Time-Crystalline Topological Superconductors

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Time crystals form when arbitrary physical states of a periodically driven system spontaneously break discrete time-translation symmetry. We introduce one-dimensional time-crystalline topological superconductors, for which time-translation symmetry breaking and topological physics intertwine—yielding anomalous Floquet Majorana modes that are not possible in free-fermion systems. Such a phase exhibits a bulk magnetization that returns to its original form after two drive periods, together with Majorana end modes that recover their initial form only after four drive periods. We propose experimental implementations and detection schemes for this new state.

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Introduction.—Periodically driven quantum systems evade certain constraints imposed in equilibrium. For instance, “time crystals” that spontaneously break time-translation symmetry in the sense envisioned in Refs. [1,2] cannot arise in equilibrium [3], yet can emerge with periodic driving. In periodically driven time crystals any physical (i.e., noncat) state evolves with a subharmonic of the drive frequency [4–6]. The canonical realization consists of disordered Ising spins that collectively flip after each drive period, thereby requiring two periods to recover their initial state. Experiments have detected signatures of time crystallinity both in driven cold atoms [7,8] and solid-state spin systems [9–11].

As a second, deeply related example, consider a one-dimensional (1D) free-fermion topological superconductor hosting Majorana end modes [12], each described by a Hermitian operator $\gamma$. If $\gamma$ adds energy $E$ then $\gamma^\dagger$ adds $-E$, while Hermiticity requires that these be equivalent. In equilibrium the unique solution is $E = 0$—corresponding to the well-studied Majorana zero modes. Periodically driving with frequency $\Omega$ additionally permits “Floquet Majorana modes” carrying $E = \Omega/2$ since energy is then only conserved mod $\Omega$ [13]. Floquet Majorana modes have been proposed to facilitate more efficient quantum information processing compared to equilibrium systems [14–16]. Moreover, they encode a topological flavor of time-translation symmetry breaking in that Floquet Majorana operators change sign each drive cycle, thus also requiring two periods to recover their initial form.

We merge the phenomena above by exploring periodically driven 1D topological superconductors generated upon coupling Cooper-paired electrons to doubled-periodicity time-crystalline Ising spins. Such “time-crystalline topological superconductors” intertwine bulk time-translation symmetry breaking and topological physics—yielding anomalous quadrupled-periodicity Floquet Majorana modes that categorically cannot arise in free-fermion platforms. We propose implementation via quantum-dot arrays (see Fig. 1) reminiscent of setups utilized in Refs. [17–19] for engineering equilibrium Majorana zero modes. We derive and analyze an exactly solvable, physically intuitive model for time-crystalline topological superconductivity and show that probing junctions between time-crystalline and static topological superconductors reveals the Floquet Majorana modes’ quadrupled periodicity.

Model and setup.—Time-crystalline topological superconductors closely relate to equilibrium topological superconductors that spontaneously violate electronic time-reversal symmetry $T$, which importantly satisfies $T^2 = -1$. We thus begin by modeling the latter. Our setup,
sketched in Fig. 1, consists of a superconductor coupled to a
chain of quantum dots indexed by sites \( j \), each hosting one
active spinful level described by operators \( f_{j\sigma} \) (\( \sigma = \uparrow, \downarrow \)
denotes spin, which we implicitly sum over whenever
suppressed); we assume that charging energy is quenched by
coupling to the superconductor and can thus be neglected.
A chain of Ising spins described by Pauli matrices \( m_j^z \) resides
proximate to the quantum-dot array. We model the setup
with a \( T \)-symmetric Hamiltonian \( H = H_0 + H_f \), where

\[
H_0 = \sum_j \left( -J m_j^z m_{j+1}^z - K m_j^z f_j^\dagger \sigma f_j \right),
\]

\[
H_f = \sum_j \left[ \mu f_j^\dagger f_j - t (f_j^\dagger f_{j+1} + \text{H.c.}) \right] + \alpha (f_j^\dagger \sigma f_{j+1} + \text{H.c.} + \Delta (f_j^\dagger f_{j+1} + \text{H.c.}) \right].
\]

In \( H_0 \), \( J > 0 \) ferromagnetically couples neighboring Ising
spins and \( K > 0 \) couples the Ising and dot spins. Terms in \( H_f \)
describe the chemical potential (\( \mu \)), hopping (\( t \)), spin-orbit
coupling (\( \alpha \)), and proximity-induced pairing (\( \Delta \)) for
the quantum-dot electrons.

