha_k) = -1$, and so $(\alpha_k, \beta + \alpha_i) = -2$, which implies that $\beta + \alpha_i = -\alpha_k$, contradicting that $\beta + \alpha_i$ be a positive root. This means $i \not\sim k$ which proves the last part of the lemma.

Now by hypothesis $r_i r_k B = B$ and so $\beta + \alpha_k + \alpha_i$ is in B which proves the remainder of the lemma. □

Notice that if β and $\beta + \alpha_i + \alpha_k$ are two roots in B with $(\alpha_i, \alpha_k) = 0$, $(\beta, \alpha_i) = (\beta, \alpha_k) = -1$, the hypothesis of the lemma is satisfied, and r_i maps β to $\beta + \alpha_i$ and $\beta + \alpha_i + \alpha_k$ to $\beta + \alpha_k$. Acting by r_k has the same effect except the order of the roots has been interchanged.

Lemma 3.4. *Suppose $(\mathcal{B}, <)$ is a monoidal poset for (W, R). Let $B \in \mathcal{B}$ and let $i, j \in M$ and $\beta, \gamma \in B$. Then the following assertions hold.*

- *(i) If $i \not\sim j$ and $r_i r_j B < r_i B < B$, then $r_i r_j B < r_j B < B$.*
- *(ii) If $i \not\sim j$, $B < r_i B$, $B < r_j B$, and $r_i B \neq r_j B$, then $r_i r_j B > r_i B$ and $r_i r_j B > r_j B$.*
- *(iii) If $i \sim j$, $B < r_i B$, and $B < r_j B$, then $r_i B < r_j r_i B < r_i r_j r_i B$, and $r_j B < r_i r_j B < r_j r_i r_j B$.*
- *(iv) If $i \sim j$ and $r_j r_i r_j B < r_j r_i B < r_i B < B$, then also $r_j r_i r_j B < r_i r_j B < r_j B < B$.*
- *(v) If $\alpha_i \notin B^\perp \cup B$, then either $r_i B < B$ or $r_i B > B$.*

Proof. We can refer to Proposition 3.1 for the properties of $(\mathcal{B}, <)$.

(i). If $r_j B = r_i r_j B$, then also $r_i B = B$, a contradiction. Suppose $r_i r_j B > r_j B$. Then, by transitivity $B > r_j B$. Also $B > r_i B$ by hypotheses. Notice if $r_i B = r_j B$,

$r_ir_jB = r_i^2B = B$ but $r_ir_jB < B$. Now $r_ir_jB < r_jB$ by Proposition 3.1(iii), a contradiction. Hence, by Proposition 3.1(i), $r_ir_jB < r_jB$.

If $r_jB = B$, then also $r_iB = r_ir_jB$, a contradiction. Suppose $r_jB > B$. Then, by transitivity, $r_jB > r_ir_jB$. Therefore, Proposition 3.1(iii) gives $r_ir_jB > r_jr_ir_jB = r_iB$, a contradiction. Hence, by Proposition 3.1(i), $r_jB < B$. But also $r_iB < B$, so Proposition 3.1(iii) gives $r_ir_jB < r_jB$ (and $r_ir_jB < r_iB$).

(ii). If $r_ir_jB = r_iB$, then $r_jB = B$, a contradiction. If $r_ir_jB < r_iB$, then, by Proposition 3.1(iii) applied to r_iB we have $r_jB < B$, a contradiction. Hence by Proposition 3.1(i), $r_ir_jB > r_iB$. The proof of $r_ir_jB > r_jB$ is similar.

(iii). Suppose $r_ir_jB = r_jB$. If $\alpha_i \in (r_jB)^\perp$, then, as r_j lowers r_jB, by Proposition 3.1(ii) r_i lowers $r_jr_jB = B$ which is a contradiction. This means $\alpha_i \in r_jB$ and so $\alpha_i + \alpha_j \in B$. Notice neither α_i nor α_j are in B as they are not orthogonal to $\alpha_i + \alpha_j$. As both r_i and r_j raise B, there must be k, l, with $i \sim k$ and $j \sim l$ with α_k and α_l in B. Neither are orthogonal to $\alpha_i + \alpha_j$ and this is impossible. This means $r_ir_jB \neq r_jB$.

Suppose $r_ir_jB < r_jB$. We can't have $r_ir_jB = r_jr_jB = B$ by Lemma 3.3. Now Proposition 3.1(iv) gives $r_iB < B$, a contradiction. Hence $r_ir_jB > r_jB$. The roles of i and j are symmetric, so similarly we find $r_ir_jB > r_iB$.

If $r_ir_jr_iB = r_ir_jB$ then $B = r_iB$, a contradiction. Suppose $r_jr_ir_jB < r_ir_jB$. As also $r_jB < r_ir_jB$, Proposition 3.1(iv) gives $r_iB < B$, a contradiction, because $\alpha_i + \alpha_j \in r_ir_jB$ would imply $\alpha_j \in r_jB$ whence $r_jB = B$.

Similarly, it can be shown that $r_jr_ir_jB > r_ir_jB$.

(iv). If $r_jB = B$, then $r_jr_ir_jB = r_jr_iB$, a contradiction. If $r_jB < B$, then the result follows from Proposition 3.1(iv) because $\alpha_i + \alpha_j \in B$ would imply $\alpha_j \in r_iB$ whence $r_jr_iB = r_iB$.

Suppose therefore $r_jB > B$. If $r_jB = r_ir_jB$, then $r_jr_iB = r_jr_ir_jB$, a contradiction. If $r_jB > r_ir_jB$, then by Proposition 3.1(iv) $r_jr_ir_jB > r_jr_iB$, a contradiction because $\alpha_i + \alpha_j \in r_jB$ would imply $\alpha_i \in B$ whence $r_iB = B$.

Hence $r_jB < r_ir_jB$. But then by transitivity $r_ir_jB > r_jr_ir_jB$, and, since $r_ir_jB > r_jB$, gives Proposition 3.1(iv) $r_jr_ir_jB > r_ir_jB$ (for otherwise $\alpha_i + \alpha_j \in r_ir_jB$, implying $\alpha_j \in r_jB$ so $r_jB = B$), a final contradiction.

