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FAIRNESS AND EFFICIENCY FOR PROBABILISTIC ALLOCATIONS WITH ENDOWMENTS

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Abstract

We propose to use endowments as a policy instrument in market design, so as to ensure that agents have the right to enjoy certain resources. For example, in school choice, endowments can guarantee children a chance of admission to high-quality schools. We introduce a notion of justified envy adapted to allocation problems with endowments, and show that fairness (understood as the absence of justified envy) can be obtained together with efficiency and individual rationality.

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1 Introduction

School choice is the problem of allocating children to schools when we want to take into account children's (or their parents') preferences. Several large US school districts have in the last 15 years implemented school choice programs that follow economists' recommendation and are based on economic theory.[1] Practical implementation of school choice programs presents us with a number of lessons and challenges.

The first lesson is that school choice should be guided by fairness, or *justified fairness*. When given the choice of implementing either a fair or an efficient outcome, school districts have consistently chosen fairness (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005; Abdulkadiroğlu, Pathak, and Roth, 2005). One reason could be that district administrators are concerned with litigation: if Alice prefers the school that student Bob was allocated to, meaning that she envies Bob's allocation, then the district can invoke justified envy to argue as a defense that Bob had a higher priority than Alice at the school in question. It is also likely that district administrators, and society as a whole, have an intrinsic preference for fairness. Such a preference for fairness is important enough to outweigh efficiency.

The second lesson is that school districts have a strong preference for controlling the racial and socio-economic composition of their schools: so-called *controlled school choice*. A common critique of the experience in last 15 years is that the new school choice programs have led to undesirable school compositions. For example, in Boston, schools have been left with too few neighborhood children, which has motivated a move away from the system recommended by economists (Dur, Kominers, Pathak, and Sönmez, 2017). In New York City, the new school choice system exhibits high degrees of racial segregation. Segregation in NYC schools is arguably not new, but the complaint is that the new school choice program may have made it worse, and certainly has not helped.

[1] Boston (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005), New York (Abdulkadiroğlu, Pathak, and Roth, 2005), and Chicago (Pathak and Sönmez, 2013) are the leading examples.

In the words of a recent New York Times article "...school choice has not delivered on a central promise: to give every student a real chance to attend a good school. Fourteen years into the system, black and Hispanic students are just as isolated in segregated high schools as they are in elementary schools — a situation that school choice was supposed to ease.[2]" The article points to a dissatisfaction with school composition, and access to the best schools.

The situation in NYC has reached a point where there are talks of doing away with school priorities, and instead instituting a lottery. In fact, Professor Eric Nadelstern at Columbia University, who served as deputy school chancellor when the new school choice system was implemented, has recently proposed that children be allowed to apply to any school, and have a lottery decide the allocations.[3]

Our paper seeks to make Nadelstern's approach compatible with school choice. We imagine that there is a lottery that gives an *initial* probabilistic allocation of children to schools. The lottery could be as simple as giving each child the same chance of attending any school. It could also reflect different objectives in controlled school choice, such as giving each child a higher chance of attending his or her neighborhood school, or giving each minority child a chance (literally, a positive probability) of attending the highest-ranked schools. Our model takes as primitive an *arbitrary and given* probabilistic allocation of children to schools.

The initial allocation is typically not the final allocation, because we want preferences to play a role. Therefore, we construct an exchange economy by regarding the initial probabilistic allocation of each child as his or her endowment. Our main finding is that, under reasonably general conditions, we can always find a probabilistic allocation that takes into account children's preferences so as to exhaust all the possible gains from trade (efficiency), gives each child an allocation that is as least as good as his or her initial endowment (individual rationality), and guarantees that no child justifiably envies another (no justified-envy).

Two aspects of our paper should be emphasized. First, we assume that the preference relation of each child over probabilistic allocations is represented by a continuous utility function and sometimes by an expected utility function. Most papers in the literature often only assume ordinal preferences, and probabilistic allocations are compared by stochastic dominance. Second, we formalize a new notion of fairness that is meaningful in probabilistic allocation problems with endowments. The standard definition of fairness in the literature is often based on priorities, and makes no sense in environments with endowments. Moreover, priority-based fairness is often incompatible with the objective of efficiency.

The idea behind our fairness notion is simple. In our model, each child has the right to obtain her initially endowed probabilistic allocation. Any deviation from the initial

[2] "The Broken Promises of Choice in New York City Schools", *New York Times*, May 5th, 2017.
[3] See "Confronting Segregation in New York City Schools", *New York Times*, May 15th, 2017.

endowment must reflect her preferences. Endowments enshrine *property rights*: no child would accept an allocation that she regards as worse than her initial endowment. Now, consider an allocation in which a child, say Alice, prefers, or envies, the assignment of another child, say Bob, to her assignment. We say that Alice's envy towards Bob is not justified if, if they were to switch assignments, then Bob's property rights would be violated. That is, if Bob prefers his initial endowment to Alice's assignment.

There are two motivations behind our fairness notion. First, our fairness notion is analogous to the standard definition of fairness based on priorities, because priorities can also be thought of as granting property rights. In the presence of priorities, Alice's envy towards Bob is regarded as not justified if Bob has a higher priority than Alice at the school he is assigned to. Let us consider the effect of switching Alice's and Bob's assignments in a model with priorities. If the switch makes them both better off, then the allocation in the market must not be efficient. However, recall that our no-justified envy is compatible with efficiency. So to make a proper analogy let us consider an efficient allocation. If Alice envies Bob, then efficiency demands that Bob must regard Alice's assignment as worse than his own. This means that Bob ranks his assignment at which he has a high priority over Alice's. Think of Bob's higher priority over Alice as a property right that Bob has relative to Alice for his assignment. This means that the switch would give Bob an assignment that is worse than the school at which Bob has property right. The argument is therefore analogous to our notion of fairness.