Suppose that the \( K \) term dominates and energetically
enforces alignment of each electron spin with the nearest
Ising spin. Only one of the two spinful levels in each dot
remains active at low energies—effectively creating a
system of spinless fermions described by operators

\[
c_j = \frac{1}{2} \left( (1 + m_j^z) f_j^\dagger + (1 - m_j^z) f_{j+1} \right).
\]

as Fig. 1 illustrates. Time-reversal \( T \) sends \( m_j^z \rightarrow -m_j^z \) and
c \( \rightarrow m_j^z c_j \), thus satisfying time-reversal symmetry. This
intertwining between spinless fermions and Ising spins is
unavoidable; without it, \( c_j \) has no way of acquiring the
required minus sign upon two applications of \( T \).

In the Supplemental Material [20], we project \( H \) onto the
spinless-fermion subspace by integrating out high-energy
fermionic modes, yielding an effective Hamiltonian

\[
H_{\text{eff}} = \sum_j \left[ -J m_j^z m_{j+1}^z - \mu' c_j^\dagger c_j + \left( t'_{m_j^z m_{j+1}^z} c_j^\dagger c_{j+1} + \Delta'_{m_j^z m_{j+1}^z} c_j c_{j+1} + \text{H.c.} \right) \right].
\]

Here \( \mu' = -(K + \mu) \) is a renormalized chemical
potential, while \( t'_{m_j^z m_{j+1}^z} = a + \alpha m_j^z m_{j+1}^z \) and \( \Delta'_{m_j^z m_{j+1}^z} = \beta m_j^z - b m_j^z m_{j+1}^z \) denote Ising-spin-dependent effective hopping and
\( p \)-wave pairing amplitudes, with \( a = -(t + i\alpha)/2 \) and
\( b = (-t + i\alpha)\Delta/(K - \mu) \). The real part of \( a \) sets the
hopping strength between sites with aligned Ising spins,
which is directly inherited from spin-conserving tunneling
in Eq. (2); the imaginary part similarly fixes the hopping
when Ising spins antialign, which is instead mediated by
spin-orbit coupling \( \alpha \). Pairing in \( H_{\text{eff}} \) follows from second-
order processes that involve virtual excitations out of the
spinless-fermion subspace—hence the \( K - \mu \) energy
denominator in \( b \). Depending on the Ising configuration,
either spin-conserving hopping or spin-orbit coupling
virtually creates a doubly occupied site of \( f \) fermions that
then Cooper pair via the original \( s \)-wave \( \Delta \) term, effectively
mediating \( p \)-wave pairing of spinless fermions.

Phase diagram.—Equation (4) describes a strongly
interacting system of Ising spins and fermions. Nevertheless,
for any given Ising configuration the model reduces to free
fermions. Consider first uniformly polarized all-up or all-
down Ising spins. Here Eq. (4) maps to the familiar Kitaev
chain \[12\] with uniform hopping strength \( 2 |a| \cos \phi_a \) and
pairing \( \pm 2i |b| \sin \phi_b \), where \( a = \frac{1}{2} |a|e^{i\phi_a} \) and \( b = \frac{1}{2} |b|e^{i\phi_b} \).
(Our derivation above yielded \( \phi_a = \phi_b \), though it will be
useful to now keep these phases independent.) Accordingly,
the chain hosts edge Majorana zero modes provided the
chemical potential intersects the band and pairing is finite,
\( i.e. \), for \( |\mu'| < 4 |a| |\cos \phi_a| \) and \( \sin \phi_b \neq 0 \) as sketched in
Fig. 2(a).

