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FAIRNESS AND EFFICIENCY FOR PROBABILISTIC ALLOCATIONS WITH ENDOWMENTS

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Abstract

We propose to use endowments as a policy instrument in market design. Endowments give agents the right to enjoy certain resources. For example in school choice, one can ensure that low-income families have a shot at high-quality schools by endowing them with a chance of admission.

We introduce two new criteria in resource allocation problems with endowments. The first adapts the notion of justified envy to a model with endowments, while the second is based on market equilibrium. Using either criteria, we show that fairness (understood as the absence of justified envy, or as a market outcome) can be obtained together with efficiency and individual rationality.

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1 Introduction

Motivation: school choice and lotteries. School choice is the problem of allocating children to schools when we want to take into account children's (or their parents') preferences. Several large US school districts have in the last 15 years implemented school choice programs that follow economists' recommendation and are based on economic theory.[1] Practical implementation of school choice programs presents us with a number of lessons and challenges.

The first lesson is that school choice should be guided by fairness, or *justified fairness*. When given the choice of implementing either a fair or an efficient outcome, school districts have consistently chosen fairness (Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005; Abdulkadiroğlu, Pathak, and Roth, 2005). One reason could be that district administrators are concerned with litigation: if Alice prefers the school that student Bob was allocated to, meaning that she envies Bob's allocation, then the district can invoke justified envy to argue as a defense that Bob had a higher priority than Alice at the school in question. It is also likely that district administrators, and society as a whole, have

[1] Boston (Abdulkadiroğlu and Sönmez, 2003; Abdulkadiroğlu, Pathak, Roth, and Sönmez, 2005), New York (Abdulkadiroğlu, Pathak, and Roth, 2005), and Chicago (Pathak and Sönmez, 2013) are the leading examples.

an intrinsic preference for fairness. Such a preference for fairness is important enough to outweigh efficiency.

The second lesson is that school districts have a strong preference for controlling the racial and socio-economic composition of their schools: so-called *controlled school choice*. A common critique of existing school choice programs is that they have led to undesirable school compositions. For example, in Boston, schools have been left with too few neighborhood children, which has motivated a move away from the system recommended by economists (Dur, Kominers, Pathak, and Sönmez, 2017). In New York City, the new school choice system exhibits high degrees of racial segregation. Segregation in NYC schools is not new, but the complaint is that the new school choice program may have made it worse, and certainly has not helped. In the words of a recent New York Times article "...school choice has not delivered on a central promise: to give every student a real chance to attend a good school. Fourteen years into the system, black and Hispanic students are just as isolated in segregated high schools as they are in elementary schools — a situation that school choice was supposed to ease.[2]" The article points to a dissatisfaction with school composition, and access to the best schools.

The situation in NYC has reached a point where there are talks of doing away with school priorities, and instead instituting a lottery. In fact, Professor Eric Nadelstern at Columbia University, who served as deputy school chancellor when the new school choice system was implemented, has recently proposed that children be allowed to apply to any school, and have a lottery decide the allocations.[3]

Our paper seeks to make Nadelstern's approach compatible with school choice. We imagine that there is a lottery that gives an *initial* probabilistic allocation of children to schools. The lottery could be as simple as giving each child the same chance of attending any school. It could also reflect different objectives in controlled school choice, such as giving each child a higher chance of attending his or her neighborhood school, or giving each minority child a chance (literally, a positive probability) of attending the

[2] "The Broken Promises of Choice in New York City Schools", *New York Times*, May 5th, 2017.

[3] See "Confronting Segregation in New York City Schools", *New York Times*, May 15th, 2017.

highest-ranked schools. Our model takes as primitive an *arbitrary and given* probabilistic allocation of children to schools.

The initial allocation is typically not the final allocation, because we want preferences to play a role. Therefore, we construct an exchange economy by regarding the initial probabilistic allocation of each child as his or her endowment. Our main finding is that, under reasonably general conditions, we can always find a probabilistic allocation that takes into account children's preferences so as to exhaust all the possible gains from trade (efficiency), gives each child an allocation that is as least as good as his or her initial endowment (individual rationality), and guarantees that no child justifiably envies another, according to our notions of justified envy.

Our contribution. We investigate the meaning of fairness in the presence of endowments. Endowments mean that agents start from unequal initial positions, and this inequality may be reflected in the final outcome. We do not want to say that an outcome is unfair if its unfairness can be traced to differences in the allocation of endowments. For example, suppose that Alice and Bob are endowed with seats at two schools. The final allocation of seats depends on agents' preferences. If Alice envies Bob's final allocation, but her endowment was worse than Bob, then we may be willing to tolerate her envy. Such inequity could simply be the product of an unequal starting point: Alice had a much worse endowment than Bob to start with. In other words, the challenge is to make sense of *fairness among unequally endowed agents.* We say that envy, or inequality, among agents is justified if it can be traced to the agents' differing endowments.

We investigate two different notions of fairness. The first notion, termed *no justified envy*, is based on property rights defined by agents' endowments: an agent has the right to be at least as well off as she would be by consuming her endowments. The starting point is the standard notion of fairness in matching and school choice. In the standard notion of fairness, school priorities define property rights. We present a notion of no justified envy where property rights defined by endowments play the role of the property rights defined by priorities.

Our second notion uses markets, and the price mechanism, as a basis for determining fair allocations. When agents have the same endowments, market equilibria are fair, in the sense that no agent envies another agent (Varian, 1974). We extend the idea that market equilibrium prices guarantee fair outcomes to a model where agents' endowments may not be the same.

Importantly, our notions of fairness are compatible with efficiency. In the standard model of school choice and matching, fairness and efficiency cannot generally both be obtained. This will not be true in our model. We now proceed to explain our notions of fairness in some more detail, and to briefly describe our main results.

Absence of justified envy is, as said, based on our understanding of endowments as guaranteeing property rights. Each agent has the right to obtain her initially endowed probabilistic allocation, and any deviation from her initial endowment must reflect her preferences. Endowments therefore enshrine certain property rights: no agent would accept an allocation that she regards as worse than her initial endowment. Now, consider an allocation in which Alice envies Bob, meaning that she prefers Bob's assignment to her own. We say that Alice's envy towards Bob is not justified if, if they were to switch assignments, then Bob's property rights would be violated. That is, Alice's envy towards Bob is not justified if Bob prefers his initial endowment to Alice's assignment.

Our fairness notion is analogous to the standard definition of fairness based on priorities, because priorities can also be thought of as granting property rights. In the presence of priorities, Alice's envy towards Bob is regarded as not justified if Bob has a higher priority than Alice at the school he is assigned to. Let us consider the effect of switching Alice's and Bob's assignments in a model with priorities. If the switch makes them both better off, then the allocation in the market must not be efficient. However, recall that our no-justified envy is compatible with efficiency. So to make a proper analogy let us consider an efficient allocation. If Alice envies Bob, then efficiency demands that Bob must regard Alice's assignment as worse than his own. This means that Bob ranks his assignment, at which he has a higher priority than Alice, over Alice's. Think of Bob's higher priority over Alice as a property right that Bob has relative to Alice for his assignment. This means that the switch would give Bob an assignment that is worse than the school at which Bob has property rights. The notion of fairness with priorities is therefore analogous to our notion of fairness.[4]

Our second notion of fairness is based on markets and prices. When all endowments are equal, a market equilibrium is fair because no agent can envy any other agent. We extend this idea to an economy with endowments. In equilibrium, Alice may envy Bob, but only if Bob's endowment is more valuable (at equilibrium market prices) than Alice's. In particular, when all endowments are the same, they have to be equally valuable, and therefore no envy is possible. But more generally, we may think of the market process itself as being fair because any envy can be traced to how endowments are valued. We also provide a connection between market prices and agents' preferences: we show that, under certain hypotheses, if Alice envies Bob then a coalition of agents (a coalition that includes Alice) would rather have more of Bob's endowments than of Alice's.

Our market equilibrium solution is a hybrid of the standard equal-income market solution, and classical Walrasian equilibrium. Agents' expenses in our market must be debited against a budget constraint that is a weighted average of a fixed income, and an income derived from selling endowments at market prices. Say that the weight on the fixed income is $\alpha \in [0, 1]$ and the weight on the income from endowments is $1 - \alpha$. When α is zero, the market is a textbook Walrasian exchange economy, in which agents derive income purely from selling their endowment at market prices. Unfortunately, the $\alpha = 0$

[4] Fairness also serves as a defense against litigation. If Alice envies Bob and wants to bring the matter to court, the most plausible remedy she could offer is for the two of them to switch assignments: the school district should give Alice's assignment to Bob, and Bob's assignment to Alice. In an environment with priorities, the district would counter-argue that Bob has a higher priority than Alice at Bob's assignment. In the absence of priorities, our fairness notion enables district administrators to argue that such a switch is not possible because Alice's assignment is not acceptable to Bob. Note that the standard for rejecting the switch is independent of the allocation proposed: Bob's endowment was fixed before the allocation was determined, and the district has to respect that Bob has a right to insist on his endowment.