Second, as discussed above, fairness serves as a defense against litigation. If Alice envies Bob and wants to bring the matter to court, the most plausible remedy she could offer is for the two of them to switch assignments: the district should give Alice's assignment to Bob, and Bob's assignment to Alice. In an environment with priorities, the district would counter-argue that Bob has a higher priority than Alice at Bob's assignment. In the absence of priorities, our fairness notion enables district administrators to argue that such a switch is not possible because Alice's assignment is not acceptable to Bob. Note that the standard for rejecting the switch is independent of the allocation proposed: Bob's endowment was fixed before the allocation was determined, and the district has to respect that Bob has a right to insist on his endowment. Therefore, Bob cannot receive Alice's assignment, which is worse for him than his endowment.[4]

In addition to no justified envy, we define a weaker notion called *no strong justified envy* and a stronger notion called *no ϵ-justified envy* for any $\epsilon > 0$. Our main theorem states that (1) when children's utility functions are continuous, there exists a weak Pareto optimal allocation that is individually rational and satisfies no strong justified envy, or is ϵ-individually rational and satisfies no ϵ-justified envy for any $\epsilon > 0$; (2) when children's utility functions are expected utilities, weak Pareto optimality can be strengthened to Pareto optimality in the above result. The gap between strong and weak Pareto optimality is particularly important in our environment, for reasons that we explain below. The proof is an application of the Knaster-Kuratowski-Mazurkiewicz lemma, which has

[4]Of course, it is possible that the initial assignment of endowments is itself subject to litigation, but that is a possibility in the model with priorities as well.

long tradition in economic theory, but has not be used (to our knowledge) extensively in matching theory.

Before moving to the related literature section we want to emphasize that controlling school choice by way of endowment has the benefit of being very transparent in the possibilities that it guarantees for each child. Lotteries are familiar objects, and they are easy to interpret. It will be clear to the families participating in the market that they can opt for their endowed probabilistic allocation. It is, however, a change of focus from the standard ideas in controlled school choice, where the final composition of the school is the focus.

Our model is about access to schools; arguably about equality of opportunities. In that sense, it is telling that the New York Times article we quoted from earlier talks about giving students a chance to attend the best schools. It views the undesirable school composition as a reflection of the lack of chances. In our model, the final composition of the school may differ from the initial allocations. Probably in most cases, it will differ substantially from the initial allocation. Arguably the difference is desirable because it reflects efficiency and children's preferences, while respecting individual rationality and fairness.

Related literature. The problem of controlled school choice was first analyzed formally by Abdulkadiroğlu and Sönmez (2003), the paper that introduced school choice as a mechanism design problem. The literature continued with Kojima (2012), Hafalir, Yenmez, and Yildirim (2013), Ehlers, Hafalir, Yenmez, and Yildirim (2014), and Echenique and Yenmez (2015). None of these papers, however, use endowments as a way to control school choice. By using endowments we take the position that what matters is *access*, not the final composition of the schools. It may be that the final outcome is more segregated than desired by the district, but the segregation would be the result of agents' preferences.

Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017) is the only paper we are aware of that also uses initial endowments to control school choice. They assume that each child owns one seat of some school as endowment. Their goal is to design strategy-proof allocation mechanisms to meet the distributional constraint in the market and individual rationality constraint of each child. Since they consider deterministic endowments and ordinal preferences, and their fairness notions are based on priorities, their results are unrelated to ours.

Our notion of justified envy is analogous to the fairness notion of Yılmaz (2010). Yilmaz uses first-order stochastic dominance instead of utility functions, and says that Alice justifiably envies Bob if she does not regard her allocation as first-order stochastically dominating Bob's, while any object that she obtains with positive probability in her allocation is regarded by Bob as acceptable. An important difference between Yilmaz's paper and ours is that endowments are deterministic in his model and probabilistic in ours. Finally, Yilmaz studies the probabilistic serial rule (Bogomolnaia and Moulin, 2001), and as a consequence his results are simply unrelated to ours.

Our paper is related in spirit to studies of allocation using markets and prices, such as Varian (1974), Hylland and Zeckhauser (1979), Budish (2011), Ashlagi and Shi (2015), He, Miralles, Pycia, Yan, et al. (2015) and He, Li, and Yan (2015). These authors explore markets with exogenously given budgets. When all agents have equal budgets, there can be no envy in a competitive equilibrium. But equal budgets of course eliminate any role for the initial endowments in the same blow as they eliminate envy. Our paper instead allows for agents to retain the option of consuming their (maybe unequal) initial endowments, and can be seen as an attempt to endogenize budgets. Indeed, our no justified-envy allocations can be obtained as market equilibrium outcomes after budget transfers (in an application of the second welfare theorem). But transfers, and therefore budgets, are set endogenously so as to achieve individual rationality and absence of justified envy.

2 The model

Our model is essentially the standard model of an exchange economy in general equilibrium theory. The difference with the standard model is that agents consume lotteries: consumption bundles cannot add up to more than one. This difference is far from minor. For example, it results in the non-existence of Walrasian equilibrium, even for economies that are otherwise well-behaved (Hylland and Zeckhauser, 1979). In fact, there are no known equilibrium existence results that apply to our model. We shall also argue that it means that strong Pareto optimality is a substantally stronger property than weak Pareto optimality. In contrast, under standard assumptions in general equilibrium theory, the two properties are equivalent.