(v). The hypotheses imply that there exists $\beta \in B$ with $(\alpha_i, \beta) = \pm 1$. Then $r_i\beta = \beta \pm \alpha_i$, which is not orthogonal to β. As the elements of B are orthogonal by definition, $r_i\beta$ does not belong to B, so $r_iB \neq B$, and the conclusion follows from Proposition 3.1(i). □

Pick B_0 a maximal element of $\mathcal{B}$. This means r_iB_0 is either B_0 or lowers B_0. This is possible as $\mathcal{B}$ is finite. We need more properties of the poset determined by $>$. To begin with this we consider certain Weyl group elements, w, for which $wB_0 = B$ for a fixed element $B \in \mathcal{B}$. In particular we let $w = r_{i_1}r_{i_2}\ldots r_{i_s}$ be such that $B_0 > r_{i_s}B_0 > r_{i_{s-1}}r_{i_s}B_0 > \cdots > r_{i_2}r_{i_3}\cdots r_{i_s}B_0 > r_{i_1}r_{i_2}r_{i_3}\cdots r_{i_s}B_0 = B$. If there is such an expression for w, then there is one of minimal length. We let $\mathcal{B}'$ be the set of $B \in \mathcal{B}$ which are of this form. We will show that in fact $\mathcal{B}' = \mathcal{B}$.

Lemma 3.5. *In the notation just above, $\mathcal{B}' = \mathcal{B}$.*

Proof. Notice that B_0 is in $\mathcal{B}'$ by definition using w the identity. Recall that r_iB_0 is either B_0 or lower. In particular nothing raises B_0. We show first that if $B \in \mathcal{B}'$ and $r_jB > B$ then $r_jB \in \mathcal{B}'$. We prove this by induction on the minimal length of a chain from B_0 to wB which satisfies the descending property of the definition of $\mathcal{B}'$. In particular $w = r_{i_1} r_{i_2} \dots i_s$ and $B_0 > r_{i_s} B_0 > r_{i_{s-1}} r_{i_s} B_0 > \cdots > r_{i_2} r_{i_3} \cdots r_{i_s} B_0 > r_{i_1} r_{i_2} r_{i_3} \cdots r_{i_s} B_0 = B$. We say this chain has length s, the length of w. We in fact show that there is a chain from B_0 to $r_j B$ of length less than or equal to $s-1$. We have seen that no r_i raises B_0. Suppose that $B = r_i B_0$ and $r_j B > B$. If $r_j B = B_0$ the induction assumption is true. If $r_j B \neq B_0$ we can use Lemma 3.4(ii) or (iii) to see that $r_j r_i B > B_0$ a contradiction. In particular the induction assumption is true for $s \leq 1$.

We can now assume $s \geq 2$. Pick a B with a chain of length s and assume the result is true for any $B' \in \mathcal{B}'$ with a shorter chain length. Suppose $r_j B$ is not in $\mathcal{B}'$ and $r_j B > B$. Notice $r_{i_1} B = r_{i_2} r_{i_3} \cdots r_{i_s} B_0 > B$ by the hypothesis. Clearly $r_j B \neq r_{i_1} B$ as $r_{i_1} B$ is in $\mathcal{B}'$ using the element $r_{i_2} r_{i_3} \cdots r_{i_s}$. In particular we can use Lemma 3.4(ii) or (iii). In either case $r_j r_{i_1} B > r_{i_1} B$ and by our choice of s and the induction assumption, $r_j r_{i_1} B$ is in $\mathcal{B}'$ and has a chain of length at most $s-1$ from B_0 to it.

Suppose first $i_1 \not\sim j$ and use Lemma 3.4(ii). By the induction assumption there is a chain down to $r_j r_{i_1} B$ of length at most $s-2$ and then by multiplying by r_{i_1} gives a chain down to $r_j B$ of length at most $s-1$ and the induction gives $r_j B \in \mathcal{B}'$.

Suppose now $i_1 \sim j$ and use Lemma 3.4(iii). Again $r_j r_{i_1} B$ is in $\mathcal{B}'$ by the induction hypothesis and has a chain down to it of length at most $s-2$. Using the induction again, and the hypothesis of the minimality of s, we see also $r_{i_1} r_j r_{i_i} B$ is in $\mathcal{B}'$ and has a chain to it of length at most $s-3$. Now using this as $r_j r_{i_1} r_j B$, multiplying by r_j and then by r_{i_1} gives a chain to $r_j B$ of length at most $s-1$ and we are done with this part.

In particular, if $B \in \mathcal{B}'$ and $r_j B > B$, then $r_j B$ is in $\mathcal{B}'$. If $B \in \mathcal{B}'$ and $r_j B = B$ of course $r_j B \in \mathcal{B}'$. Suppose $r_j B < B$. Then the sequence to B and then $r_j B$ gives a sequence to $r_j B$ and $r_j B$ is in $\mathcal{B}'$. We see that $\mathcal{B}'$ is closed under the action of W and as $\mathcal{B}$ is an orbit, $\mathcal{B}' = \mathcal{B}$. □

Corollary 3.6. *There is a unique maximal element B_0 in $\mathcal{B}$.*

Proof. We have just shown that for every element B in $\mathcal{B}$ except B_0 there is a sequence lowering to B and so B_0 is the only maximal element. □

See Example 4.5 for a listing of some of the B_0.

This shows that each $B \in \mathcal{B}$ has a level associated with it, namely the smallest s for which B can be obtained from B_0 as above with a Weyl group element w of length s. Namely the smallest s for which there is a reduced expression $w = r_{i_1} r_{i_2} \cdots r_{i_s}$ with $wB_0 = B$ for which $B_0 > r_{i_s} B_0 > r_{i_{s-1}} r_{i_s} B_0 > \cdots > r_{i_2} r_{i_3} \cdots r_{i_s} B_0 > B$. In particlar B_0 has level 0 and if $r_j B_0 < B_0$ it has level 1. The next lemma says that this s is the shortest length of any word w for which $wB_0 = B$.

Lemma 3.7. *Suppose w is an element of W of the smallest length for which $wB_0 = B$. Then this length, s, is the length of the shortest word defining B as an element of $\mathcal{B}'$. In particular if the word is $r_{i_1} r_{i_2} \cdots r_{i_s}$, then $B_0 > r_{i_s} B_0 > r_{i_{s-1}} r_{i_s} \cdots > r_{i_1} r_{i_2} \cdots r_{i_s} B_0 = B$ and this is the shortest which does this. It is reduced.*

Proof. Suppose w is an element of W for which $wB_0 = B$ and for which as in the definition of $\mathcal{B}'$, we have $w = r_{i_1} r_{i_2} \cdots r_{i_s}$ and $r_{i_s} B_0 > r_{i_{s-1}} r_{i_s} B_0 > \cdots > r_{i_1} r_{i_2} \cdots r_{i_s} B_0 = B$ with this the shortest possible. Suppose w' is any other Weyl group element with $wB_0 = B$. If $w' = r_{j_1} r_{j_2} \cdots r_{j_t}$ is a reduced decomposition of length t, then t is at most s and we get a sequence B_0, $r_{j_t} B_0$, $r_{j_{t-1}} r_{j_t} B_0$, $\cdots$, $r_{j_1} r_{j_2} \cdots r_{j_t} B_0 = B$. If any of these differences do not have the relation $>$ between them, the level of B would be strictly smaller than s, contradicting the minimality of s. Hence, $t = s$ and the sequence corresponding to w' is also a chain. In particular, w is reduced and any other reduced expression gives a descending sequence of the same length. This shows there is a reduced word with this length taking B_0 to B and any word doing this of shorter or the same length, has to be descending at each step. This proves the lemma. □

Lemma 3.8. *Suppose that* $(\mathcal{B}, <)$ *is a monoidal poset for* (W, R).