To examine the fermionic ground state with random
Ising spins—which is our main interest—we compute the
correlation length \( \xi \) using the transfer-matrix technique;
see, e.g., Ref. [21] and the Supplemental Material [20].
This method allows us to map out phase boundaries by
numerically searching for diverging \( \xi \) as we vary \( \phi_a, b \); for
our purposes a regular \( 400 \times 400 \) grid of \( \phi_a \) and \( \phi_b \) values in
the interval \( [-\pi/2, \pi/2] \) is sufficient. [Exploiting

FIG. 2. Phase diagram for Eq. (4) assuming (a) fully polarized and
(b) random Ising spins. In (a) a nonzero chemical potential
\( \mu' = |a| \) generates the trivial phase, and the system is gapless
along the thick black lines. Data in (b) were generated from
transfer-matrix simulations at \( \mu' = |b| = |a|/4 \) with \( 10^6 \) sites.
Data points indicate sharp peaks in the localization length, as
expected at a topological phase transition. The red diagonal line
\( \phi_a = \phi_b \) is relevant for the physical quantum-dot setup from
Fig. 1. As the dashed arrow illustrates, the topological phase
along this line can be deformed to the zero-correlation-length
limit with \( \phi_a = \pi/4 \), \( \phi_b = -\pi/4 \) (and also \( a = b \), \( \mu' = 0 \))
without crossing a phase boundary. Increasing the magnitude
of \( \mu' \) tends to thicken the trivial regions, while altering the relative
magnitudes of \( |a| \) and \( |b| \) shifts the boundaries separating the
topological and trivial phases.


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even when the zero-mode wave functions extend over
modes the Supplemental Material [20] shows that Eq. (7) holds
Majorana zero modes. The topological degeneracy of the
other reflects topological degeneracy encoded in the
factor of 2 arises because
states are at least fourfold degenerate in this limit: one
indicate local maxima where \( \xi \) is typically of order \( 10^2 \)
or larger, while it is of order unity elsewhere. We expect
these peaks to represent true divergences in \( \xi \) when \( \phi_a \) or
\( \phi_b \) are tuned continuously in the thermodynamic limit.
Topological regions are easily identified via exact diagno-
salization on smaller systems and confirming the presence
of edge Majorana zero modes. In the Supplemental Material
we analytically capture the topological phase for a restricted window of \( \phi_{a,b} \) via the Born approximation.

For our quantum-dot setup, we expect \( \phi_a = \phi_b \) [red line in
Fig. 2(b)] and also \( |a| \gg |b| \) since \( p \)-wave pairing
encoded in \( b \) appears at second order in perturbation
theory. Starting from the topological phase in this physical
regime, Fig. 2(b) strongly suggests that we can deform
parameters to \( \phi_a = \pi/4 \) and \( \phi_b = -\pi/4 \), \( |a| = |b| \), and
\( \mu' = 0 \) without encountering a divergent \( \xi \). (See the
Supplemental Material [20] for additional evidence.)
This special point corresponds to the model’s zero-correla-
tion-length-limit. Here it is convenient to decompose the
spinless fermions in terms of Majorana operators \( \eta_{A,B} \) via
\[
c_j = e^{-i(\pi/4)m_j} (\eta_{Bj} + i\eta_{Aj}),
\]
whereupon Eq. (4) becomes
\[
H'_{\text{eff}} = \sum_j \left( -Jm_j^2 m_{j+1}^2 - ik\delta m_j m_{j+1} (\eta_{Aj} + \eta_{Bj+1}) \right),
\]
with \( s_m m_j = (1 - m_i + m_j + m_j m_j)/2 = \pm 1 \) and \( \kappa = 4 \sqrt{2} |a| \).
For any choice of \( m_j \)s the Majorana operators dimerize
nontrivially as shown in Fig. 1, yielding Majorana zero
modes
\[
\begin{align*}
\gamma_1 &\equiv \eta_{B1} = e^{i(\pi/4)m_1} c_1 + \text{H.c.}, \\
\gamma_2 &\equiv \eta_{A1} = -i e^{i(\pi/4)m_1} c_1 + \text{H.c.},
\end{align*}
\]
at the leftmost and rightmost sites. Notice the spin-fermion
intertwinement inherent in the zero modes, which con-
sequently evolve under \( T \) via
\[
\gamma_1 \to m_1^2 \gamma_1, \quad \gamma_2 \to -m_1^2 \gamma_2,
\]
again consistent with \( T^2 = -1 \). All Hamiltonian eigen-
states are at least fourfold degenerate in this limit: one
factor of 2 arises because \( T \) flips all Ising spins, while
the other reflects topological degeneracy encoded in the
Majorana zero modes. The topological degeneracy of the
fermionic ground states given a static Ising configuration
persists even away from the special limit examined above,
due to the finite gap for fermionic excitations. Moreover,
the Supplemental Material [20] shows that Eq. (7) holds
even when the zero-mode wave functions extend over
many sites.