Notation. When X is a finite set of cardinality n, $\Delta_-(X) \subseteq \mathbf{R}^n$ denotes the set $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j \leq 1\}$ while $\Delta(X) \subseteq \mathbf{R}^n$ denotes the simplex $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j = 1\}$.

Model. A *discrete allocation problem* is a tuple $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$, where:

- $S = \{s_k\}_{k=1}^m$ is a set of indivisible objects.
- $I = \{i\}_{i=1}^n$ is a set of agents, each of whom demands exactly one copy of an object.
- $Q = \{q_s\}_{s \in S}$ is a capacity vector, and $q_s \in \mathbb{N}$ is the number of copies of object s. For simplicity, we assume that $\sum_{s \in S} q_s = n$, i.e., the number of copies of objects is equal to the number of agents.
- For each agent i, $u^i : \Delta_-(S) \to \mathbf{R}$ is a continuous utility function defined on $\Delta_-(S)$. The function u^i is *expected utility* if it is linear.

- For each agent i, $\omega^i \in \Delta(S)$ is i's endowment vector such that ω^i_s is the fraction of object s owned by i. We assume that all objects are owned by agents. So $\sum_{i=1}^n \omega^i = Q$.

Allocations and Pareto optimality. An *allocation* is a vector $x \in \mathbf{R}_+^{mn}$, which we write as $x = (x^i)_{i=1}^n$, with $x^i \in \mathbf{R}_+^m$, such that

$$\sum_{i \in I} x^i_s \leq q_s \text{ and } \sum_{s \in S} x^i_s \leq 1$$

for all $i \in I$ and all $s \in S$. When $x^i_s \in \{0, 1\}$ for all i and all s, x is a deterministic allocation. The Birkhoff-von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953) states that every allocation is a convex combination of deterministic allocations.

An allocation x is *acceptable* to agent i if $u^i(x^i) \geq u^i(\omega^i)$; x is *individually rational* (IR) if it is acceptable to all agents. We also define a notion of approximate individual rationality: for any $\epsilon > 0$, x is *ϵ-individually rational* (ϵ-IR) if $u^i(x^i) \geq u^i(\omega^i) - \epsilon$ for all i.

The notion of efficiency comes in two flavors: An allocation x is *weak Pareto optimal* (wPO) if there is no allocation y such that $u^i(y^i) > u^i(x^i)$ for all i, and is *Pareto optimal* (PO) if there is no allocation y such that $u^i(y^i) \geq u^i(x^i)$ for all i and $u^j(y^j) > u^j(x^j)$ for some j. In our model, the difference between wPO and PO is very significant because of the constraint that each x^i cannot add up to more than 1. This means that wPO is compatible with wasteful situations where we can use existing resources to make some agents strictly better off, but cannot construct an allocation that makes all agents strictly better off because there are agents that have achieved the largest possible quantities of their most preferred goods.

Fairness. As introduced before, we regard agents as having the right to consume their endowments. So agents have the right to be at least as well off as they would be by consuming their endowments. Our fairness notion is based on the idea that if an agent i envies another agent j in an allocation x (that is, i prefers x^j to x^i), then switching their allocations must violate the right of j mentioned above (that is, j prefers ω^j to x^i). As discussed in Introduction, this fairness notion parallels the standard definition of fairness in priority-based allocation problems, and provides an argument for social planner to defend any possible complaint from any agent.

Formally, we say an agent i has *justified envy* towards another agent j at an allocation x if

$$u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq u^j(\omega^j).$$

We say x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x.

We explore some simple implications of NJE. In an IR and NJE allocation x, if $u^i = u^j$ and $u^i(\omega^i) \geq u^j(\omega^j)$, then it must be that $u^i(x^i) \geq u^j(x^j)$.[5] That is, if two agents i, j have equal preferences and i's endowment is weakly better than j's, then i's allocation in x is also weakly better than j's. In particular, if $u^i = u^j$ and $u^i(\omega^i) = u^j(\omega^j)$, then it must be that $u^i(x^i) = u^j(x^j)$. So NJE and IR imply *equal treatment of equals* (also called symmetry by Zhou, 1990).

We also define two variants of justified envy: one is stronger than justified envy, while the other is weaker. We say i has a *strong justified envy* (SJE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j)$. For any $\epsilon > 0$, we say i has an *ϵ-justified envy* (ϵ-JE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j) - \epsilon$. *No strong justified envy* (NSJE) and *no ϵ-justified envy* (NϵJE) are defined similarly as before. It is easy to see that

$$\text{no } \epsilon\text{-justified envy} \Longrightarrow \text{no justified envy} \Longrightarrow \text{no strong justified envy}$$

Finally, a *Walrasian equilibrium with transfers* is a tuple (x, p, t) such that:

1. x is an allocation, $p = (p_s)_{s \in S} \in \mathbf{R}^m_+$ is a price vector, and $t = (t^i)_{i \in I} \in \mathbf{R}^n$ is a transfer vector;
2. x^i maximizes i's utility within his budget:
$$x^i \in \text{argmax}\{u^i(y^i) : y^i \in \mathbf{R}^m_+,\ \sum_{s \in S} y^i_s \leq 1,\ p \cdot y^i \leq p \cdot \omega^i + t^i\};$$
3. total transfers are balanced: $\sum_{i \in I} t^i = 0$.

3 Main Result

Theorem 1. *I. Let $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$ be a discrete allocation problem.*

(a) There exists an allocation that is individually rational, weak Pareto optimal and has no strong justified envy;

(b) For any $\epsilon > 0$, there exists an allocation that is ϵ-individually rational, weak Pareto optimal and has no ϵ-justified envy.