(i) For each $B \in \mathcal{B}$ *and each element* $w \in W$ *of minimal length such that* $B = wB_0$ *and node* i *of* M *such that* $l(r_i w) < l(w)$, *we have* $r_i B > B$.

(ii) For each $B \in \mathcal{B}$, *if* $w, w' \in W$ *are of minimal length such that* $B = wB_0 = w'B_0$, *then* $l(w) = l(w')$ *and, for each node* i *such that* $r_i B > B$, *there is* $w'' \in W$ *of length* $l(w)$ *such that* $B = w''B_0$ *and* $l(r_i w'') < l(w'')$.

Proof. For (i) we use the characterization in Lemma 3.7 and realize that any of the equivalent expressions also give a descending sequence. In particular if $l(r_i w) < l(w)$, an equivalent word can be chosen to start with r_i and so $r_i B$ is one step above B in the chain to B from this word and so $r_i B > B$.

For (ii) again use Lemma 3.7 and so $l(w) = l(w')$. If $r_i B > B$ for some i, there is a sequence from $r_i B$ to B_0. If $w' B_0 = r_i B$ accomplishes this in the minimal number of steps, $w'' = s_i w'$ satisfies the conclusion of the lemma. □

4. The positive monoid

We now turn our attention to the Artin group A associated with the Coxeter system (W, R). We recall that the defining presentation of A has generators s_i corresponding to the fundamental reflections $r_i \in R$ and *braid relations* $s_i s_j s_i = s_j s_i s_j$ if $i \sim j$ and $s_i s_j = s_j s_i$ if $i \not\sim j$. The monoid A^+ given by the same presentation is known ([10]) to embed in A. For each admissible W-orbit of a set of mutually commuting reflections, we shall construct a linear representation of A^+. To this end, we need a special element $h_{B,i}$ of A^+ for each pair (B, i) consisting of a set B of mutually commuting reflections and a node i of M whose reflection r_i does not belong to B but commutes with each element of B. As in the previous section, we shall represent reflections by positive roots.

We now define the elements $h_{B,i}$. As in [5] we do this by defining reduced words $v_{B,i} \in A$ and letting $h_{B,i} = v_{B,i}^{-1} s_i v_{B,i}$. Later we shall consider the image of these elements in a certain Hecke algebra.

We make definitions of $v_{B,i}$ which depend on certain chains from B_0 to B and show in an early lemma that conjugating s_i by any of them gives the same element. Furthermore, this element corresponds to a fundamental generator of A commuting with every reflection having its positive root in B_0.

Definition 2. *Suppose* $(\mathcal{B}, <)$ *is a monoidal poset of* (W, R), *with maximal element* B_0. *Let* (B, i) *be a pair with* $B \in \mathcal{B}$ *and* i *a node of* M *such that* $\alpha_i \in B^\perp$.

Choose a node j *of* M *with* $r_j B > B$. *If* $j \not\sim i$ *let* $v_{B,i} = s_j v_{r_j B, i}$ *and if* $i \sim j$ *let* $v_{B,i} = s_j s_i v_{r_i r_j B, j}$. *We define* $v_{B_0, i}$ *as the identity.*

Furthermore, set $h_{B,i} = {v_{B,i}}^{-1} s_i v_{B,i}$.

Notice this definition makes sense as a nondeterministic algorithm assigning an element of A to each pair (B,i) as specified because

- if $\alpha_i \in B^\perp$ and $i \not\sim j$, then $\alpha_i \in (r_jB)^\perp$;
- If $i \sim j$, then $\alpha_i + \alpha_j \in (r_jB)^\perp$ and $\alpha_j \in (r_ir_jB)^\perp$.

By Lemma 3.7, $v_{B,i}$ will be a reduced expression whose length is the length of a chain from B to B_0. The elements $v_{B,i}$ are not uniquely determined, but we will show that the elements $h_{B,i}$ are.

Lemma 4.1. *For $\mathcal{B}$ and (B,i) as in Definition 2, suppose that $v_{B,i}$ and $v'_{B,i}$ both satisfy Definition 2. Then the elements $h_{B,i}$ of A defined by each are the same, i.e., $h_{B,i} = {v_{B,i}}^{-1}s_iv_{B,i} = {v_{B,i}'}^{-1}s_i{v_{B,i}}'$.*

Furthermore, each $h_{B,i}$ is a fundamental generator s_j of A whose root α_j is orthogonal to every root of B_0.

Proof. We use induction on the height from B_0. The case of B_0 is trivial.

We first dispense with the case in which $r_jB = r_{j'}B$. We know from Lemma 3.3 that in B there are two elements β and $\beta+\alpha_j+\alpha_{j'}$ with $j \not\sim j'$. As $\alpha_i \in B^\perp$ we know $(\beta,\alpha_i) = 0$ and also $(\beta+\alpha_j+\alpha_{j'},\alpha_i) = 0$. It follows that $(\alpha_j,\alpha_i) = (\alpha_{j'},\alpha_i) = 0$ as the inner products of fundamental roots are 0 or -1. In particular using r_j we get $v_{B,i} = s_jv_{r_jB,i}$, and ${v_{B,i}}^{-1}s_iv_{B,i} = {v_{r_jB,i}}^{-1}s_j^{-1}s_is_jv_{B,i}$. As $s_j^{-1}s_is_j = s_i$ this is ${v_{r_jB,i}}^{-1}s_iv_{B,i} = h_{r_jB,i}$. The same is true for $r_{j'}$ and we are assuming $r_jB = r_{j'}B$. Now we can use induction.