\textbf{Adiabatic cycle.}—Next we generalize Eq. (1) to
\[
H'_0 = \sum_j \left[ -J(\hat{n} \cdot \hat{m}_j)(\hat{n} \cdot \hat{m}_{j+1}) - K(\hat{n} \cdot \hat{m}_j)f_j^\dagger n \cdot \sigma f_j \right],
\]
where \( \hat{m}, \sigma \) denote vectors of Pauli matrices and the unit
vector \( \hat{n} \equiv \cos \theta \hat{z} + \sin \theta \hat{y} \) determines the easy axis for the
Ising spins. At either \( \theta = 0 \) or \( \pi \), \( H'_0 \) reduces to Eq. (1).

Suppose that we again deform to the zero-correla-
tion-length limit (which is possible for any \( \theta \)) and then imple-
ment the following cycle: (i) start with an arbitrary Ising
spin configuration at \( \theta = 0 \), (ii) initialize the fermions into
one of the topological-superconductor ground states, and
finally (iii) adiabatically rotate the easy axis by winding \( \theta 
\)
from 0 to \( \pi \).

Although the Hamiltonian returns to its original form,
the wave functions do not. Rather, the cycle slowly rotates
all Ising spins by \( \pi \), while the fermions follow their
instantaneous minimum-energy configuration given the
adiabaticity. The initial ground state thereby transforms
into its time-reversed counterpart. One rotation sends
\( m_j^+ \to -m_j^+ \), \( f_j \to e^{i(\pi/2)\sigma} f_j \), and hence \( c_j \to i c_j \). Majorana
zero modes thus transform as \( \gamma_1 \to m_1^2 \gamma_1 \) and \( \gamma_2 \to m_1^2 \gamma_2 \),
similar to the action of \( T \). Interestingly, two cycles return
the Ising spins to their original form whereas four cycles are
required to recover the initial zero-mode operators, e.g.,
\[
\gamma_1 \to m_1^2 \gamma_1 \to -\gamma_1 \to -m_1^2 \gamma_1 \to \gamma_1.
\]

\textbf{Time-crystalline topological superconductivity and
detection.}—We now promote the adiabatic ground-state
phenomenon described above to a dynamic phenomenon
applicable to \textit{arbitrary} physical states. To this end we apply
a variation of the preceding cycle periodically with period
\( T \), thus generating time-crystalline topological supercon-
ductivity. We specifically consider a binary drive such that
the Floquet operator that evolves the system over a single
period reads
\[
U_T = e^{-i(\pi/2-\epsilon)(m_1^+ c_1^\dagger)} e^{-iH_{\text{eff}}^\text{dis}}.
\]
The right exponential evolves the system with respect to a
disorder, static Hamiltonian \( H_{\text{eff}}^\text{dis} \) that is the same as
Eq. (4) but with \( J, a, b \) replaced with random site-
dependent couplings \( J_j, a_j, b_j \). We neglect randomness
in the phases of \( a_j, b_j \) and treat \( J_j, a_j, b_j \) as independent
random variables with magnitudes drawn from uniform
distributions \( [\tilde{J} - \delta J, \tilde{J} + \delta J], [a - \delta a, a + \delta a], [b - \delta b, b + \delta b] \).
Disorder crucially introduces many-body localization (MBL)
into the dynamics and prevents heating to infinite
water\textsuperscript{22}. The left exponential in Eq. (10)
performs an instantaneous “kick” that (at least approxi-
ately) flips the Ising spins via a transverse magnetic field.
response, whereas the Floquet Majorana modes globally flips all Ising spins, yielding doubled-periodicity bulk response, whereas the Floquet Majorana modes exhibit quadrupled-periodicity response that can be probed in the junction with the static topological superconductor on the right. The inner Majorana modes hybridize with coupling strength \( \lambda \). Since \( \gamma_3 \) is static while \( \gamma_2 \) evolves nontrivially after each period \( T \), the junction’s energy inherits the latter’s quadrupled periodicity.