Walrasian model may not posses an equilibrium (see our discussion in Section 5.3), and may have Pareto dominated equilibria (see our Section 5.2). We show, however, that when $\alpha > 0$ equilibrium always exists, and an equilibrium can be found that is Pareto optimal. Moreover, by choosing $\alpha > 0$ to be arbitrarily small, we can come as close a desired to respecting individual rationality. Finally, as long as $\alpha < 1$, the model allows endowments to matter and play a role in the final allocation. As a consequence, in equilibrium, if Alice envies Bob, her envy must be the reflection of Bob's endowment being more valuable than Alice's, and (under some additional conditions) by a coalition of agents wanting more of Bob's endowments and less of Alice's.

Before moving to the related literature, we want to emphasize that controlling school choice by way of endowments has the benefit of being very transparent in the possibilities that it guarantees for each child. Lotteries are familiar objects, and they are easy to interpret. It will be clear to the families participating in the market that they can opt for their endowed probabilistic allocation. It is, however, a change of focus from the standard ideas in controlled school choice, where the final composition of the school is the focus.

Our model is about access to schools; arguably about equality of opportunities. In that sense, it is telling that the New York Times article we quoted from earlier talks about giving students a chance to attend the best schools. It views the undesirable school composition as a reflection of lack of chances, or opportunities. In our model, the final composition of the school may differ from the initial allocations; and probably in most cases, it will differ substantially from the initial allocation. Arguably, such difference is desirable because it reflects efficiency and childrens' preferences, while respecting individual rationality and fairness.

2 Related literature.

The problem of controlled school choice was first analyzed formally by Abdulkadiroğlu and Sönmez (2003), the paper that introduced school choice as a mechanism design problem. The literature continued with, among others, Kojima (2012), Hafalir, Yenmez, and Yildirim (2013), Ehlers, Hafalir, Yenmez, and Yildirim (2014), and Echenique and Yenmez (2015). None of these papers, however, use endowments as a way to control school choice. By using endowments we take the position that what matters is access and opportunity, not the final composition of the schools. It may be that the final outcome is more segregated than desired by the district, but the segregation would be the result of agents' preferences.

Hamada, Hsu, Kurata, Suzuki, Ueda, and Yokoo (2017) is the only paper we are aware of that also uses initial endowments to control school choice. They assume that each child owns one seat of some school as endowment. Their goal is to design strategy-proof allocation mechanisms to meet the distributional constraint in the market and individual rationality constraint of each child. Since they consider deterministic endowments and ordinal preferences, and their fairness notions are based on priorities, their results are unrelated to ours.

Our notion of justified envy is analogous to the fairness notion of Yılmaz (2010). Yilmaz uses first-order stochastic dominance instead of utility functions, and says that Alice justifiably envies Bob if she does not regard her allocation as first-order stochastically dominating Bob's, while any object that she obtains with positive probability in her allocation is regarded by Bob as acceptable. An important difference between Yilmaz's paper and ours is that endowments are deterministic in his model and probabilistic in ours. Finally, Yilmaz studies the probabilistic serial rule (Bogomolnaia and Moulin, 2001), and as a consequence his results are simply unrelated to ours.

Hylland and Zeckhauser (1979) were the first to propose markets over lottery shares to solve centralized allocation problems; they assume a fixed income for each agent, independent of prices. Hylland and Zeckhauser make the point, which we elaborate on in Section 5.3, that a model with endowments would not work because equilibrium may not exist. They also emphasize that equilibrium may not be efficient, and introduce the "cheapest bundle" property that we employ as well in our version of the first welfare theorem. It should be clear that allowing for endowments is a stark departure from the model in Hylland and Zeckhauser (1979), and poses significant challenges. Many other papers have followed Hylland and Zeckhauser in analyzing competitive equilibria as solutions in market design. For example, Budish (2011), Ashlagi and Shi (2015), He, Miralles, Pycia, Yan, et al. (2015) and He, Li, and Yan (2015). These authors explore markets with exogenously given budgets: $\alpha = 1$ in our model. When all agents have equal budgets, there can be no envy in a competitive equilibrium. But equal budgets of course eliminate any role for the initial endowments in the same blow as they eliminate envy.

Our market solution is a hybrid of the standard "competitive equilibrium with equal incomes" of Varian (1974), and the textbook model of a Walrasian exchange economy. The exchange economy model allows for endowments to play a role justifying envy, but equilibrium, as we have emphasized, may not exist. There may also exist Pareto ranked Walrasian equilibria (see Section 5.2).

3 The model

Our model is essentially the standard model of an exchange economy in general equilibrium theory. The difference with the standard model is that agents consume lotteries: consumption bundles cannot add up to more than one. This difference is far from minor. For example, it results in the non-existence of Walrasian equilibrium, even for economies that are otherwise well-behaved, and in the presence of Pareto-ranked Walrasian equilibria (see our discussions in 5.3 and 5.2 below). In fact, no known equilibrium existence results apply to our model.

Notation and preliminary definitions. The simplex $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j = 1\}$ in $\mathbf{R}^n$ is denoted by $\Delta^n \subseteq \mathbf{R}^n$, while the set $\{x \in \mathbf{R}^n_+ : \sum_{j=1}^n x_j \leq 1\}$ is denoted by $\Delta^n_- \subseteq \mathbf{R}^n$. When n is understood, we simply use the notation Δ and Δ_-.

A function $u : \Delta^n_- \to \mathbf{R}$ is

- *concave* if, for any $x, z \in \Delta_-$, and $\lambda \in (0,1)$, $\lambda u(z) + (1-\lambda)u(x) \leq u(\lambda z + (1-\lambda)x)$;
- *quasi-concave* if, for each $x \in \Delta_-$, the set $\{z \in \Delta_- : u(z) \geq u(x)\}$ is convex.
- *semi-strictly quasi-concave* if, for any $x, z \in \Delta_-$, $u(z) < u(x)$ and $\lambda \in (0,1)$ imply that $u(z) < u(\lambda z + (1-\lambda)x)$.
- *expected utility* if it is linear. In this case we identify u with a vector $u \in \mathbf{R}^n$ and denote $u(x)$ as $u \cdot x$.
- C^1 if it can be extended to a continuously differentiable function defined on an open set that contains Δ_-.

Model. A *discrete allocation problem* is a tuple $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$, where:

- $S = \{s_k\}_{k=1}^L$ represents a set of indivisible objects.
- $I = \{a_i\}_{i=1}^N$ represents a set of agents, each of whom demands exactly one copy of an object.
- $Q = \{q_s\}_{s \in S}$ is a capacity vector, and $q_s \in \mathbf{N}$ is the number of copies of object s. For simplicity, we assume that $\sum_{s \in S} q_s = N$, i.e., the number of copies of objects is equal to the number of agents.
- For each agent i, $u^i : \Delta^L_- \to \mathbf{R}$ is a continuous utility function defined on Δ^L_-.
- For each agent i, $\omega^i \in \Delta^L$ is i's endowment vector such that ω^i_s is the fraction of object s owned by i. We assume that all objects are owned by agents. So $\sum_{i=1}^N \omega^i = Q$.

Allocations and Pareto optimality. An *allocation* in Γ is a vector $x \in \mathbf{R}_+^{LN}$, which we write as $x = (x^i)_{i=1}^N$, with $x^i \in \Delta_-^L$, such that

$$\sum_{i \in I} x_s^i = q_s$$

for all $i \in I$ and all $s \in S$. When $x_s^i \in \{0,1\}$ for all i and all s, x is a deterministic allocation. The Birkhoff-von Neumann theorem (Birkhoff, 1946; Von Neumann, 1953) implies that every allocation is a convex combination of deterministic allocations.

An allocation x is *acceptable* to agent i if $u^i(x^i) \geq u^i(\omega^i)$; x is *individually rational* (IR) if it is acceptable to all agents. We also define a notion of approximate individual rationality: for any $\epsilon > 0$, x is *ϵ-individually rational* (ϵ-IR) if $u^i(x^i) \geq u^i(\omega^i) - \epsilon$ for all i.