We next suppose $r_jB > B$ and $r_{j'}B > B$ with $r_jB \neq r_{j'}B$. There will be two cases depending on whether $j \not\sim j'$ or $j \sim j'$. Suppose first $j \not\sim j'$. We use Lemma 3.4(ii) to see that $r_jr_{j'}B > r_{j'}B$ and $r_{j'}r_jB > r_jB$. Suppose first $i \not\sim j$ and $i \not\sim j'$. For the chain starting with r_j we can follow it with $r_{j'}$ and if we start with $r_{j'}$ we can follow it with r_j. In each case with these choices we get $v_{B,i} = s_js_{j'}v_{r_jr_{j'}B,i}$ as $s_js_{j'} = s_{j'}s_j$ and $\alpha_i \in (r_jB)^\perp$ and $\alpha_i \in r_{j'}B)^\perp$. The induction is used for $v_{r_jB,i}$ and for $v_{r_{j'}B,i}$ in order to take the chain we have chosen and then also for $v_{r_jr_{j'}B,i}$. In each case we get $h_{r_jr_{j'}B,i}$.

Suppose next that $i \sim j$ but $i \not\sim j'$. Using the chain for r_j we get $B < r_jB < r_ir_jB$ by Proposition 3.1(ii). As above by Lemma 3.4(ii) we get $r_jr_{j'}B > r_{j'}B$ and now again by Proposition 3.1(ii) using $\alpha_i \in (r_{j'}B)^\perp$ we get $r_ir_jr_{j'}B > r_jr_{j'}B$. Now for the $r_{j'}$ chain continue through r_j and then r_i to reach $v_{B,i} = s_{j'}s_js_iv_{r_ir_jr_{j'}B,j}$. Through the r_j chain which goes through r_ir_jB add $r_{j'}$ for which $j' \not\sim i$. Here we get $v_{B,i} = s_js_is_{j'}v_{r_{j'}r_ir_jB,j}$. Again use induction at all the levels to get the needed result. Notice $r_ir_jB \neq r_{j'}r_jB$ as $r_ir_jr_{j'}B > r_jr_{j'}B$ as above and so r_i raises $r_jr_{j'}B$.

The final case in which $j \not\sim j'$ is with $j \sim i \sim j'$, see Figure 1. For this we again use Lemma 3.4(ii) and (iii) and Proposition 3.1. In particular $r_jB > B$ and $r_ir_jB > r_jB$. Also as $r_jB \neq r_{j'}B$ we have $r_{j'}r_jB > r_jB$. Now by Lemma 3.4(iii) we have $r_ir_{j'}r_ir_jB > r_{j'}r_ir_jB > r_ir_jB$. We know $\alpha_j \in (r_ir_jB)^\perp$ and also in $(r_{j'}r_ir_jB)^\perp$ as $j \not\sim j'$. Now using Proposition 3.1(ii) we see $r_jr_ir_{j'}r_ir_jB > r_ir_{j'}r_ir_jB$. Notice $r_ir_jB \neq r_{j'}r_jB$ by Lemma 3.3 as $i \sim j'$. Following the trail of α_i we see it is in $(r_jr_ir_{j'}r_ir_jB)^\perp$. Following this chain after r_ir_j and using induction we see $v_{B,i} = s_js_is_{j'}s_is_jv_{r_jr_ir_{j'}r_ir_jB,i}$. Going up through $r_ir_{j'}$ gives the same result as $s_js_is_{j'}s_is_j = s_js_{j'}s_is_{j'}s_j$ is similar to $s_{j'}s_is_js_is_{j'} = s_{j'}s_js_is_js_{j'}$. In particular the result is true again using induction at all the higher levels.

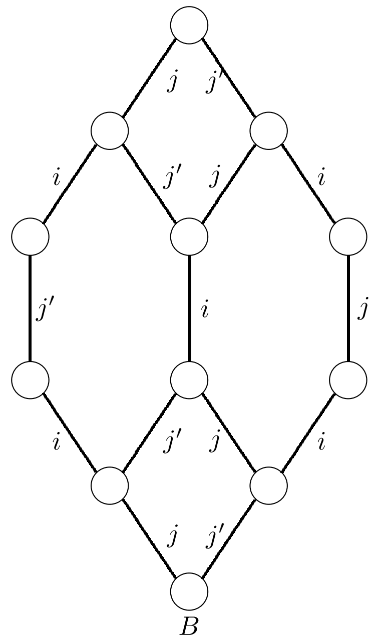

Figure 1

We now consider the cases in which $j \sim j'$. As before, the easiest case is when $i \not\sim j$ and $i \not\sim j'$. In this case as before use Lemma 3.4(iii) to obtain $r_{j'}r_jr_{j'}B > r_jr_{j'}B > r_{j'}B$. As α_i is orthogonal to α_j and $\alpha_{j'}$ we obtain $v_{B,i} = s_{j'}s_js_{j'}v_{r_{j'}r_jr_{j'}B,i}$. The same result holds for the other order and the result follows by induction.

The only other possibility is $i \sim j$ and $i \not\sim j'$ as i could not be adjacent to both j and j' (for otherwise there would be a triangle in the Dynkin diagram). For this we use the familiar six sides diagram generated by r_j and $r_{j'}$ using Lemma 3.4(iii), see Figure 2. At r_jB we may also act by r_i which we know raises r_jB. It is clear that $r_{j'}r_jB \neq r_ir_jB$ as α_j is in $(r_ir_jB)^\perp$ and so r_j could not raise (or even lower) it. We can proceed by Lemma 3.4(ii) to $r_{j'}r_jr_ir_jB$. Notice $r_jr_{j'}r_jB \neq r_ir_{j'}r_jB$ by Lemma 3.3 as $i \sim j$. Now proceed by Lemma 3.4(iii) to $r_ir_jr_ir_{j'}r_jB = r_ir_jr_{j'}r_ir_jB$. By following the perpendicularities we see $\alpha_{j'} \in (r_ir_jr_{j'}r_ir_jB)^\perp$. Using this chain which starts with r_j and continues with r_i, we find $v_{B,i} = s_js_is_{j'}s_js_iv_{s_ir_jr_{j'}r_ir_jB,j'}$. Using the other direction starting with $r_{j'}$ then r_j, then r_i, we can continue with $r_{j'}$ and r_j to get $r_jr_{j'}r_ir_jr_{j'}B$ and conclude using this direction $v_{B,i} = s_{j'}s_js_is_{j'}s_jv_{r_jr_{j'}r_ir_jr_{j'}B,j'}$. At the juncture $r_jr_{j'}B$ we act by r_i or by $r_{j'}$. These two could not be equal as again α_j is in $(r_ir_jr_{j'}B)^\perp$ and so r_j could not move it. However, if they were equal, it lowers it to $r_{j'}r_jB$. These words are equivalent and we can use induction as usual for the last time.