pulse and applies a potential to thespinless fermions—thereby mimicking evolution from our adiabatic cycle without the adiabaticity requirement.

The dynamics is analytically tractable at \( \epsilon = 0 \) and when \( H_{\text{eff}}^{\text{dis}} \) reduces to Eq. (5) with random couplings \( J_j, \kappa_j \). Starting from any Ising configuration, the “perfect” kick in \( U_T \) sends \( m_i^j \rightarrow -m_i^j \) and thus flips all spins, signifying period-doubling time crystallinity in the spin sector. In the fermionic sector, \( \gamma_{1,2} \) in Eq. (6) continue to commute with \( H_{\text{eff}}^{\text{dis}} \) despite the randomness. The kick, however, nontrivially transforms the Majorana edge operators so that \( U_T \gamma_1 U_T^\dagger = m_i^j \gamma_1 \) and \( U_T \gamma_2 U_T^\dagger = m_i^j \gamma_2 \). Precisely as illustrated in Eq. (9), \( \gamma_{1,2} \) therefore require four drive periods to recover their initial form, i.e., they form the hallmark quadrupled-periodicity Floquet Majorana modes. Shaded regions of Fig. 3 summarize the evolution.

Quadrupled periodicity can be experimentally probed in junctions between time-crystalline and static topological superconductors as in the right side of Fig. 3, wherein \( \gamma_3 \) and \( \gamma_4 \) denote time-independent Majorana zero modes. Electron tunneling across the junction couples \( \gamma_2 \) with \( \gamma_3 \), producing a Hamiltonian term \( H_{23} = \lambda \gamma_2 \gamma_3 \) for some \( \lambda \) that may depend on the adjacent Ising spins. Consequently, the junction’s energy density (among other local properties) directly manifests the quadrupled-periodicity built into the anomalous Floquet Majorana mode \( \gamma_2 \).

FIG. 3. Time evolution for the time-crystalline topological superconductor generated by Eq. (10) at \( \epsilon = 0 \). Each period \( T \) globally flips all Ising spins, yielding doubled-periodicity bulk response, whereas the Floquet Majorana modes exhibit quadrupled-periodicity response that can be probed in the junction with the static topological superconductor on the right. The inner Majorana modes hybridize with coupling strength \( \lambda \). Since \( \gamma_3 \) is static while \( \gamma_2 \) evolves nontrivially after each period \( T \), the junction’s energy inherits the latter’s quadrupled periodicity.

FIG. 4. Fourier transform of the quantities shown in the legend following time evolution via Eq. (10) with \( \epsilon = 0.2 \) and parameters specified in the main text. Data are normalized by setting the maximum of each Fourier spectrum to 1, and frequency \( \omega \) on the horizontal axis is normalized by \( \Omega = 2\pi/T \), with \( T \) the drive period. Here \( m_{i10} \) represents an Ising spin at the center of the chain, \( c_0 \) is an auxiliary zero-energy static fermion that enables probing the Floquet Majorana mode periodicity, and \( c_1 \) is the fermion at the left end of the quantum-dot chain. For initialization we use random Ising configurations and random fermionic states that entangle \( c_0 \) with the rest of the system. Runs were repeated 150 times for disorder averaging with maximum bond dimension \( \chi = 50 \); similar results were obtained with \( \chi = 25 \). For \( aT = 2 \) sharp peaks persist at \( \Omega/2 \) and \( 3\Omega/4 \)—despite “imperfect” driving generated by \( \epsilon \neq 0 \)—indicating “rigid” doubled-periodicity Ising spins and quadrupled-periodicity Floquet Majorana modes characteristic of time-crystalline topological superconductivity. For \( aT = 0.2 \), the imperfect drive pushes the peak frequencies away from these quantized values, indicating a loss of rigid time crystallinity.