II. Let $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$ be a discrete allocation problem in which all utility functions u^i are expected utility.

(a) There exists an allocation x that is individually rational, Pareto optimal and has no strong justified envy;

[5] If $u^i(x^i) < u^j(x^j)$, then i's envy towards j is justified because $u^j(x^i) = u^i(x^i) \geq u^i(\omega^i) \geq u^j(\omega^j)$.

(b) *For any $\epsilon > 0$, there exists an allocation x that is ϵ-individually rational, Pareto optimal and has no ϵ-justified envy.*

Moreover, there are a price vector $p \in \mathbf{R}^m_+$ and a transfer vector $t \in \mathbf{R}^n$ such that (x, p, t) is a Walrasian equilibrium with transfers.

In Part II of the theorem, when a desirable allocation x is found, the existence of a price vector p and a transfer vector t to support x as a Walrasian equilibrium allocation with transfers is an instance of the second welfare theorem. The difference with the classical result is that budget sets here are subject to the satiation constraint $\sum_{s \in S} x^i_s \leq 1$.[6] Miralles and Pycia (2014) have recently shown a version of the second welfare theorem for a more general setting than ours. We state the result here because it provides economic content for our result, and because the proof in our setting is particularly simple.

We should emphasize that transfers must be carefully chosen so as to obtain individual rationality and no justified envy. One could use transfers so that all agents have the same incomes, and therefore face the same budget sets. In an equal-income market equilibrium, there would be no envy (Varian, 1974), justified or not. The equal-income approach would, however, eliminate any role for initial endowments: see our discussion in page 5.

Finally, while transfers ensure individual rationality, an agent's endowment may not be affordable in his post-transfer budget set. The following example shows that the presence of transfers is necessary.

Example 1. *There are four agents $\{1, 2, 3, 4\}$ and four objects $\{s_1, s_2, s_3, s_4\}$. Each object has one copy. The von-Neumann-Morgenstern utilities are as shown in the following table:*

	$u^i_{s_1}$	$u^i_{s_2}$	$u^i_{s_3}$	$u^i_{s_4}$
1	*1*	*2*	*3*	*100*
2	*2*	*1*	*3*	*100*
3	*100*	*3*	*1*	*2*
4	*100*	*3*	*2*	*1*

Suppose that each agent i is endowed with a probability 1 of attending school s_i. Consider the allocation defined by:

	$x^i_{s_1}$	$x^i_{s_2}$	$x^i_{s_3}$	$x^i_{s_4}$
1	*0*	*0*	*1/2*	*1/2*
2	*0*	*0*	*1/2*	*1/2*
3	*1/2*	*1/2*	*0*	*0*
4	*1/2*	*1/2*	*0*	*0*

[6] As remarked in the Introduction, this is far from a minor deviation from the standard model of an exchange economy.

Note that the allocation x is individually rational, Pareto optimal, and satisfies no strong justified envy. Suppose that we want to support it as a competitive equilibrium allocation without transfers. Let p_k be the equilibrium price of s_k, then we have $1/2p_3 + 1/2p_4 \leq p_1, p_2$ and $1/2p_1 + 1/2p_2 \leq p_3, p_4$. Therefore, $p_1 = p_2 = p_3 = p_4$. However, with these prices agents $1, 2$ would spend all of their budgets on s_4, while agents $3, 4$ would spend all of their budgets on s_1. So (x, p) is not a competitive equilibrium.

If transfers are allowed, it is easy to support x as an competitive equilibrium allocation: let $p_1 = p_4 = 2$, $p_2 = p_3 = 0$, $t^1 = t^4 = -1$, and $t^2 = t^3 = 1$. Observe that in this case, agent 1 cannot afford ω^1, as his post-transfer income is 1 and $p \cdot \omega^1 = 2$. The allocation is, nevertheless, individually rational.

4 Application to School Choice

In this section we discuss the application of our model to school choice. We show that by properly designing the initial endowments of students, we can achieve many goals in school choice. This is in contrast to the mainstream approach of designing priorities in the literature.

Egalitarian school choice. If a school district wants to implement an egalitarian school choice in which no student is favored ex-ante, then a natural solution is to give students equal fractions of the seats of each school as initial endowments. That is, each student i owns an endowment vector $\omega^i = (\frac{q_s}{|I|})_{s \in S}$. Then there exits an allocation with the desirable properties stated in Theorem 1. In particular, individual rationality here implies *equal-division lower bound*.[7] Egalitarianism here refers to equality of opportunities. The allocation after preferences being taken into account can be very different from a uniform distribution.

Respecting neighborhood priority. Suppose in a school district each student lives in the neighborhood of one school, and the number of seats of each school equals the number of students in its neighborhood. If the district wants to guarantee that each student is able to attend his neighborhood school if he or she wants, then a natural solution is to give each student a seat in his neighborhood school as initial endowment. That is, for each student i, $\omega^i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where $\omega^i_s = 1$ if and only if s is the neighborhood school of i.

This special endowment structure may remind the reader of the Top Trading Cyle (TTC) mechanism. Here we emphasize that the allocation of TTC may not satisfy NJE. For example, suppose there are three students i, j, k with distinct endowments. i, k most

[7] See Thomson (1987); an allocation x satisfies *equal division lower bound* if $u^i(x^i) \geq u^i(e)$ where $e = (\frac{q_s}{|I|})_{s \in S}$, exactly as stated here.

prefer j's endowment, i least prefers his own endowment, and j most prefers i's endowment. TTC will let i, j trade their endowments and let k keep his endowment. However, in this allocation k has justified envy towards i since his endowment is acceptable to i.