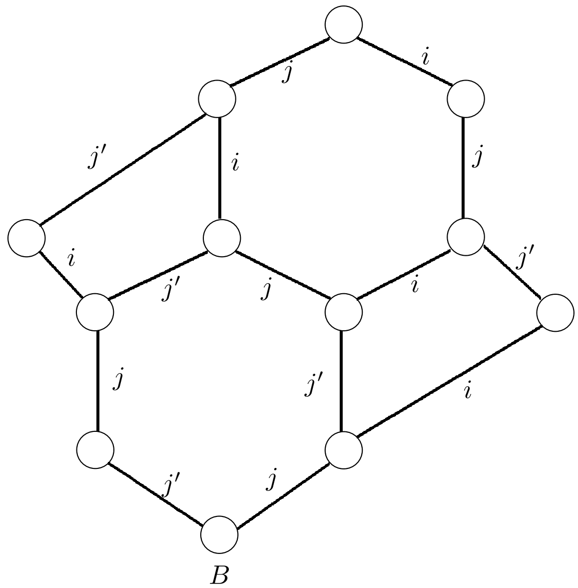

Figure 2

This finishes all cases and shows the words have the same effect under conjugation on s_i. □

We finish this section by exhibiting relations that hold for the $h_{B,i}$. Since we are actually interested in their images in the Hecke algebra H of type M under the natural morphism $\mathbb{Q}(m)[A] \to H$, we phrase the result in terms of elements of this algebra.

Proposition 4.2. *Suppose that* $(\mathcal{B}, <)$ *is a monoidal poset with maximal element* B_0*. Let* C *be the set of nodes of* M *such that* α_i *is orthogonal to* B_0 *and denote* Z *the Hecke algebra over* $\mathbb{Q}(m)$ *of the type* C*. Then the images of the elements* $h_{B,i} \in A$ *in the Hecke algebra of type* M *under the natural projection from the group algebra of* A *over* $\mathbb{Q}(m)$ *actually are fundamental generators of* Z *and satisfy the following properties.*

(i) $h_{B,i}^2 = 1 - m h_{B,i}$.
(ii) $h_{B,i} h_{B,j} = h_{B,j} h_{B,i}$ *if* $i \not\sim j$.
(iii) $h_{B,i} h_{B,j} h_{B,i} = h_{B,j} h_{B,i} h_{B,j}$ *if* $i \sim j$.
(iv) $h_{r_j B,i} = h_{B,i}$ *if* $i \not\sim j$.
(v) $h_{r_i r_j B,j} = h_{B,i}$ *if* $i \sim j$ *and* $(\alpha_j, B) \neq 0$.

Proof. By [6] we can identify the Hecke algebra of type C with the subalgebra of the Hecke algebra H generated by the s_i for $i \in C$. As described above we define $h_{B,i} = {v_{B,i}}^{-1} s_i v_{B,i}$ where we consider this element in the Hecke algebra. By Lemma 4.1, it is a fundamental generator of Z.

(i). This clearly follows from the quadratic Hecke algebra relations we are assuming.

(ii). Assume first $r_k B > B$ and both α_i and α_j are orthogonal to α_k. We are assuming here $i \not\sim j$. Then we can take $v_{B,i} = s_k v_{r_k B,i}$ and $v_{B,j} = s_k v_{r_k B,j}$. Now $h_{B,i} = (v_{r_k B,i})^{-1} s_k^{-1} s_i s_k v_{r_k B,i}$. This is $h_{r_k B,i}$ and we can use induction.

Suppose $i \sim k$ but $j \not\sim k$. Then we can take $v_{B,i} = s_k s_i v_{r_i r_k B,k}$ and we can take $v_{B,j} = s_k s_i v_{r_i r_k B,j}$. Then $h_{B,j} = h_{r_i r_k B,j}$ and $h_{B,i} = h_{r_i r_k B,k}$. Now as above we can again use induction.

The final case with $i \not\sim j$ is when $i \sim k \sim j$. Now $v_{B,i} = s_k s_i v_{r_i r_k B,k}$ and $v_{B,j} = s_k s_j v_{r_j r_k B,k}$. Suppose that $r_i r_k B = r_j r_k B$. If so $v_{B,i} = s_k s_i w'$ with $v_{B,j} = s_k s_j w'$ and $w' = v_{r_i r_k B,k}$. Now $v_{B,i}{}^{-1} s_i v_{B,i} = w'^{-1} s_i^{-1} s_k^{-1} s_i s_k s_i w' = w'^{-1} s_k w'$. Doing the same with r_j gives the same thing and so they commute. This means $r_j r_k B \neq r_i r_k B$ and we can use Lemma 3.4(ii) to get $r_i r_j r_k B > r_i r_k B$ and $r_i r_j r_k B > r_j r_k B$. Now applying this with $v_{B,i}$ gives $v_{B,i} = s_k s_j s_i s_k v_{r_k r_i r_j r_k B,j}$ and $v_{B,j} = s_k s_i s_j s_k v_{r_k r_j r_i r_k B,i}$. Let $B' = r_k r_i r_j r_k B$. Now $h_{B,i} = h_{B',j}$ and $h_{B,j} = h_{B',i}$. Now use induction.

(iii). Suppose $i \sim j$. We wish to show $h_{B,i} h_{B,j} h_{B,i} = h_{B,j} h_{B,i} h_{B,j}$. Suppose first $k \not\sim i$ and $k \not\sim j$. In this case $v_{B,i} = s_k v_{r_k B,i}$ and $v_{B,j} = s_k v_{r_k B,j}$. This means $h_{B,i} = h_{r_k B,i}$ and $h_{B,j} = h_{r_k B,j}$. Now use induction.

We are left with the case where $k \sim i \sim j$. Then $j \not\sim k$ as there are no triangles in the Dynkin diagram. Notice on the chain from α_i we start with r_k, apply r_i and can then if we wish add r_j provided r_j raises $r_i r_k B$. The chain from α_j is r_k which fixes α_j, and then we can continue with r_i and then r_j which forces $r_j r_i r_k B > r_i r_k B$ by Proposition 3.1(ii). Now $v_{B,i} = s_k s_i s_j v_{r_j r_i r_k B,k}$ and $v_{B,j} = s_k s_i s_j v_{r_j r_i r_k B,i}$. Now check that if $B' = r_j r_i r_k B$ that $h_{B,i} = h_{B',k}$ and $h_{B,j} = h_{B',i}$. Now use induction.

(iv). Suppose $r_j B > B$. Then $v_{B,i} = s_j v_{r_j B,i}$. Now conjugating r_i by $v_{B,i}$ has the same effect as conjugating $v_{r_j B,i}$ as $s_j^{-1} s_i s_j = s_i$. If $r_j B < B$, use the same argument on $r_j B$ which is raised by r_j.