The notion of efficiency comes in three flavors: An allocation x is *weak Pareto optimal* (wPO) if there is no allocation y such that $u^i(y^i) > u^i(x^i)$ for all i; *ε-weak Pareto optimal* (ε-PO), for $\varepsilon > 0$, if there is no allocation y such that $u^i(y^i) > u^i(x^i) + \varepsilon$ for all i; and *Pareto optimal* (PO) if there is no allocation y such that $u^i(y^i) \geq u^i(x^i)$ for all i and $u^j(y^j) > u^j(x^j)$ for some j. In our model, the difference between wPO and PO is significant because of the constraint that each x^i cannot add up to more than 1. This means that wPO is compatible with wasteful situations where we can use existing resources to make some agents strictly better off, but cannot construct an allocation that makes all agents strictly better off because there are agents that have achieved the largest possible quantities of their most preferred goods.

Walrasian equilibrium. Let $\alpha \in [0,1]$. An *α-slack Walrasian equilibrium* is a pair (x,p) such that $x \in \Delta_-^N$, and $p = (p_s)_{s \in S} \in \mathbf{R}_+^L$ is a price vector such that

1. $\sum_{i=1}^N x^i = \sum_{i=1}^N \omega^i$; and
2. x^i maximizes i's utility within his α-modified budget:

$$x^i \in \text{argmax}\{u^i(z^i) : z^i \in \Delta_- \text{ and } p \cdot z^i \leq \alpha + (1-\alpha) p \cdot \omega^i\};$$

Property 1 means that x is an allocation, or that demand equals supply and all markets clear.

A *Walrasian equilibrium* is a 0-slack Walrasian equilibrium. The following result is well-known (Hylland and Zeckhauser, 1979).

Proposition 1. *There are economies in which all agents utility functions are expected utility, that posses no Walrasian equilibria.*

The proposition re-appears, in context, as Proposition 3. Thus, Walrasian equilibria may not exist in our model, even for very well behaved utility functions, and we are not aware of any general existence results for our model. Section 5.3 elaborates further.

Fairness. As discussed before, we present two notions of fairness. The first tries to parallel the standard definition of justified envy in the model of school choice with priorities. The second is based on market equilibrium, and says that if an agent i envies j then i's endowment at equilibrium prices must be worth more than j's. In turns, this means that society values j's endowment more than i's endowment (see Theorem 4 below).

Our first notion of fairness relies on the idea of property rights. We regard agents as having the right to consume their endowments. So agents have the right to be at least as well off as they would be by consuming their endowments. Our fairness notion is based on the idea that if an agent i envies another agent j in an allocation x (that is, i prefers x^j to x^i), then switching their allocations must violate the property rights of j. That is, j must prefer ω^j to x^i. As discussed in the Introduction, this fairness notion parallels the standard definition of fairness in priority-based allocation problems, and provides an argument for social planner to defend any possible complaint from any agent who feels envy towards another.

Formally, we say an agent i has *justified envy* towards another agent j at an allocation x if

$$u^i(x^j) > u^i(x^i) \text{ and } u^j(x^i) \geq u^j(\omega^j).$$

We say that x has *no justified envy* (NJE) if no agent has justified envy towards any other agent at x.

We explore some simple implications of NJE. In an IR and NJE allocation x, if $u^i = u^j$ and $u^i(\omega^i) \geq u^j(\omega^j)$, then it must be that $u^i(x^i) \geq u^j(x^j)$.[5] That is, if two agents i and j have equal preferences and both agree that i's endowment is weakly better than j's, then both agree that i's allocation in x is also weakly better than j's. In particular, if $u^i = u^j$ and $u^i(\omega^i) = u^j(\omega^j)$, then it must be that $u^i(x^i) = u^j(x^j)$. So NJE and IR imply *equal treatment of equals* (also called symmetry by Zhou, 1990).

We also define two variants of justified envy: one is stronger than justified envy, while the other is weaker. We say that i has a *strong justified envy* (SJE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j)$. For any $\epsilon > 0$, we say i has an *ϵ-justified envy* (ϵ-JE) towards j at x if $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j) - \epsilon$. *No strong justified envy* (NSJE) and *no ϵ-justified envy* (NϵJE) are defined similarly as before. It is easy to see that

$$\text{no } \epsilon\text{-justified envy} \Longrightarrow \text{no justified envy} \Longrightarrow \text{no strong justified envy}$$

Our second notion of fairness is based on α-slack Walrasian equilibrium. If (x, p) is a α-slack Walrasian equilibrium for $\alpha \in (0, 1)$, and i envies j, then it must be the case that $p \cdot \omega^j > p \cdot \omega^i$. In other words, i's envy is not justified, because j's endowment is more valuable at market prices than i's. This means (in a sense that is made precise below) that society values j's endowment more than i's. Similarly to NJE, α-slach Walrasian equilibrium also ensures an equal treatment property: if $\omega^i = \omega^j$, then i cannot envy j in equilibrium.[6]

4 Main Results

Let $\Gamma = \{S, I, Q, (u^i, \omega^i)_{i \in I}\}$ be a discrete allocation problem.

[5] If $u^i(x^i) < u^j(x^j)$, then i's envy towards j is justified because $u^j(x^i) = u^i(x^i) \geq u^i(\omega^i) \geq u^j(\omega^j)$.

[6] Of course, this is the idea in the competitive equilibrium from equal incomes of Varian (1974).

Theorem 1. *Suppose that agents' utility functions in* Γ *are concave.*

1. *For any* $\varepsilon > 0$*, there exists an allocation that is* ε*-individually rational,* ε*-Pareto optimal and has no* ε*-justified envy;*

2. *There exists an allocation that is individually rational, weak Pareto optimal and has no strong justified envy.*

Theorem 2. *Suppose that agents' utility functions in* Γ *are quasi-concave. For any* $\alpha \in (0, 1]$*, there exists an* α*-slack Walrasian equilibrium* (x, p)*. Moreover, if agents' utility functions are semi-strictly quasi-concave, then* x *is Pareto optimal.*

Theorem 3. *Suppose that agents' utility functions in* Γ *are semi-strictly quasi-concave. For any* $\varepsilon > 0$ *there is* $\alpha \in (0, 1]$ *and an* α*-slack Walrasian equilibrium* (x, p)*, such that* x *is Pareto optimal and*

$$\max\{u^i(y) : y \in \Delta_- \text{ and } p \cdot y \leq p \cdot \omega^i\} - u^i(x) < \varepsilon.$$

In particular, x *is* ε*-individually rational.*

The next result clarifies why we think of α-Walras equilibria as justifying envy among agents.

Theorem 4. *Suppose that agents' utility functions in* Γ *are concave and* C^1*. Let* (x, p) *be an* α*-slack Walrasian equilibrium. Denote by* $S = \{i : u^i(x^i) = \max\{u^i(z^i) : z^i \in \Delta_-\}\}$ *the set of satiated consumers, and by* $U = [N] \setminus S$*. Suppose that* $\sum_{i \in U} x^i \gg 0$ *and that* i *envies* j *in* x $(u^i(x^j) > u^i(x^i))$*.*

Then there exists a set of welfare weights $\theta \in \mathbf{R}^U_{++}$ *such that if*

$$v(t) = \sup\{\sum_{i \in U} \theta^i u^i(\tilde{x}^i) : (\tilde{x}^i) \in \Delta^U_- \text{ and } \sum_{i \in U} \tilde{x}^i \leq \bar{\omega} + t(\omega^i - \omega^j) - \sum_{i \in S} x^i\},$$

then $(x^i)_{i \in U}$ *solves the problem for* $v(0)$*, and* $v(t) < 0$ *for all* t *small enough.*

The meaning of Theorem 4 is that if an agent i envies j then i's endowment is more valuable to a coalition of players U (a coalition that includes i!) in a specific sense. It is more valuable to U in the sense that there are welfare weights for the members of U such that a change in agents' endowment towards having more of i's endowment and less of j's leads to a worse weighted utilitarian outcome. The results requires that $\sum_{i\in U} x^i \gg 0$ simply to ensure that when we subtract ω^j we do not force some agent to consume negative quantities of some good.

5 Discussion

5.1 Scope of application of our results

In this section we discuss some applications of our model. The first remark we should make is that it is possible to generalize our model to an environment where the total amount of consumption of an agent is bounded above by some arbitrary $T^i > 0$, and where i's endowment also sums up to T^i. This allows us to capture the phenomenon of *time banks*, where agents exchange labor. One example of time banks is child care cooperatives.

The rest of our applications are to school choice. We argue that by properly designing the initial endowments of students, we can achieve many goals in school choice. This is in contrast to the more common approach of designing priorities in the literature.