Affirmative action. Suppose there are two types of students: majority and minority. If a school district wants to implement affirmative action for minority students, it can give each minority student some fractions of popular schools in their initial endowments. This guarantees that minority students have chances to attend popular schools if they so desire, and if some of them give up their chances, they do so in exchange for more favorable allocations.

Distributional constraints. Some districts may have distributional goals in the composition of its schools. For example, in an ideal composition of each school, each racial or ethnic group may have a given percentage in the target composition. As we stated before, such a goal is hard to achieve through our approach. While the initial endowment may reflect group quotas, the final allocation results from students exchanging allocations may be quite different from the initial endowment.

5 An example of envy between agents with identical endowment.

We present an example of a discrete allocation problem in which all agents have expected utility preferences, together with an allocation that is individually rational, Pareto optimal, and satisfies no strong justified envy. In the example, one agent envies another agent even though they have equal endowments.

The example matters for two reasons. First, because one may think that no-envy among agents with equal endowments is intrinsically desirable. After all, we have tied the notion of justified envy to endowments; we have insisted on fairness by "controlling for endowments." The idea behind the example, the explanation for what makes the example work is, however, straightforward, and illustrates that endowments are not the end of the story. The two agents in question have equal endowments, but they have different preferences. Through their preferences, the two agents play very different roles in the economy. Other agents "trade" with the two agents in question, and the outcome can be explained through such trades. Because the agents' preferences are different, they interact with the remaining agents in very different ways. Hence it results in envy. Put differently, an agent can be valuable to others because she has a very desirable endowment, or because she is willing to trade in ways that enhance the welfare of others. The example we present in this section illustrates the role of preferences in generating value.

The second reason for why the example is important is that it suggests that our notion of fairness may fail to be incentive compatible. We have not specified a selection mechanism, and opted not to discuss incentives and strategy-proofness, but the example conveys some insights. One agent envies another even though they have equal endowments. This fact suggests that one agent may want to pretend to be the agent that he envies. In a large economy, in which the number of agents who report each type of preference does not change very much after a misreport, it stands to reason that such a misreport would not be profitable. Of course, the example we present here falls short of proving that if we were to define a fair mechanism it would not be strategy proof.

Example 2. *In the example there are five agents, labeled* $i = 1, \ldots, 5$, *and three schools,* s_1, s_2 *and* s_3. *There are two copies (seats) of schools* s_2 *and* s_3. *There is only one copy of school* s_1. *In the example, all the "action" involves agents 1 and 2. The remaining three agents are, in a sense, residual; they are also identical.*

The agents' von-Neumann-Morgenstern utilities are as described in the following table:

i	$u^i_{s_1}$	$u^i_{s_2}$	$u^i_{s_3}$
1	*3*	*1*	*2*
2	*3*	*2*	*1*
3	*2*	*3*	*1*
4	*2*	*3*	*1*
5	*2*	*3*	*1*

The agents' endowments are:

i	$\omega^i_{s_1}$	$\omega^i_{s_2}$	$\omega^i_{s_3}$
1	*0*	*1*	*0*
2	*0*	*1*	*0*
3	*1/3*	*0*	*2/3*
4	*1/3*	*0*	*2/3*
5	*1/3*	*0*	*2/3.*

Observe that agents 1 and 2 have identical endowments.

Finally, consider the following allocation x:

i	$x^i_{s_1}$	$x^i_{s_2}$	$x^i_{s_3}$
1	*0*	*0*	*1*
2	*1/2*	*0*	*1/2*
3	*1/6*	*2/3*	*1/6*
4	*1/6*	*2/3*	*1/6*
5	*1/6*	*2/3*	*1/6*

Observe that agent 1 envies agent 2, as

$$u^1 \cdot x^1 = 2 < 3/2 + 2/2 = u^1 \cdot x^2.$$

The envy is not justified, however, as

$$u^2 \cdot x^1 = 1 < 2 = u^2 \cdot \omega^2.$$

In fact, it is easy to see that x has no strong justified envy.

It is also easy to see that the allocation x is individually rational and Pareto optimal. In any PO allocation y, we cannot have $y^1_{s_2} > 0$, as agent 1 and any agent $j \in \{3, 4, 5\}$ are willing to trade school 2 for any other school. So y^1 must be a convex combination of $(1, 0, 0)$ and $(0, 0, 1)$. To make agent 1 better off then we would need to give agent 1 some shares in school 3, but these can only come at the expense of agent 2. To make agent 2 better off, she would need to get more shares in school 3, but these can only come at the expense of agents 3,4 and 5. These agents could only exchange shares in school 3 for shares in school 2, which agent 2 does not have. All agents 2, 3, 4 and 5 rank schools 3 and 1 in the same way.

6 Proof

We first present the proof of Theorem 1 (IIa) and (IIb) by assuming that every agent's utility function is an expected utility function. Then we comment on how the proof can be adapted to prove Theorem 1 (Ia) and (Ib).

We define some notations. Let Δ denote the simplex in $\mathbf{R}^n$, that is, $\Delta = \{\lambda \in \mathbf{R}^n : \sum_{i=1}^n \lambda^i = 1, \lambda^i \geq 0\}$. For any $\lambda \in \Delta$, let $\text{supp}(\lambda) = \{i \in \{1, \dots, n\} : \lambda^i > 0\}$. Let Δ^o be the interior of Δ, that is, $\Delta^o = \{\lambda \in \Delta : \lambda^i > 0 \text{ for all } i \in \{1, \dots, n\}\}$. Let $\mathcal{A}$ be the set of allocations. It is easy to see that $\mathcal{A}$ is compact and convex.