(v). Assume first that $r_j B > B$. Then $v_{B,i} = s_j s_i v_{r_i r_j B,j}$. Notice $s_i^{-1} s_j^{-1} s_i s_j s_i = s_j$ and conjugating s_i by $v_{B,i}$ has the same effect as conjugating s_j by $v_{r_i r_j B,j}$ and the result follows. If $r_j B < B$, then $r_i r_j B < jB$ by Proposition 3.1(ii). Now apply the above to $r_i r_j B$. As $(\alpha_j, B) \neq 0$, we know $r_j B \neq B$ by Lemma 3.4(v).

All cases have been completed. □

Remark 4.3. For the definition of $v_{B,i}$ we have used chains (and their labels) from B to B_0 depending on i. In particular for $r_j B > B$ and $i \not\sim j$ we use $s_j v_{r_j B,i}$ and for $j \sim i$ we use $s_j s_i v_{r_i r_j B,i}$. If we were to use just any chain we would not get this unique element without some further work. For instance, if $M = \mathrm{D}_5$ and $B = \{\varepsilon_3 + \varepsilon_4, \varepsilon_1 + \varepsilon_2\}$, both α_1 and α_3 are in $B^\perp$ and $r_2 r_1$ and $r_2 r_3$ both take B to $B_0 = \{\varepsilon_1 + \varepsilon_4, \varepsilon_2 + \varepsilon_3\}$. If we use the definition here, with $v_{B,1} = s_2 s_1$, we find $h_{B,1} = (s_2 s_1)^{-1} s_1 (s_2 s_1) = s_2$. However, if we would use $v_{B,1} = s_2 s_3$, corresponding to a non-admitted chain, we find $s_3^{-1} s_2^{-1} s_1 s_2 s_3$ instead of s_2 and we would need a proper quotient of the Hecke algebra for $h_{B,1}$ to be well defined.

Corollary 4.4. *Let $(\mathcal{B}, <)$ be a monoidal poset. Retain the notation of the previous proposition. Denote C the set of all nodes j of M such that $(\alpha_j, B_0) = 0$ and Z the Hecke algebra whose type is the diagram M restricted to C. Then, for each node j in C, there is a minimal element B of $(\mathcal{B}, <)$ and a node k of M such that $(\alpha_k, B) = 0$ and $h_{B,k} = s_j$, the image of the fundamental generator of A in Z.*

Proof. The following proof is similar to the one of Lemma 3.8 of [5]. Let j be a node of C. Then $h_{B_0,j} = s_j$. Let $B \in \mathcal{B}$ be minimal such that there exists a node k with $(\alpha_k, B) = 0$ and $h_{B,k} = s_j$. Suppose there is a node i such that $r_i B < B$. If $i \not\sim k$ then by Proposition 4.2(iv) $h_{r_i B,k} = h_{B,k} = s_j$. If $i \sim k$ then by Proposition

4.2(v) $h_{r_k r_i B,i} = h_{B,k} = s_j$ and by Proposition 3.1(ii), $r_k r_i B < B$. Both cases contradict the minimal choice of B, so B must be a minimal element of $(\mathcal{B}, <)$. □

Example 4.5. Suppose M is a connected simply laced diagram. Then the type of C as defined in Corollary 4.4 is given in Table 2. We deal with two series in particular.

If $M = \mathrm{A}_{n-1}$ and $\mathcal{B}$ is the W-orbit of $\{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}\}$, then

$$\begin{aligned} B_0 &= \{\varepsilon_1 - \varepsilon_{n-p+1}, \varepsilon_2 - \varepsilon_{n-p+2}, \ldots, \varepsilon_p - \varepsilon_n\} \text{ and} \\ C &= \{\alpha_{p+1}, \alpha_{p+2}, \ldots, \alpha_{n-p-1}\}. \end{aligned}$$

Therefore, the Hecke algebra Z is of type A_{n-2p-1}.

If $M = \mathrm{D}_n$ and $\mathcal{B}$ is the W-orbit of $\{\alpha_1, \alpha_3, \ldots, \alpha_{2p-1}\}$, then

$$\begin{aligned} B_0 &= \{\varepsilon_1 + \varepsilon_{2p}, \varepsilon_2 + \varepsilon_{2p-1}, \ldots, \varepsilon_p + \varepsilon_{p+1}\} \text{ and} \\ C &= \{\alpha_p, \alpha_{2p+1}, \alpha_{2p+2}, \ldots, \alpha_n\}. \end{aligned}$$

The Hecke algebra Z has type $\mathrm{A}_1\mathrm{D}_{n-2p}$ (where D_1 is empty and $\mathrm{D}_2 = \mathrm{A}_1\mathrm{A}_1$).

5. The Monoid Action

Let $\mathcal{B}$ be an admissible W-orbit of sets of mutually orthogonal positive roots, let $(\mathcal{B}, <)$ be the corresponding monoidal poset (cf. Proposition 3.1), let B_0 be the maximal element of $(B, <)$ (cf. Corollary 3.6), and let C be the set of nodes i of M with $\alpha_i \in B_0^{\perp}$. As before (Proposition 4.2), Z is the Hecke algebra over $\mathbb{Q}(m)$ of type C. These are listed in Table 1 under column C. In analogy to the developments in [5] we define a free right Z-module V with basis x_B indexed by the elements B of $\mathcal{B}$. By Lemma 4.1 the linear transformations τ_i of (1) are completely determined. We are ready to prove the main theorem.

Proof of Theorem 1.1. Let M be connected (see a remark following the theorem). We need to show that the braid relations hold for τ_i and τ_j, that is, they commute if $i \not\sim j$ and $\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j$ if $i \sim j$.

Take $B \in \mathcal{B}$. By linearity, it suffices to check the actions on x_B. We first dispense with the case in which either $\tau_i x_B$ or $\tau_j x_B$ is 0. This happens if B contains α_i or α_j. If both roots are in B both images are 0 and the relations hold.