Egalitarian school choice. If a school district wants to implement an egalitarian school choice in which no student is favored ex-ante, then a natural solution is to give students equal fractions of the seats of each school as initial endowments. That is, each student i owns an endowment vector $\omega^i = (\frac{q_s}{|I|})_{s\in S}$. Then there exits an allocation with the desirable properties stated in Theorem 1. In particular, individual rationality here implies *equal-division lower bound*.[7] Egalitarianism here refers to equality of opportunities. The allocation after preferences being taken into account can be very different from a uniform distribution.

[7] See Thomson (1987); an allocation x satisfies *equal division lower bound* if $u^i(x^i) \geq u^i(e)$ where $e = (\frac{q_s}{|I|})_{s\in S}$, exactly as stated here.

Respecting neighborhood priority. Suppose in a school district each student lives in the neighborhood of one school, and the number of seats of each school equals the number of students in its neighborhood. If the district wants to guarantee that each student is able to attend his neighborhood school if he or she wants, then a natural solution is to give each student a seat in his neighborhood school as initial endowment. That is, for each student i, $\omega^i = (0, \ldots, 0, 1, 0, \ldots, 0)$ where $\omega^i_s = 1$ if and only if s is the neighborhood school of i.

This special endowment structure may remind the reader of the Top Trading Cyle (TTC) mechanism. Here we emphasize that the allocation of TTC may not satisfy NJE. For example, suppose there are three students i, j, k with distinct endowments. i, k most prefer j's endowment, i least prefers his own endowment, and j most prefers i's endowment. TTC will let i, j trade their endowments and let k keep his endowment. However, in this allocation k has justified envy towards i since his endowment is acceptable to i.

Affirmative action. Suppose there are two types of students: majority and minority. If a school district wants to implement affirmative action for minority students, it can give each minority student some fractions of popular schools in their initial endowments. This guarantees that minority students have chances to attend popular schools if they so desire, and if some of them give up their chances, they do so in exchange for more favorable allocations.

Distributional constraints. Some districts may have distributional goals in the composition of its schools. For example, in an ideal composition of each school, each racial or ethnic group may have a given percentage in the target composition. As we stated before, such a goal is hard to achieve through our approach. While the initial endowment may reflect group quotas, the final allocation results from students exchanging allocations may be quite different from the initial endowment.

5.2 Efficiency

The first welfare theorem is not true in our model. Walrasian equilibria, and even α-slack Walrasian equilibria with $\alpha > 0$, may fail to be Pareto efficient. Example 1 below illustrates the point by exhibiting Pareto-ranked Walrasian equilibria.

The finding in Theorem 2 relies on a property of the α-slack Walrasian equilibria that we find under semi-strictly quasiconcave utilities: a α-slack Walrasian equilibrium (x, p) satisfies the *cheapest-bundle property* if, for each i, x^i minimizes expenditure $p \cdot z^i$ among all the $z^i \in \Delta_-$ for which $u^i(z^i) = u^i(x^i)$. The notion of a cheapest bundle, and its role in the first welfare theorem, was already established by Hylland and Zeckhauser (1979).

Example 1. *Given is an economy with two agents and two schools. Agents have expected utilities given by the following vNM indexes:*

i	$u^i_{s_1}$	$u^i_{s_2}$
1	*1*	*1*
2	*1*	*100*

And endowments $\omega^i = (1/2, 1/2)$*.*

Consider the allocations $x = ((1,0),(0,1))$ *and* $y = ((1/2,1/2),(1/2,1/2))$*. Note that* x *Pareto dominates* y*.*

The following table summarizes how both x *and* y *may be supported as Walrasian equilibria, both with* $\alpha > 0$ *and* $\alpha = 0$*. The first welfare theorem fails because agents have satiated preferences, not because we focus on equilibria with slack.*

α	*allocation*	p	$\alpha + (1-\alpha)p \cdot \omega^i$
0	x	$(1,1)$	1
1/2	x	$(1,1)$	1
0	y	$(0,1)$	1/2
1/2	y	$(0,2)$	1

The table is hopefully obvious, but it may be useful to detail why y is an equilibrium allocation with $\alpha = 0$. Note that income with prices $(0, 1)$ is $1/2$ for each agent. Agent 1 is happy to spend his income purchasing $x^1 = (1/2, 1/2)$ for a (global) utility maximum. Agent 2 spends all his income on school s_2 and purchases a $1/2$ share in s_2, but optimizes by also hitting his add-to-one constraint and purchasing a $1/2$ share in s_1.

Theorem 5. *Any Walrasian equilibrium with slack is weakly Pareto optimal, and any Walrasian equilibrium with slack and the cheapest-bundle property is Pareto optimal.*

5.3 The Hylland and Zeckhauser example

A Walrasian equilibrium (a 0-slack equilibrium) may not exist in our model. We present a non-existence example originally due to Hylland and Zeckhauser (1979), and we show how the symmetric Pareto optimal allocation in this example can be sustained as a α-slack Walrasian equilibrium with any $\alpha \in (0, 1]$ (Proposition 3).

Given is an economy with three agents and two schools, A and B. School B has two seats. We can model this as there being three schools: $\{s_1, s_2, s_3\}$ with s_2 and s_3 being copies of school B with a capacity of one.

Agents have expected utilities given by the following vNM indexes:

i	$u^i_{s_1}$	$u^i_{s_2}$	$u^i_{s_3}$
1	100	1	1
2	100	1	1
3	1	100	100

Endowments are $\omega^i = (1/3, 1/3, 1/3)$ for $i = 1, 2, 3$.

Proposition 2. *There is no Walrasian equilibrium.*

Proof. Suppose (towards a contradiction) that (x, p) is a Walrasian equilibrium. To simplify, let a B subindex indicate s_2 or s_3 and an A subindex indicate s_1. Since two copies of B are identical, they have wlog the same price in equilibrium.

Suppose first that $p_B > 0$. Normalize P_B to one. Then all agents have the same positive budget. If $p_A = 0$, then 1 and 2 would each buy one copy of A, which is a contradiction. So p_A must be positive. This further implies that A has no excess supply. The preferences of agents imply that 1 and 2 must each obtain a half of A. Therefore, $1/3p_A + 2/3 \geq 1/2p_A$, and we obtain $p_A \leq 4$. However, if $p_A < 4$, 1 and 2 would spend all of their budgets on A, and each obtain more than a half of A, which is a contradiction. So it must be that $1/3p_A + 2/3 = 1/2p_A$ and $p_A = 4$. But this means that at most 3 demands B and B must have excess supply, which contradicts the positive price of B.

Now suppose $p_B = 0$ and $p_A > 0$. Then 3 must obtain one copy of B, that is, $x_3 = (0, 1)$. Since p_A is positive, 1 and 2 must each obtain a half of A. However, their budget of $1/3p_A$ cannot afford such an allocation. □

Consider the allocation x defined by:

i	$x^i_{s_1}$	$x^i_{s_2}$	$x^i_{s_3}$
1	1/2	1/2	0
2	1/2	1/2	0
3	0	0	1

Proposition 3. *For any $\alpha \in (0, 1]$, there is a Walrasian equilibrium with α-slack that supports the allocation x.*

Proof. Let $\alpha \in (0, 1]$ and

$$p = (\frac{6\alpha}{1 + 2\alpha}, 0, 0).$$

Then $p \cdot \omega^i = \frac{2\alpha}{1+2\alpha}$ and

$$\alpha + (1 - \alpha)p \cdot \omega^i = \frac{\alpha + 2\alpha^2 + (2\alpha - 2\alpha^2)}{1 + 2\alpha} = \frac{3\alpha}{1 + 2\alpha} = p \cdot x^i,$$

for $i = 1, 2$.

Agents 1 and 2 can improve by purchasing more s_1, but they cannot afford any more. They can only afford a 1/2 share in s_1 and buy 1/2 in s_2 for free. They can improve by purchasing more s_2 at the zero price, but that would not be feasible in Δ_-. Agent 3 is optimizing by choosing $x^3_{s_3} = 1$ for a price of zero. □

Note that in an equilibrium supporting x, the value of agents 1 and 2's endogenous income ($p \cdot \omega^i$) in equilibrium is $2\alpha/(1+2\alpha)$. So the value of the α-slack (the exogenous part of the budget) relative to the value of the endogenous $p \cdot \omega^i$ is

$$\frac{1+2\alpha}{2} \to \frac{1}{2}$$

as $\alpha \to 0$. While α shrinks to zero, the value of the exogenous income is not negligible. In the same spirit, the following proposition shows that the average endogenous budget will always be below the exogenous budget of one.