Define a correspondence $\psi : \Delta \rightrightarrows \mathcal{A}$ with

$$\psi(\lambda) = \text{argmax}\{\sum_{i=1}^n \lambda^i u^i(x^i) : (x^i)_{i=1}^n \in \mathcal{A}\}.$$

That is, $\psi(\lambda)$ is the set of allocations that maximize the sum of agents' utilities weighted by λ. Then in the following lemma we prove that an allocation x is Pareto optimal if and only if $x \in \psi(\lambda)$ for some $\lambda \in \Delta^o$.

Lemma 1. *$\psi(\Delta^o)$ is a closed set and it coincides with the set of Pareto optimal allocations.*

Proof. We first show that $\overline{\psi(\Delta^o)} \subseteq \psi(\Delta^o)$. Let x_ℓ be a sequence in $\psi(\Delta^o)$ with $x_\ell \to x \in \mathcal{A}$. Choose $\lambda_\ell \in \Delta^o$ with $x_\ell \in \psi(\lambda_\ell)$. Note that $\sum_{i=1}^n x^i_{s_k,\ell} = q_k$ and $\sum_{k=1}^m x^i_{s_k,\ell} = 1$ for all i, s_k, and ℓ. The Kuhn-Tucker conditions for x_ℓ to maximize $\sum_{i=1}^n \lambda^i_\ell u^i(x^i_\ell)$ require that there exist $\theta_\ell \in \mathbf{R}^n_+$ and $\eta_\ell \in \mathbf{R}^m_+$ for each ℓ such that

$$\lambda^i_\ell u^i_{s_k} - \theta^i_\ell - \eta_{s_k,\ell} \begin{cases} = 0 & \text{if } x^i_{s_k,\ell} > 0, \\ \leq 0 & \text{if } x^i_{s_k,\ell} = 0. \end{cases}$$

For each i and s_k such that $x^i_{s_k} > 0$, let $N(i, s_k)$ be such that if $\ell \geq N(i, s_k)$ then $x^i_{s_k,\ell} > 0$. Let N be such that

$$N > \max\{N(i, l) : x^i_{s_k} > 0\}.$$

Let $\lambda = \lambda_N$, $\theta = \theta_N$ and $\eta = \eta_N$. Then x satisfies the Kuhn-Tucker conditions for a solution to maximize $\sum_{i=1}^n \lambda^i u^i(x^i)$. Therefore, $x \in \psi(\lambda)$. Since $\lambda \gg 0$, $\lambda \in \Delta^o$.

Now we prove that $\psi(\Delta^o)$ is the set of Pareto optimal allocations. It is obvious that all allocations in $\psi(\Delta^o)$ are Pareto optimal, so we need to prove that if x is a Pareto optimal allocation, then there is some $\lambda \in \Delta^o$ with $x \in \psi(\lambda)$.

If x is Pareto optimal, for any agent i, x solves the problem of maximizing $u^i(y^i)$ subject to 1) $u^j(y^j) \geq u^j(x^j)$ for all $j \neq i$, and 2) $y \in \mathcal{A}$. Then by the Kuhn-Tucker theorem there exist $\mu^t(i) \geq 0$ for each agent t, $\theta(i) \in \mathbf{R}^n_+$ and $\eta(i) \in \mathbf{R}^m_+$ such that

$$\mu^t(i)u^t_{s_k} - \theta^t(i) - \eta_{s_k}(i) \begin{cases} = 0 & \text{if } x^t_{s_k} > 0, \\ \leq 0 & \text{if } x^t_{s_k} = 0, \end{cases}$$

where $\mu^i(i) = 1$. Then by taking the sum of the above inequalities, for each agent t we have

$$u^t_{s_k} \sum_{i=1}^n \mu^t(i) - \sum_{i=1}^n \theta^t(i) - \sum_{i=1}^n \eta_{s_k}(i) \begin{cases} = 0 & \text{if } x^t_{s_k} > 0, \\ \leq 0 & \text{if } x^t_{s_k} = 0. \end{cases} \tag{1}$$

Define

$$\lambda^t = \frac{\sum_{i=1}^n \mu^t(i)}{\sum_{t=1}^n \sum_{i=1}^n \mu^t(j)}.$$

Since $\mu^t(t) = 1$, $\lambda^t > 0$ and therefore $\lambda \in \Delta^o$. Then by choosing $\theta^t = \frac{\sum_{i=1}^n \theta^t(i)}{\sum_{t=1}^n \sum_{i=1}^n \mu^t(j)}$ and $\eta_{s_k} = \frac{\sum_{i=1}^n \eta_{s_k}(i)}{\sum_{t=1}^n \sum_{i=1}^n \mu^t(j)}$, Equation 1 becomes the Kuhn-Tucker conditions for x to maximize $\sum_{i=1}^n \lambda^i u^i(x^i)$. So $x \in \psi(\lambda)$. □

Define

$$\mathcal{A}^* = \{x \in \psi(\Delta^o) : x \text{ is individually rational}\}.$$

Since the set of individually rational allocations is closed, by Lemma 1 $\mathcal{A}^*$ is nonempty and compact.[8] Similarly, for any given $\epsilon > 0$, define

$$\mathcal{A}^{**}_\epsilon = \{x \in \psi(\Delta^o) : x \text{ is } \epsilon\text{-individually rational}\}.$$

$\mathcal{A}^{**}_\epsilon$ is also nonempty and compact.

[8]It is easy to show that $\mathcal{A}^*$ is nonempty. The endowment allocation w is individually rational. If it is not Pareto optimal, it must be dominated by a Pareto optimal allocation x. Then x is individually rational.