Rigidity against “imperfect” drives is a crucial feature of time-crystalline phases [4–6,27]. Here, such imperfection arises from taking \( \epsilon \neq 0 \) and \( H_{\text{eff}}^{\text{dis}} \) away from the zero-correlation-length limit, which spoils exact solvability and prompts us to turn to numerics.

**Numerics.**—We employ time-evolving block decimation (TEBD), using a maximum bond dimension of \( \chi = 50 \), on a 20-site system with random Ising spins and parameters appropriate for our quantum-dot setup: \( \phi_a = \phi_b = \pi/8 \), \( b = \bar{a}/2 \), \( \bar{J} = \bar{a}/4 \), \( \mu' = 0 \), \( \delta a = \delta b = \delta \bar{a}/8 \). Our simulations incorporate a decoupled, static zero-energy fermion \( c_0 \) that functions similarly to the static topological superconductor in Fig. 3. We initialize into a state that entangles the static fermion with the rest of the system. We then simulate the Floquet operator in Eq. (10) with \( aT = 2 \) and \( \bar{a}T = 0.2 \), and with the kick shifted away from commensurability by \( \epsilon = 0.2 \) [28]. Despite the rather small system size, in both cases the bond dimension quickly saturated, and the truncation error was relatively coarse. To check robustness of our numerics we repeated the computations for maximum bond dimension \( \chi = 25 \), and the results agreed with those at \( \chi = 50 \).
Over a run of 60 Floquet evolutions and 150 disorder averages, we measure the Ising spin \( \langle m_j^{(10)} \rangle \) in the middle of the system as well as \( \langle c_0^+c_1 \rangle \), where \( c_1 \) corresponds to the leftmost quantum dot. The former probes bulk time crystallinity while the latter probes the Floquet Majorana modes. Figure 4 plots the Fourier transform of both quantities as a function of frequency \( \omega \) normalized by \( \Omega = 2\pi/T \). For \( \bar{a}T = 2 \) the data show the rigidity characteristic of a time crystal: despite the imperfect drive, the bulk magnetization and edge fermion bilinear, respectively, remain peaked at \( \omega = \Omega/2 \) and \( \omega = 3\Omega/4 \) (as expected for doubled-periodicity Ising spins and quadrupled periodicity Floquet Majorana modes). By contrast, in our \( \bar{a}T = 0.2 \) simulations both peaks clearly shift due to nonzero \( \epsilon \), indicating an absence of rigid time-crystallinity for this case. We also ran exact numerics on a seven-site system and measured the level-spacing statistics of the \( U_T \) eigenvalues. At \( \bar{a}T = 2 \) the mean level spacing was approximately 0.39, close to the Poisson value 0.386 expected for MBL [29].

Discussion.—The admixture of symmetry breaking and topology is known to generate new physics in static systems; examples include \( 8\pi \)-periodic Josephson effects [30,31] and enrichment of Majorana braiding and fusion [32]. Our work establishes that driven systems can be similarly enriched by “decorating” topological phases with spontaneous time-translation symmetry breaking. We specifically showed that 1D time-crystalline topological superconductors engineered from quantum-dot arrays host novel Floquet Majorana modes that display anomalously long periodicity not possible with free fermions. Exotic states of this type are not captured by the cohomology classification of interacting topological Floquet phases [33–37]. Our work opens up the possibility of harnessing time crystals to enrich other “designer” phases of matter. One could envision promoting spinless fermions to spinful fermions coupled to magnetic degrees of freedom in systems such as driven spinless 2D \( p + ip \) superconductors [38–40]. Subtleties regarding MBL in two dimensions can be avoided by focusing on prethermal regimes, possibly leading to new higher-dimensional adiabatic cycles and time-crystalline topological phases.

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[28] Calculations were performed using the ITensor Library, http://itensor.org.