Proposition 4. *If (x, p) is a Walrasian equilibrium with slack $\alpha \in (0, 1]$ then*

$$\frac{1}{n}\sum_{i=1}^{n} p \cdot \omega^i \leq 1$$

Proof. Note that $p \cdot (x^i - \omega^i) \leq \alpha(1 - p \cdot \omega^i)$. Sum over i to obtain:

$$0 = p \cdot \left(\sum_i x^i - \bar{\omega}\right) \leq \alpha(n - p \cdot \bar{\omega}).$$

□

Proposition 4 puts an upper bound on the average endogenous income. It cannot exceed the exogenous income of 1. In particular this means that the economy needs outside "money."

Proposition 4 reveals more than the proof of Theorem 2, which bound prices by the inequality:

$$\frac{p_l(\min_{l\in[L]} \bar{\omega}_l - \varepsilon)}{N} \leq 1.$$

6 An example of envy between agents with identical endowment.

We present an example of a discrete allocation problem in which all agents have expected utility preferences, together with an allocation that is individually rational, Pareto optimal, and satisfies no strong justified envy. In the example, one agent envies another agent even though they have equal endowments.

The example matters for two reasons. First, because one may think that no-envy among agents with equal endowments is intrinsically desirable. After all, we have tied the notion of justified envy to endowments; we have insisted on fairness by "controlling for endowments." The idea behind the example, the explanation for what makes the example work is, however, straightforward, and illustrates that endowments are not the end of the story. The two agents in question have equal endowments, but they have different preferences. Through their preferences, the two agents play very different roles in the economy. Other agents "trade" with the two agents in question, and the outcome can be explained through such trades. Because the agents' preferences are different, they interact with the remaining agents in very different ways. Hence it results in envy. Put differently, an agent can be valuable to others because she has a very desirable endowment, or because she is willing to trade in ways that enhance the welfare of others. The example we present in this section illustrates the role of preferences in generating value.

The second reason for why the example is important is that it suggests that our notion of fairness may fail to be incentive compatible. We have not specified a selection mechanism, and opted not to discuss incentives and strategy-proofness, but the example conveys some insights. One agent envies another even though they have equal endowments. This fact suggests that one agent may want to pretend to be the agent that he envies. In a large economy, in which the number of agents who report each type of preference does not change very much after a misreport, it stands to reason that such a misreport would not be profitable. Of course, the example we present here falls short of proving that if we were to define a fair mechanism it would not be strategy proof.

Example 2. *In the example there are five agents, labeled* $i = 1, \ldots, 5$*, and three schools,* s_1*,* s_2 *and* s_3*. There are two copies (seats) of schools* s_2 *and* s_3*. There is only one copy of school* s_1*. In the example, all the "action" involves agents 1 and 2. The remaining three agents are, in a sense, residual; they are also identical.*

The agents' von-Neumann-Morgenstern utilities are as described in the following table:

i	$u^i_{s_1}$	$u^i_{s_2}$	$u^i_{s_3}$
1	*3*	*1*	*2*
2	*3*	*2*	*1*
3	*2*	*3*	*1*
4	*2*	*3*	*1*
5	*2*	*3*	*1*

The agents' endowments are:

i	$\omega^i_{s_1}$	$\omega^i_{s_2}$	$\omega^i_{s_3}$
1	*0*	*1*	*0*
2	*0*	*1*	*0*
3	*1/3*	*0*	*2/3*
4	*1/3*	*0*	*2/3*
5	*1/3*	*0*	*2/3.*

Observe that agents 1 and 2 have identical endowments.

Finally, consider the following allocation x*:*

i	$x^i_{s_1}$	$x^i_{s_2}$	$x^i_{s_3}$
1	*0*	*0*	*1*
2	*1/2*	*0*	*1/2*
3	*1/6*	*2/3*	*1/6*
4	*1/6*	*2/3*	*1/6*
5	*1/6*	*2/3*	*1/6*

Observe that agent 1 envies agent 2, as

$$u^1 \cdot x^1 = 2 < 3/2 + 2/2 = u^1 \cdot x^2.$$

The envy is not justified, however, as

$$u^2 \cdot x^1 = 1 < 2 = u^2 \cdot \omega^2.$$

In fact, it is easy to see that x has no strong justified envy.

It is also easy to see that the allocation x is individually rational and Pareto optimal. In any PO allocation y, we cannot have $y^1_{s_2} > 0$, as agent 1 and any agent $j \in \{3,4,5\}$ are willing to trade school 2 for any other school. So y^1 must be a convex combination of $(1,0,0)$ and $(0,0,1)$. To make agent 1 better off then we would need to give agent 1 some shares in school 3, but these can only come at the expense of agent 2. To make agent 2 better off, she would need to get more shares in school 3, but these can only come at the expense of agents 3,4 and 5. These agents could only exchange shares in school 3 for shares in school 2, which agent 2 does not have. All agents 2, 3, 4 and 5 rank schools 3 and 1 in the same way.

7 Proof of Theorem 1

For given $\varepsilon > 0$, define

$$\mathcal{A}^* = \{x \text{ is } \varepsilon\text{-individually rational and } \varepsilon\text{-Pareto optimal}\}.$$

It is easy to see that $\mathcal{A}^*$ is nonempty and compact.[8]

[8] $\mathcal{A}^*$ is nonempty since the endowment allocation w is individually rational and any allocation strictly Pareto dominating w is individually rational. Let $\{x_n\} \subseteq \mathcal{A}^*$ and $x_n \to x$. It is obvious that x is an allocation since the set of allocations is closed. Since $u^i(x^i_n) \geq u^i(\omega^i) - \varepsilon$ for all n, in the limit $u^i(x^i) \geq u^i(\omega^i) - \varepsilon$. So x is ε-individually rational. Suppose x is not ε-Pareto optimal. Then there exists an allocation y such that $u^i(y^i) > u^i(x^i) + \varepsilon$ for all i. For big enough n, it must be that $u^i(y^i_n) > u^i(x^i_n) + \varepsilon$ for all i, which contradicts the ε-Pareto optimality of x_n.

For any $\lambda \in \Delta$, define

$$\phi(\lambda) = \text{argmax}\{\sum_{i=1}^{n} \lambda^i u^i(x^i) - \delta \sum_{i=1}^{n} \|x^i - (1, \ldots, 1)\| : (x^i)_{i=1}^n \in \mathcal{A}^*\},$$

where $\delta > 0$ is small enough such that

$$\delta \max_{x \in \mathcal{A}^*} \sum_{i=1}^{n} \|x^i - (1, \ldots, 1)\| < \varepsilon.$$

Since all u^i are continuous and concave and $\sum_{i=1}^n \|x^i - (1, \ldots, 1)\|$ is continuous and strictly convex, the objective function $\sum_{i=1}^n \lambda^i u^i(x^i) - \delta \sum_{i=1}^n \|x^i - (1, \ldots, 1)\|$ is continuous and strictly concave. Also, $\mathcal{A}^*$ is compact. Thus $\phi : \Delta \to \mathcal{A}^*$ is a function (singleton-valued), and, by the Maximum Theorem, continuous.

For any agent i, define

$$C^i = \{\lambda \in \Delta : \nexists j \in I \text{ s.t } i \text{ has an } \varepsilon\text{-justified envy towards } j \text{ at } \phi(\lambda)\}$$

In the following two lemmas we prove that $\{C^i\}_{i=1}^n$ is a Knaster-Kuratowski-Mazurkiewicz (KKM; see Theorem 5.1 in Border (1989)) covering of the simplex Δ.

Lemma 1. *For every $i \in I$, C^i is closed.*

Proof. Let λ_n be a sequence in C^i such that $\lambda_n \to \lambda \in \Delta$. Let $x_n = \phi(\lambda_n)$. By continuity of ϕ, $x_n \to x = \phi(\lambda) \in \mathcal{A}^*$. Now we prove that $\lambda \in C^i$, that is, i does not have an ε-justified envy towards any other agent. Suppose that there is an agent j such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(\omega^j) - \varepsilon$. Since u^i and u^j are continuous, for n large enough we have $u^i(x_n^j) > u^i(x_n^i)$ and $u^j(x_n^i) > u^j(\omega^j) - \varepsilon$, which contradicts that i has no ε-justified envy at x_n. Therefore, $\lambda \in C^i$ and C^i is closed. $\square$

Lemma 2. *For every $\lambda \in \Delta$, $\lambda \in \cup_{i \in supp(\lambda)} C^i$.*

Proof. Suppose, towards a contradiction, that for some $\lambda \in \Delta$, $\lambda \notin \cup_{i \in \text{supp}(\lambda)} C^i$. Let $x = \phi(\lambda)$. Then for every $i \in \text{supp}(\lambda)$ there exists some j such that $u^i(x^j) > u^i(x^i)$ and $u^j(x^i) > u^j(w^j) - \varepsilon$.