By Berg's Maximum Theorem, there exist a continuous function $\phi^* : \Delta \to \mathcal{A}^*$ and a continuous function $\phi^{**}_\epsilon : \Delta \to \mathcal{A}^{**}_\epsilon$ such that[9]

$$\phi^*(\lambda) \in \text{argmax}\{\sum_{i=1}^{n} \lambda^i u^i(x^i) : (x^i)_{i=1}^n \in \mathcal{A}^*\},$$

$$\phi^{**}_\epsilon(\lambda) \in \text{argmax}\{\sum_{i=1}^{n} \lambda^i u^i(x^i) : (x^i)_{i=1}^n \in \mathcal{A}^{**}_\epsilon\}.$$

For any agent i, define

$$C^{i*} = \{\lambda \in \Delta : \nexists j \in I \text{ s.t } i \text{ has a strong justified envy towards } j \text{ at } \phi^*(\lambda)\},$$
$$C^{i**}_\epsilon = \{\lambda \in \Delta : \nexists j \in I \text{ s.t } i \text{ has an } \epsilon\text{-justified envy towards } j \text{ at } \phi^{**}_\epsilon(\lambda)\}.$$

In the following two lemmas we prove that $\{C^{i*}\}_{i=1}^n$ and $\{C^{i**}_\epsilon\}_{i=1}^n$ are two KKM coverings of the simplex Δ.

Lemma 2. *For every $i \in I$, C^{i*} and C^{i**}_ϵ are closed.*

Proof. We first prove that C^{i*} is closed. Let λ_n be a sequence in C^{i*} such that $\lambda_n \to \lambda \in \Delta$. Let $x_n = \phi^*(\lambda_n)$. By continuity, $x_n \to x = \phi(\lambda) \in \mathcal{A}^*$. Now we prove that $\lambda \in C^{i*}$, that is, i does not have a strong justified envy towards any other agent. Suppose to the contrary that there is an agent j such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j)$. Since u^i and u^j are continuous, for n large enough we have $u^i(x^j_n) > u^i(x^i_n)$ and $u^j(x^i_n) > u^j(\omega^j)$, which contradicts that i has no strong justified envy at x_n.

Similarly, let λ_n be a sequence in C^{i**}_ϵ such that $\lambda_n \to \lambda \in \Delta$. Let $x_n = \phi(\lambda_n)$, then $x_n \to x = \phi(\lambda) \in \mathcal{A}^*$. Suppose there is an agent j such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j) - \epsilon$. Since u^i and u^j are continuous, for n large enough we have $u^i(x^j_n) > u^i(x^i_n)$ and $u^j(x^i_n) > u^j(\omega^j) - \epsilon$, which contradicts that i has no ϵ-justified envy at x_n. Therefore, $\lambda \in C^{i**}$. □

Lemma 3. *For every $\lambda \in \Delta$, $\lambda \in \cup_{i \in supp(\lambda)} C^{i*}$ and $\lambda \in \cup_{i \in supp(\lambda)} C^{i**}_\epsilon$.*

Proof. (1) We prove that $\lambda \in \cup_{i \in \text{supp}(\lambda)} C^{i*}$. Suppose to the contrary that for some $\lambda \in \Delta$, $\lambda \notin \cup_{i \in \text{supp}(\lambda)} C^{i*}$. Let $x = \phi^*(\lambda)$. Then for every $i \in \text{supp}(\lambda)$ there exists some j such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(w^j)$.

[9]More formally, if we define $\Phi(\lambda) = \text{argmax}\{\sum_{i=1}^n \lambda^i u^i(x^i) : (x^i)_{i=1}^n \in \mathcal{A}^*\}$, then Berg's Maximum theorem implies that Φ is upper hemi-continuous. We state that there exists a continuous function ϕ^* such that $\phi^*(\lambda) \in \Phi(\lambda)$. ϕ^* is continuous if and only if for any λ and any $\epsilon > 0$, there exists $\delta > 0$ such that if $\| \lambda - \lambda' \| < \delta$, then $\| \phi(\lambda) - \phi(\lambda') \| < \epsilon$. Suppose ϕ^* does not exist, then it means that for any $\delta > 0$ and $\| \lambda - \lambda' \| < \delta$, we have $\| x - x' \| > \epsilon$ for any $x \in \Phi(\lambda)$ and $x' \in \Phi(\lambda')$, which contradicts the upper hemi-continuity of Φ. The existence of ϕ^{**}_ϵ can be proved similarly.

Suppose first that there exists some j mentioned above such that $j \notin \text{supp}(\lambda)$. Then consider an allocation y in which i, j exchange their allocations in x, and the other agents keep their allocations in x. Then y is individually rational and $\sum_{i=1}^n \lambda^i u^i(x^i) < \sum_{i=1}^n \lambda^i u^i(y^i)$. By definition of ϕ^*, y cannot be Pareto optimal. So there is a Pareto optimal allocation $z \in \mathcal{A}$ that Pareto dominates y. So $z \in A^*$ and $\sum_{i=1}^n \lambda^i u^i(x^i) < \sum_{i=1}^n \lambda^i u^i(z^i)$, which contradicts the definition of x.

Therefore, every $i \in \text{supp}(\lambda)$ has a strong justified envy towards some $j \in \text{supp}(\lambda)$. Given that the set of agents in $\text{supp}(\lambda)$ is finite, there is a cycle $i_1, \ldots i_K$ in $\text{supp}(\lambda)$ such that i_1 has a strong justified envy towards i_2, i_2 has a strong justified envy towards i_3, and so on until i_K has a strong justified envy towards i_1. Then by letting the agents in the cycle exchange their allocations, we obtain a new allocation that Pareto dominates x, which is a contradiction.