Suppose then that α_i is in B but α_j is not in B. Consider first the case in which $i \not\sim j$. Then $\tau_i x_B = 0$ and so $\tau_j\tau_i x_B = 0$. Now $\tau_j x_B$ is in the span of x_B and $x_{r_j B}$. Notice as $(\alpha_i, \alpha_j) = 0$ that α_i is in $r_j B$ as well as B and so $\tau_i\tau_j x_B = 0$ also. Suppose $i \sim j$. Clearly $\tau_i\tau_j\tau_i x_B = 0$ as $\tau_i x_B = 0$. As $\alpha_i \in B$, the root $\alpha_i + \alpha_j$ belongs to $r_j B$. Also r_j raises B as a height one element, α_i, becomes height 2. This means $\tau_j x_B = x_{r_i B} - m x_B$. If r_i lowers $r_j B$, $\tau_i x_{r_i B} = x_{r_i r_j B}$. But $r_i r_j B$ contains $r_i(\alpha_i + \alpha_j) = \alpha_j$, so $\tau_j x_{r_i r_j B} = 0$. Also $\tau_i x_B = 0$ as $\alpha_i \in B$. This proves the result unless τ_i raises $\tau_j B$. We know τ_i takes the root $\alpha_i + \alpha_j$ to α_j and so lowers a root of height 2. The only way r_i could raise $r_j B$ is if $r_j B$ contained an α_k with $k \sim i$. This would be $r_j\beta$ for $\beta \in B$. If $r_j\beta = \beta$ we would have $\alpha_k \in B$ but all elements of B except α_i are orthogonal to α_i. This means α_k is not orthogonal to α_j and we have $j \sim k$, $j \sim i$, and $i \sim k$ a contraction as there are no triangles in the Dynkin diagram. We conclude that the braid relations hold if either τ_i or τ_j annihilates x_B.

We now consider the cases in which $i \not\sim j$ with neither α_i nor α_j being in B. We wish to show $\tau_i\tau_j = \tau_j\tau_i$.

We suppose first that both α_i and α_j are in $B^\perp$. This means that $\tau_i x_B = x_B h_{B,i}$ and that $\tau_j x_B = x_B h_{B,j}$. We need only ensure that $h_{B,i}$ and $h_{B,j}$ commute, which is Proposition 4.2(ii).

Suppose now that α_i is in $B^\perp$ and α_j is not in $B^\perp$. In this case $\tau_j\tau_i x_B = \tau_j x_B h_{B,i}$. Also $\tau_j x_B = x_{r_jB} - \delta m x_B$ where δ is 0 or 1. This gives

$$\tau_j\tau_i x_B = (x_{r_jB} - \delta m x_B)h_{B,i}.$$

We also get $\tau_i\tau_j x_B = \tau_i x_{r_jB} - \delta m \tau_i x_B$. Notice $\alpha_i \in B^\perp$ and $i \not\sim j$ imply $\alpha_i \in (r_jB)^\perp$. In particular

$$\tau_i\tau_j x_B = x_{r_jB}h_{r_jB,i} - \delta m x_B h_{B,i}.$$

In order for this to be $\tau_j\tau_i x_B$ we need $h_{r_jB,i} = h_{B,i}$, which is satisfied by Proposition 4.2(iv).

We are left with the case in which neither α_i nor α_j is in B or in $B^\perp$. In this case the relevant actions are τ_i on x_B and τ_j on x_B. If $r_iB = r_jB$ it is clear τ_i and τ_j commute. This gives the table

τ_i on x_B	τ_j on x_B	$\tau_i\tau_j x_B = \tau_j\tau_i x_B$
lower	lower	$x_{r_ir_jB}$
lower	raise	$x_{r_ir_jB} - m x_{r_iB}$
raise	raise	$x_{r_ir_jB} - m x_{r_iB} - m x_{r_jB} + m^2 x_B$

Notice that $\alpha_i \notin (r_jB)^\perp$ as $\alpha_i \notin B^\perp$. Similarly $\alpha_j \notin (r_iB)^\perp$.

Suppose first that τ_i and τ_j both lower B. By Proposition 3.1(iii) this means τ_i also lowers r_jB and τ_j lowers r_iB. Now

$$\tau_i\tau_j x_B = \tau_i x_{r_iB} = x_{r_ir_jB}.$$

The same result occurs in the reverse order as r_i and r_j commute.

Suppose next that τ_i and τ_j both raise B. Then by Lemma 3.4(ii), τ_i raises r_jB and τ_j raises r_iB. In particular we have

$$\tau_j\tau_i x_B = \tau_j(x_{r_iB} - m x_B) = x_{r_jr_iB} - m x_{r_iB} - m x_{r_jB} + m^2 x_B.$$

The same is true for the reverse order.

Suppose then τ_i lowers B and τ_j raises B. By Lemma 3.4(i), applied to $\{\tau_iB < B < \tau_jB\}$, the reflection r_i also lowers r_jB and r_j raises r_iB. This means

$$\tau_i\tau_j x_B = \tau_i(x_{r_jB} - m x_B) = x_{r_ir_jB} - m x_{r_iB}.$$

In the other order

$$\tau_j\tau_i x_B = \tau_j x_{r_iB} = x_{r_jr_iB} - m x_{r_iB}.$$

These are the same. Notice here the assumptions imply $r_iB \neq r_jB$ and $r_ir_jB \neq B$. We conclude that τ_i and τ_j commute whenever $i \not\sim j$.

We now suppose $i \sim j$ and wish to show $\tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j$. Suppose first α_i and α_j are in $B^\perp$. Then $\tau_i x_B = x_B h_{B,i}$ and $\tau_j x_B = x_B h_{B,j}$. The condition needed is $h_{B,i}h_{B,j}h_{B,i} = h_{B,j}h_{B,i}h_{B,j}$, which is Proposition 4.2(iii).

Suppose now $i \sim j$ and $\alpha_i \in B^\perp$ but $\alpha_j \notin B^\perp$. We are still asssuming neither α_i nor α_j is in B. The relevant data here are the actions of r_j on B and r_i on r_jB. The table below handles the cases where r_j lowers B and r_i lowers r_jB as well as those where r_j raises B and r_i raises r_jB. The other cases, of r_i raising r_jB when

r_j lowers B and of r_i lowering r_jB when r_i raise B, are ruled out by Condition (ii) of Proposition 3.1.

r_i on r_jB	r_j on B	$\tau_i\tau_j\tau_i x_B$
lower	lower	$x_{r_ir_jB}h_{r_ir_jB,j} = x_{r_ir_jB}h_{B,i}$
raise	raise	$x_{r_ir_jB}h_{r_ir_jB,j} - mx_B - x_{r_jB}h_{B,i} + m^2x_Bh_{B,i}$

Notice that $\alpha_i \notin (r_jB)^\perp$ as if $(\alpha_j, \beta) \neq 0$, then $(\alpha_i, r_j\beta) = (\alpha_i, \beta - (\alpha_j, \beta)\alpha_j) = (\alpha_j, \beta) \neq 0$.

Suppose first r_j lowers B and r_i lowers r_jB as in the first row. Then

$$\tau_j\tau_i\tau_j x_B = \tau_j\tau_i x_{r_jB} = \tau_j x_{r_ir_jB} = x_{r_ir_jB}h_{r_ir_jB,j}.$$

Note here $\alpha_j \in (r_ir_jB)^\perp$ by application of r_ir_j to $\alpha_i \in B^\perp$. Also

$$\tau_i\tau_j\tau_i x_B = \tau_i\tau_j x_B h_{B,i} = \tau_i x_{r_jB}h_{B,i} = x_{r_ir_jB}h_{B,i}.$$

Now the braid relation is satisfied according to Proposition 4.2(v).