Suppose first that there exists some i and j in the aforementioned situation such that $j \notin \text{supp}(\lambda)$. Then consider an allocation y in which i, j exchange their allocations in x, and the other agents keep their allocations in x (that is, $y^i = x^j$, $y^j = x^i$, and $y^h = x^h$ for all $h \notin \{i, j\}$). Then y is ε-individually rational and $\sum_{i=1}^n \lambda^i u^i(x^i) < \sum_{i=1}^n \lambda^i u^i(y^i)$. Note that $\sum_{h \in I} \|x^h - \mathbf{1}\| = \sum_{h \in I} \|y^h - \mathbf{1}\|$. So

$$\sum_{i=1}^n \lambda^i u^i(x^i) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{i=1}^n \lambda^i u^i(y^i) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\|.$$

By definition of ϕ, then, $y \notin \mathcal{A}^*$. Since y is an ε-individually rational allocation, it cannot be ε-Pareto optimal. So there is a ε-Pareto optimal allocation z such that $u^i(z^i) > u^i(y^i) + \varepsilon$ for all i. Then z must be ε-individually rational and belong to $\mathcal{A}^*$. By our choice of δ,

$$\sum_{i=1}^n \lambda^i u^i(y^i) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\| < \sum_{i=1}^n \lambda^i u^i(z^i) - \delta \max_{x \in \mathcal{A}^*} \sum_{h \in I} \|x^h - \mathbf{1}\| \leq \sum_{i=1}^n \lambda^i u^i(z^i) - \delta \sum_{h \in I} \|z^h - \mathbf{1}\|.$$

Therefore,

$$\sum_{i=1}^n \lambda^i u^i(x^i) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{i=1}^n \lambda^i u^i(z^i) - \delta \sum_{h \in I} \|z^h - \mathbf{1}\|,$$

which contradicts the definition of $x = \phi(\lambda)$.

The above argument means that every $i \in \text{supp}(\lambda)$ has an ε-justified envy towards some $j \in \text{supp}(\lambda)$. Then, since the set of agents in $\text{supp}(\lambda)$ is finite, there is a cycle $i_1, \dots i_K$ in $\text{supp}(\lambda)$ such that i_1 has an ε-justified envy towards i_2, i_2 has an ε-justified envy towards i_3, and so on until i_K has an ε-justified envy towards i_1. We can construct a new allocation y by letting agents in the cycle exchange their allocations. As before, we have that $\sum_{h \in I} \|x^h - \mathbf{1}\| = \sum_{h \in I} \|y^h - \mathbf{1}\|$ because y obtained from x by a permutation of the assignments that the agents obtain. Then we have

$$\sum_{i=1}^n \lambda^i u^i(x^i) - \delta \sum_{h \in I} \|x^h - \mathbf{1}\| < \sum_{i=1}^n \lambda^i u^i(y^i) - \delta \sum_{h \in I} \|y^h - \mathbf{1}\|.$$

As before, y is ε-individually rational but cannot be ε-Pareto optimal. Then as before we can find an allocation $z \in \mathcal{A}^*$ that results in a contradiction. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The proof is an application of the KKM lemma: see Theorem 5.1 in Border (1989).

By Lemmas 1 and 2, $\{C^i\}_{i=1}^n$ is a KKM covering of Δ. So there exists $\lambda_\varepsilon^* \in \cap_{i=1}^n C^i$. Let $x_\varepsilon^* = \phi(\lambda_\varepsilon^*)$. Then x_ε^* is ε-individually rational, ε-Pareto optimal and has no ε-justified envy.

Now let $\{\varepsilon_n\}$ be a sequence such that $\varepsilon_n > 0$ for all n and $\varepsilon_n \to 0$. Let x_n^* be the allocation found above for each ε_n. Since the sequence $\{x_n^*\}$ is bounded, it has a subsequence $\{x_{n_k}^*\}$ that converges to some x^*. Since the set of allocations is closed, x^* is an allocation. We prove that x^* is individually rational, weak Pareto optimal and has no strong justified envy.

Since $u^i(x_{n_k}^{*i}) > u^i(\omega^i) - \varepsilon_{n_k}$ for all n_k and all i, in the limit $u^i(x^{*i}) \geq u^i(\omega^i)$ for all i. So x^* is individually rational. Suppose x^* is not weak Pareto optimal, then there exists an allocation y such that $u^i(y^i) > u^i(x^{*i})$ for all i. For big enough n_k, $u^i(y^i) > u^i(x_{n_k}^{*i}) + \varepsilon_{n_k}$ for all i, which contradicts the ε_{n_k}-Pareto optimality of $x_{n_k}^*$. Suppose some agent i has a envy towards another agent j in x^*; that is, $u^i(x^{*j}) > u^i(x^{*i})$. Then for big enough n_k, $u^i(x_{n_k}^{*j}) > u^i(x_{n_k}^{*i})$. Since $x_{n_k}^*$ has no ε_{n_k}-justified envy, $u^j(x_{n_k}^{*i}) \leq u^j(\omega^j) - \varepsilon_{n_k}$. In the limit we have $u^j(x^{*i}) \leq u^j(\omega^j)$. That is, i does not have a strong justified envy towards j. □

8 Proof of Theorem 2.

We prove the second statement of the theorem: the existence of a α-slack Walrasian equilibrium with the cheapest bundle property. Remark 1 below outlines the difference with the proof of the first statement in the theorem.

Let

$$v^i = \max\{u^i(x) : x \in \Delta_-\}$$
$$B^i(p) = \{x \in \Delta_- : p \cdot x \leq \alpha + (1-\alpha)p \cdot \omega^i\}$$
$$d^i(p) = \text{argmax}\{u^i(x) : x \in B^i(p)\}$$
$$\underline{d}^i(p) = \text{argmin}\{p \cdot x : x \in d^i(p)\}$$
$$V^i(p) = \max\{u^i(x) : x \in B^i(p)\}$$
$$z^i(p) = \underline{d}^i(p) - \omega^i \text{ and } z(p) = \sum_{i=1}^{N} z^i(p).$$

Note that v^i is the largest utility that i can attain. B^i is the budget set, d^i is demand, $\underline{d}^i$ is cheapest-demand, V^i is i's indirect utility function. z^i is i's excess demand correspondence given the cheapest-bundle selection, and Z the aggregate excess demand.

We also use the notation $\bar{\omega} = \sum_{i \in [n]} \omega_i$.

Lemma 3. *If $V^i(p) < v^i$ then $d^i(p) = \underline{d}^i(p)$.*

Proof. Let $x \in d^i(p)$. We shall prove that $p \cdot x = \alpha + (1-\alpha)p \cdot \omega^i$, which means we are done because it implies that all bundles in $d^i(p)$ cost the same at prices p. Let $z \in \Delta_-$ be such that $u^i(z) = v^i > u^i(x)$, and note that for any $\varepsilon \in (0,1)$, $u^i(\varepsilon z + (1-\varepsilon)x) > u^i(x)$ by the semi-strict quasi-concavity of u^i. Since $\varepsilon z + (1-\varepsilon)x \in \Delta_-$, this means that $p \cdot (\varepsilon z + (1-\varepsilon)x) > \alpha + (1-\alpha)p \cdot \omega^i$ for any $\varepsilon \in (0,1)$. But this is only possible, for arbitrarily small ε, if $p \cdot x \geq \alpha + (1-\alpha)p \cdot \omega^i$. Since $x \in B^i(p)$ we have established that $p \cdot x = \alpha + (1-\alpha)p \cdot \omega^i$. □

Lemma 4. *If $V^i(p) = v^i$ then*

$$\underline{d}^i(p) = \textit{argmin}\{p \cdot x : u^i(x) = v^i \textit{ and } x \in \Delta_-\}.$$

Proof. Let $x \in \underline{d}^i(p)$. Then for any $z \in \Delta_-$ with $p \cdot z < p \cdot x$, $z \in B^i(p)$. So $u^i(z) < v^i$. Therefore, if $z \in \text{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}$, then $p \cdot z = p \cdot x \leq \alpha + (1-\alpha)p \cdot \omega^i$, and therefore

$$\underline{d}^i(p) \supseteq \text{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}.$$

The converse set inclusion follows similarly because if x is not in the righ-hand set, there would exist a $z \in \Delta_-$ with $p \cdot z < p \cdot x$ and $u^i(z) = v^i$, which is not possible as such a z^i would be in $B^i(p)$. □

Let $\varepsilon \in (0, \min_{l\in[L]} \bar{\omega}_l)$ and

$$\bar{p} = \frac{N}{\min_{l\in[L]} \bar{\omega}_l - \varepsilon} > 0.$$

Lemma 5. $\underline{d}^i$ *is upper hemi-continuous on* $[0, \bar{p}]^L$

Proof. We shall prove that $\underline{d}^i$ has a closed graph. Let $(x_n, p_n) \to (x, p)$ with $x_n \in \underline{d}^i(p_n)$ for all n.