(2) We prove that $\lambda \in \cup_{i \in \text{supp}(\lambda)} C_\epsilon^{i**}$. Suppose to the contrary that for some $\lambda \in \Delta$, $\lambda \notin \cup_{i \in \text{supp}(\lambda)} C_\epsilon^{i**}$. Let $x = \phi_\epsilon^{**}(\lambda)$. Then for every $i \in \text{supp}(\lambda)$ there exists some j such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(w^j) - \epsilon$. By using similar arguments as above we can prove that $j \in \text{supp}(\lambda)$. Then since the set of agents in $\text{supp}(\lambda)$ is finite, there is a cycle $i_1, \ldots i_K$ in $\text{supp}(\lambda)$ such that i_1 has an ϵ-justified envy towards i_2, i_2 has an ϵ-justified envy towards i_3, and so on until i_K has an ϵ-justified envy towards i_1. Then by letting the agents in the cycle exchange their allocations we obtain a new allocation that Pareto dominates x, which is a contradiction. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The proof is an application of the Knaster-Kuratowski-Mazurkiewicz lemma: see Theorem 5.1 in Border (1989). By Lemmas 2 and 3, $\{C^{i*}\}_{i=1}^n$ and $\{C_\epsilon^{i**}\}_{i=1}^n$ are two KKM coverings of Δ. So there are $\lambda^* \in \cap_{i=1}^n C^{i*}$ and $\lambda_\epsilon^{**} \in \cap_{i=1}^n C_\epsilon^{i**}$. Let $x^* = \phi^*(\lambda^*)$ and $x_\epsilon^{**} = \phi_\epsilon^{**}(\lambda_\epsilon^{**})$. Then x^* is individually rational, Pareto optimal and has no strong justified envy, and x_ϵ^{**} is ϵ-individually rational, Pareto optimal and has ϵ-justified envy. Below we prove that they can be supported as Walrasian equilibrium allocations with transfers.

Since x^* is Pareto optimal, by Lemma 1 there is $\lambda \in \Delta^o$ such that x^* is a solution to maximizing $\sum_{i=1}^n \lambda^i u^i(x^{*i})$. Then Kuhn-Tucker conditions require that there exist $\theta \in \mathbf{R}_+^n$ and $\eta \in \mathbf{R}_+^m$ such that

$$\lambda^i u_{s_k}^i - \theta^i - \eta_{s_k} \begin{cases} = 0 & \text{if } x_{s_k}^{*i} > 0, \\ \leq 0 & \text{if } x_{s_k}^{*i} = 0. \end{cases}$$

Let $\alpha^i = \theta^i / \lambda^i$, $p = \eta$, and $\beta^i = 1/\lambda^i$. Let $t_i = p \cdot x^i - p \cdot \omega^i$. Then we have

$$u_{s_k}^i - \alpha^i - \beta^i p_{s_k} \begin{cases} = 0 & \text{if } x_{s_k}^{*i} > 0, \\ \leq 0 & \text{if } x_{s_k}^{*i} = 0, \end{cases}$$

which are the Kuhn-Tucker conditions for x^* to maximize $\sum_{k=1}^m u^i_{s_k} \cdot y^i_{s_k}$ subject to (1) $p \cdot y^i \leq p \cdot w^i + t^i = p \cdot x^{*i}$ and (2) $\sum_{k=1}^m y^i_{s_k} = 1$. Moreover, since $\sum_{i=1}^n x^{*i} = \sum_{i=1}^n \omega^i$, $\sum_{i=1}^n t^i = 0$. So (x^*, p, t) is a Walrasian equilibrium with transfers. Similar arguments can be applied to x^{**}_ϵ.

For Theorem 1 (Ia) and (Ib), we redefine $\mathcal{A}^*$ and A^{**}_ϵ as

$$\mathcal{A}^* = \{x \in \mathcal{A} : x \text{ is individually rational and weak Pareto optimal}\},$$

$$\mathcal{A}^{**}_\epsilon = \{x \in \mathcal{A} : x \text{ is } \epsilon\text{-individually rational and weak Pareto optimal}\}.$$

Since the set of weak Pareto optimal allocations is closed,[10] $\mathcal{A}^*$ and A^{**}_ϵ are compact. Then $\phi^*, \phi^{**}_\epsilon$ and C^{i*}, C^{i**}_ϵ can be constructed as before. Lemma 2 still holds since it only uses the continuity of utility functions. Lemmas 3 can be easily adapted. After proving that there is a cycle $i_1, \dots i_K$ in supp(λ) such that i_1 has a strong (ϵ-) justified envy towards i_2, i_2 has a strong (ϵ-) justified envy towards i_3, and so on until i_K has a strong (ϵ-) justified envy towards i_1, we can construct a new allocation y by letting agents in the cycle exchange their allocations. Since x is (ϵ-) individually rational and weak Pareto optimal, y must be also (ϵ-) individually rational and weak Pareto optimal. But we have $\sum_{i=1}^n \lambda^i u^i(y^i) > \sum_{i=1}^n \lambda^i u^i(x^i)$, which is a contradiction. □

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[10] Let $\{x_n\}$ be a sequence of weak Pareto optimal allocations and $x_n \to x$. Suppose there is an allocation y such that $u^i(y^i) > u^i(x^i)$ for all i. Then for big enough n we have $u^i(y^i) > u^i(x^i_n)$ for all i, which is a contradiction.

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