Suppose r_j raises B and r_i raises r_jB.

$$\begin{aligned}
\tau_i\tau_j\tau_i x_B &= \tau_i\tau_j x_B h_{B,i}\\
&= \tau_i(x_{r_jB} - mx_B)h_{B,i}\\
&= (x_{r_ir_jB} - mx_{r_jB} - mx_Bh_{B,i})h_{B,i}\\
&= x_{r_ir_jB}h_{B,i} - mx_{r_jB}h_{B,i} - mx_Bh_{B,i}^2\\
&= x_{r_ir_jB}h_{B,i} - mx_{r_jB}h_{B,i} - mx_B + m^2x_Bh_{B,i}
\end{aligned}$$

Here we used $h_{B,i}^2 = 1 - mh_{B,i}$. In the other order we have

$$\begin{aligned}
\tau_j\tau_i\tau_j x_B &= \tau_j\tau_i(x_{r_jB} - mx_B)\\
&= \tau_j(x_{r_ir_jB} - mx_{r_jB} - mx_Bh_{B,i})\\
&= x_{r_ir_jB}h_{r_ir_jB,j} - mx_B - m(x_{r_jB} - mx_B)h_{B,i}\\
&= x_{r_ir_jB}h_{r_ir_jB,j} - mx_B - mx_{r_jB}h_{B,i} + m^2x_Bh_{B,i}.
\end{aligned}$$

Once again we need $h_{r_ir_jB,j} = h_{B,i}$ which is Proposition 4.2(v).

We can finally consider the case in which $i \sim j$ and neither α_i nor α_j is in $B^\perp \cup B$. Here relevant data are the actions of r_i and r_j on B, where for the first row we assume $\alpha_i + \alpha_j \notin B$ (for otherwise, each side equals zero).

r_i on B	r_j on B	$\tau_i\tau_j\tau_i x_B$
lower	lower	$x_{r_ir_jr_iB}$
lower	raise	done below
raise	raise	$x_{r_ir_jr_iB} - m(x_{r_jr_iB} + x_{r_ir_jB})$
		$+m^2(x_{r_jB} + x_{r_iB}) - (m^3 + m)x_B$

We start with the first row in which both r_i and r_j lower B. We may assume $r_iB \neq r_jB$ or τ_i and τ_j act on x_B and x_{r_iB} in the same way. By Proposition 3.1(iv) and Lemma 3.2 all the actions we encounter are lowering actions. Therefore,

$$\tau_i\tau_j\tau_i x_B = \tau_i\tau_j x_{r_iB} = \tau_i x_{r_jr_iB} = x_{r_ir_jr_iB}.$$

This gives the same result with the other product.

Next take the bottom row in which both r_i and r_j raise B. By Lemma 3.4 (iii), the actions we encounter are all raising actions.

$$\begin{aligned}\tau_i\tau_j\tau_i x_B &= \tau_i\tau_j(x_{r_iB} - mx_B)\\ &= \tau_i(x_{r_jr_iB} - mx_{r_iB} - m(x_{r_jB} - mx_B))\\ &= x_{r_ir_jr_iB} - mx_{r_jr_iB} - mx_B\\ &\quad -m(x_{r_ir_jB} - mx_{r_jB}) + m^2(x_{r_iB} - mx_B)\\ &= x_{r_ir_jr_iB} - m(x_{r_jr_iB} + x_{r_ir_jB})\\ &\quad +m^2(x_{r_jB} + x_{r_iB}) - (m^3 + m)x_B.\end{aligned}$$

This also gives the same result with the other product.

We now tackle the remaining cases. Here r_i lowers B and r_j raises B. There are two cases depending on how r_j acts on r_iB.

r_i on B	r_j on B	r_j on r_iB	$\tau_i\tau_j\tau_i x_B = \tau_j\tau_i\tau_j x_B$
lower	raise	raise	$x_{r_ir_jr_iB} - mx_{r_jr_iB} - m(x_B - mx_{r_iB})$
lower	raise	lower	$x_{r_jr_ir_jB} - mx_{r_jr_iB}$

Consider first the second row, where r_j lowers r_iB. By the Lemma 3.4(iv) applied to r_ir_jB, this means r_i raises r_jr_iB and the remaining raising and lowering actions can be determined by this. Notice $r_jr_iB \neq B$, for otherwise $r_iB = r_jB$ which is not consistent with the assumption.

$$\begin{aligned}\tau_i\tau_j\tau_i x_B &= \tau_i\tau_j x_{r_iB}\\ &= \tau_i x_{r_jr_iB}\\ &= x_{r_ir_jr_iB} - mx_{r_jr_iB}\end{aligned}$$

For the other product

$$\begin{aligned}\tau_j\tau_i\tau_j x_B &= \tau_j\tau_i(x_{r_jB} - mx_B)\\ &= \tau_j(x_{r_ir_jB} - mx_{r_iB})\\ &= x_{r_jr_ir_jB} - mx_{r_jr_iB}\end{aligned}$$

These are the same as indicated in the table.

For the first row suppose next that r_j raises r_iB. By Lemma 3.4(iii) applied to r_iB, r_j raises r_iB, r_i raises r_jr_iB, r_j lowers r_ir_jB and r_i raises r_jB. Again we use $B \neq r_jr_iB$.

$$\begin{aligned}\tau_i\tau_j\tau_i x_B &= \tau_i\tau_j x_{r_iB}\\ &= \tau_i(x_{r_jr_iB} - mx_{r_iB})\\ &= x_{r_ir_jr_iB} - mx_{r_jr_iB} - m(x_B - mx_{r_iB}).\end{aligned}$$

For the other product

$$\begin{aligned}\tau_j\tau_i\tau_j x_B &= \tau_j\tau_i(x_{r_jB} - mx_B)\\ &= \tau_j(x_{r_ir_jB} - mx_{r_jB} - mx_{r_iB})\\ &= x_{r_jr_ir_jB} - mx_B - m(x_{r_jr_iB} - mx_{r_iB}).\end{aligned}$$

This gives the same for either product.

These are also the same as indicated in the table finishing the last case. In particular Theorem 1.1 has been proven. □

We expect that the representations obtained for the positive monoid A^+ by means of our Main Theorem 1.1 will be extendible to the full Artin group A. Proving this is work in progress. For type A_n, all of them are, as is clear from the BMW algebra of that type ([2, 5]).

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