First, consider the case where $V^i(p) < v^i$. By the maximum theorem, V^i is continuous, so $V^i(p_n) < v^i$ for all n large enough. Then Lemma 3 implies that $x \in \underline{d}^i(p)$ as d^i is upper hemi-continuous.

Second, consider the case where $V^i(p) = v^i$. We know that $x \in d^i(p)$ as d^i is upper hemi continuous. Suppose (towards a contradiction) that $x \notin \underline{d}^i(p)$. Then there is $y \in d^i(p)$ with

$$p \cdot y < p \cdot x \leq \alpha + (1-\alpha) p \cdot \omega^i.$$

Then $p_n \cdot y < \alpha + (1-\alpha) p_n \cdot \omega^i$ for all n large enough. Since $y \in d^i(p)$ and $V^i(p) = v^i$, $u^i(y) = v^i$. This means that $V^i(p_n) = v^i$ for all n large enough, as $y \in B^i(p_n)$.

By, Lemma 4, then, $x_n \in \operatorname{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}$ for all n large enough. But the correspondence

$$p \mapsto \operatorname{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}.$$

is upper hemicontinous (by the maximum theorem), so $x \in \operatorname{argmin}\{p \cdot x : u^i(x) = v^i \text{ and } x \in \Delta_-\}$; a contradiction. □

Consider the correspondence $\phi : [0, \bar{p}]^L \to [0, \bar{p}]^L$ defined by

$$\phi_l(p) = \{\min\{\max\{0, \zeta_l + p_l\}, \bar{p}\} : \zeta \in z(p)\}.$$

Lemma 6. *ϕ is upper hemi-continuous, convex- and compact- valued.*

Proof. The aggregate excess demand under the cheapest selection, z, is upper hemi-continuous by Lemma 5. It is easy to see that this implies the upper hemi-continuity of ϕ. Similarly, convex and compact values are immediate. □

By Kakutani's fixed point theorem there is $p^* \in [0, \bar{p}]^L$ with $p^* \in \phi(p^*)$. We shall prove that p^* is an equilibrium price. Note that there exists $\zeta \in z(p^*)$ such that

$$p^*_l = \min\{\max\{0, \zeta_l + p^*_l\}, \bar{p}\}. \tag{1}$$

Lemma 7. *$p^* \cdot \zeta \geq 0$.*

Proof. If $p^* \cdot \zeta < 0$ then there is some good l with $p^*_l > 0$ and $\zeta_l < 0$. By Equation 1, then, $p^*_l = p^*_l + \zeta_l$, which is not possible as $\zeta_l < 0$. □

Lemma 8. *$p^*_l < \bar{p}$ for all $l \in [L]$*

Proof. Suppose towards a contradiction that there is l for which $p^*_l = \bar{p}$. Then $p^*_l > 0$, so Equation 1 means that $\bar{p} \leq \zeta_l + p^*_l = \zeta_l + \bar{p}$. Let $\zeta = \sum_i x^i - \bar{\omega}$, with $x^i \in d^i(p^*)$. The definition of $B^i(p)$ means that

$$p^* \cdot (x^i - \omega^i) \leq \alpha(1 - p^* \cdot \omega^i),$$

for all $i \in [N]$. Thus, summing over i we obtain that $p^* \cdot \zeta \leq \alpha(N - p^* \cdot \bar{\omega})$.

Now, by definition of $\bar{p}$, we have that

$$p^* \cdot \bar{\omega} \geq \bar{p}\bar{\omega}_l > \bar{p}(\min_{l \in [L]} \bar{\omega}_l - \varepsilon) = N.$$

Thus, $p^* \cdot \zeta \leq \alpha(N - p^* \cdot \bar{\omega})$ implies that $p^* \cdot \zeta < 0$, in contradiction to Lemma 7. □

Lemma 9. $\zeta = 0$

Proof. By Lemma 8 and Equation (1),

$$p_l^* = \max\{0, \zeta_l + p_l^*\} \tag{2}$$

for all $l \in [L]$.

Equation 2 implies two things. First, that $\zeta_l > 0$ is not possible for any l. Hence $\zeta \leq 0$. Second, that if $\zeta_l < 0$ then $p_l^* = 0$.

Suppose then, towards a contradiction, that that $\zeta_l < 0$ for some good l, and correspondingly that $p_l^* = 0$. Now, $\zeta_l < 0$ and $\zeta \leq 0$ means that

$$0 > \sum_l \zeta_l = \sum_l \sum_i x_l^i - \sum_l \bar{\omega}_l = \sum_i \sum_l x_l^i - N.$$

So there is some agent i for which $\sum_l x_l^i < 1$. Agent i can then increase his consumption of good l without violating the constraint that consumption lie in Δ_-. Given that $p_l^* = 0$, the increase in consumption of good l would also not violate the budget constraint. So there exist a bundle in $B^i(p)$ with strictly more of good l, and the same amount of every other good, than x^i. This contradicts the strict monotonicity of u^i, and the fact that $x^i \in d^i(p^*)$. □

Remark 1. *The proof uses semi-strict quasiconcavity only in the proof of upper hemicontinuity of* $\underline{d}^i$. *To prove existence of an equilibrium without imposing the cheapest-bundle property, observe that continuity and quasiconcavity of* u^i *is enough to ensure that* d^i *is upper hemicontinuous, and takes convex and compact valued. If* z *is defined from* d^i *in place of* d^i*, the proof as written shows the existence of a* α*-slack Walrasian equilibrium.*

9 Proof of Theorem 3

Let d_H denote the Hausdorff distance between two sets in $\mathbf{R}^L$. So,

$$d_H(A, B) = \max\{\sup\{\inf\{\|x-y\| : y \in B\} : x \in A\}, \sup\{\inf\{\|x-y\| : x \in A\} : y \in B\}\}.$$

Let $B^i(p,\alpha)$ denote the budget set given a price vector p and slack $\alpha \in [0,1]$. Let $\bar{B}^i(p,\alpha) = \{x \in \mathbf{R}^L_+ : p \cdot x = \alpha + (1-\alpha)p \cdot \omega^i$ denote the budget line. Note that $B^i(p,\alpha) = \{x \in \Delta_- : \exists y \in \bar{B}^i(p,\alpha) \text{ s.t. } x \leq y\}$.

Lemma 10. *For any $\delta > 0$ there is $\alpha > 0$ such that if p is the Walrasian equilibrium with slack α found in Theorem 2, then for any i, either $p \cdot \omega^i < 1$ or $d_H(\bar{B}^i(p,\alpha), \bar{B}^i(p,0)) < \delta$.*

Proof. Consider the price $\bar{p}$ defined in the proof of Theorem 2. Note that if p is a price obtained by application of the theorem, then $p \in [0,\bar{p}]^L$. Note also that $\bar{p}$ is independent of α.

Let $K = \sup\{\|x\| : x \in \Delta_-\}$. Now choose $\alpha \in (0,1)$ such that

$$\sup\{\left|1 - \frac{\alpha + (1-\alpha)p \cdot \omega^i}{p \cdot \omega^i}\right| K : p \in [0,\bar{p}]^L \text{ and } p \cdot \omega^i \geq 1\} < \delta$$

Let $x \in \bar{B}^i(p,0)$, then $\gamma x \in \bar{B}^i(p,\alpha)$, where

$$\gamma = \frac{\alpha + (1-\alpha)p \cdot \omega^i}{p \cdot \omega^i}.$$

Note that

$$\|x - \gamma x\| = |1-\gamma|\,\|x\| < \delta.$$

Thus $\inf\{\|x-y\| : y \in \bar{B}^i(p,\alpha)\} < \delta$, and therefore

$$\sup\{\inf\{\|x-y\| : y \in \bar{B}^i(p,\alpha)\} x \in \bar{B}^i(p,0)\} < \delta.$$

In a similar vein, we can show that

$$\sup\{\inf\{\|x-y\| : y \in \bar{B}^i(p,0)\} x \in \bar{B}^i(p,\alpha)\} < \delta,$$

and thus $d_H(B^i(p,0), B^i(p,\alpha)) < \delta$. □

To prove the theorem, let $\delta > 0$ be such that, for any $p \in [0,\bar{p}]^L$, if $d_H(B^i(p,0), B^i(p,\alpha)) < \delta$ then

$$\left|\max\{u^i(x) : x \in B^i(p,\alpha)\} - \max\{u^i(x) : x \in B^i(p,0)\}\right| < \varepsilon.$$

For such δ, let α be as in Lemma 10.

For any i, if $p \cdot \omega^i < 1$ then $B^i(p,0) \subseteq B^i(p,\alpha)$, so

$$\max\{u^i(y) : y \in \Delta_- \text{ and } p \cdot y \leq p \cdot \omega^i\} - u^i(x) < 0 < \varepsilon.$$

If, on the contrary, $p \cdot \omega^i \geq 1$, then Lemma 10 implies that $d_H(B^i(p,0), B^i(p,\alpha)) < \delta$, and the result follows from the definition of δ.

10 Proof of Theorem 5

Let $u^i(y^i) > u^i(x^i)$ for all i. Then

$$p \cdot (y^i - \omega^i) > \alpha(1 - p \cdot \omega^i) \geq p \cdot (x^i - \omega^i).$$

Sum over i to obtain:

$$p \cdot \left(\sum_i y^i - \bar{\omega} \right) > \alpha(n - p \cdot \bar{\omega}) \geq p \cdot \left(\sum_i x^i - \bar{\omega} \right) = 0.$$

Thus y cannot be an allocation.

In second place, suppose that (x,p) is an α-slack Walrasian equilibrium in which each x^i satisfies the cheapest-bundle property. Then, for any $y^i \in \Delta_-$, $u^i(y^i) \geq u^i(x^i)$ implies that $p \cdot y^i \geq p \cdot x^i$, while $u^i(y^i) > u^i(x^i)$ implies that $p \cdot y^i > p \cdot x^i$. Thus, if $(y^i)_{i \in [N]}$ Pareto dominates x, adding up gives $p \cdot \sum_i y^i > p \cdot x^i = p \cdot \bar{\omega}$, as x is an allocation. Then $(y^i)_{i \in [N]}$ cannot be an allocation.

11 Proof of Theorem 4

Our first observation establishes the relation between envy and the value of endowments at equilibrium prices.

Lemma 11. *Let (x,p) be a Walrasian equilibrium with slack $\alpha \in (0,1]$. If i envies j, then $p \cdot (x^j - x^i) > 0$ and $p \cdot (\omega^j - \omega^i) > 0$.*

Proof. Let i envy j, so $u^i(x^j) > u^i(x^i)$. Then utility maximization implies that

$$\alpha + (1-\alpha)p \cdot \omega^j \geq p \cdot x^j > \alpha + (1-\alpha)p \cdot \omega^i \geq p \cdot x^i,$$

where the strict inequality follows because $x^j \in \Delta_-$. So $p\cdot(x^j - x^i) > 0$ and $p\cdot(\omega^j - \omega^i) > 0$. $\square$

Now consider a α-slack Walrasian equilibrium (x,p). Agent i's maximization problem is:

$$\max_{x \in \mathbf{R}^L_+} u^i(x) + \lambda^i(I^i - p \cdot x) + \gamma^i(1 - \mathbf{1} \cdot x)$$

Where $I^i = \alpha + (1-\alpha)p \cdot \omega^i$, λ^i is a multiplier for the budget constraint, and γ^i for the $\sum_l x^i_l \leq 1$ constraint.

Utility functions are C^1. The first-order conditions for the maximization problems are then:

$$\partial_l u^i(x^i) - \lambda^i p_l - g^i \begin{cases} = 0 & \text{if } x^i_l > 0 \\ \leq 0 & \text{if } x^i_l = 0, \end{cases}$$

where $\partial_l u^i(x^i)$ denotes the partial derivative of u^i with respect to x^i_l.

Observe that if $p \cdot x^i < \alpha + (1-\alpha)p \cdot \omega^i$, then the budget constraint is not binding and $\lambda^i = 0$. As a consequence, $u^i(x^i) = \max\{u^i(z^i) : z^i \in \Delta_-\}$. Let $S = \{i \in [N] : p \cdot x^i < \alpha + (1-\alpha)p \cdot \omega^i\}$ be the set of *satiated* consumers. Let $U = \{i \in [N] : p \cdot x^i = \alpha + (1-\alpha)p \cdot \omega^i\}$ be the set of *unsatiated*, and observe that we can let $\lambda^i > 0$ for all $i \in U$. Consider the two stage social program:

Stage 1:

$$\max_{\tilde{y} \in (\Delta_-)^S} \textstyle\sum_{i \in S} u^i(\tilde{y}^i)$$

Stage 2:

$$\max_{\tilde{y} \in (\Delta_-)^U} \textstyle\sum_{i \in U} \frac{1}{\lambda^i} u^i(\tilde{y}^i)$$
$$\textstyle\sum_{i \in U} \tilde{y}^i \leq \bar{\omega} - \sum_{i \in S} x^i$$

Note that $(x^i)_{i\in S}$ solves Stage 1, while satisfying $\sum_{i\in S} x^i \le \bar{w}$, and that given $(x^i)_{i\in S}$, $(x^i)_{i\in U}$ solves Stage 2. That this is so follows from the fact that $(x^i)_{i\in U}$ solves the first-order conditions for the Stage 2 problem with Lagrange multiplier p for the constraint that $\sum_{i\in U} \tilde{y}^i \le \bar{\omega} - \sum_{i\notin S} x^i$.

Now use the assumption that $\sum_{i\in U} x^i \gg 0$. This means that there exists $\bar{t} > 0$ such that if $t \in (0, \bar{t}]$ then the set of $\tilde{y} \in (\Delta_-)^U$ such that $\sum_{i\in U} \tilde{y}^i \le \bar{\omega} + t(\omega^i - \omega^j) - \sum_{i\notin S} x^i$ is nonempty (and, for constraint qualification, contains an element that satisfies all constraints with slack).

Consider the problem

$$\begin{aligned}&\max_{\tilde{y}\in(\Delta_-^U)} \textstyle\sum_{i\in U} \frac{1}{\lambda^i} u^i(\tilde{y}^i)\\ &\textstyle\sum_{i\in U} \tilde{y}^i \le \bar{\omega} + t(\omega^i - \omega^j) - \sum_{i\in S} x^i\end{aligned}$$

Note that for each $t \in (0, \bar{t}]$ there exists $(\nu(t), \gamma(t), \alpha(t))$ such that

$$v(t) = \sup\{\sum_{i\in U} \frac{1}{\lambda^i} u^i \cdot \tilde{y}^i + \nu(t) \cdot (\bar{\omega} - \sum_{i\in S} \tilde{y}^i + t(\omega^i - \omega^j)) - \sum_{i\in U} \tilde{y}^i) + \sum_{i\in U} \gamma_i(t)(1 - \sum_{l\in[L]} \tilde{y}^i_l) + \sum_{i\in U} \alpha_i(t) \tilde{y}^i_l.\}$$

Here $\nu(t)$ is the Lagrange multiplier for the constraint that $\sum_{i\in U} \tilde{y}^i \le \bar{\omega} - \sum_{i\in S} x^i + t(\omega^i - \omega^j)$, while $\gamma(t)$ and $\alpha(t)$ are the Lagrange multipliers for the constraint that $(\tilde{y}^i) \in (\Delta_-)^N$. Choose a selection $(\nu(t), \gamma(t), \alpha(t))$ such that $\nu(0) = p$.

Let $\tilde{\omega} = \bar{\omega} - \sum_{i\in S} x^i$. The saddle point inequalities imply that

$$\begin{aligned}(t' - t)\nu(t) \cdot (\omega^i - \omega^j) = &\sum_{i\in U} \frac{1}{\lambda^i} u^i(x^i(t')) + \nu(t) \cdot (\tilde{\omega} + t'(\omega^i - \omega^j) - \sum_{i\in U} x^i(t'))\\ &+ \sum_{i\in U} \gamma_i(t)(1 - \sum_{l\in[L]} x^i_l(t')) + \sum_{i\in U} \alpha_i(t) x^i_l(t')\\ &- \Bigg(\sum_{i\in U} \frac{1}{\lambda^i} u^i(x^i(t')) + \nu(t) \cdot (\tilde{\omega} + t(\omega^i - \omega^j) - \sum_{i\in U} x^i(t'))\\ &+ \sum_{i\in U} \gamma_i(t)(1 - \sum_{l\in[L]} x^i_l(t')) + \sum_{i\in U} \alpha_i(t) x^i_l(t')\Bigg)\\ &\ge v(t') - v(t)\end{aligned}$$

Now recall that $\nu(0) = p$. Then Lemma 11, together with the above inequality, imply that

$$0 > p \cdot (\omega^i - \omega^j)t' \geq v(t') - v(0)$$

for all $t' > 0$ with $t' \leq \bar{t}$.

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