Noise Waves and Passive Linear Multiports

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Abstract—**A simple derivation is given for the relation between a passive circuit's noise wave correlation matrix and its scattering matrix. It is shown that this relation, referred to as Bosma's theorem, comes readily from the fundamental principle that a passive multiport in thermodynamic equilibrium with reflectionless terminations produces uncorrelated output waves.**

I. Introduction

NOISE WAVES are a powerful means for characterizing noise in microwave circuits. They permit scattering matrix and signal flow graph theory to be used for noise calculations [1], leading to advantages in microwave CAD [2]. In the wave representation of a multiport network, noise waves are outwardly directed correlated sources present at each port, representing the noise originating in the network that is deliverable to its terminations. Denoting incident and output waves by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, respectively, and the scattering matrix by $\boldsymbol{S}$, the noise waves are given by a source vector $\boldsymbol{c}$ such that

$$\boldsymbol{b} = \boldsymbol{S}\boldsymbol{a} + \boldsymbol{c}. \tag{1}$$

Characterization of the random noise wave amplitudes is made in terms of a correlation matrix $\boldsymbol{C}_s$

$$\boldsymbol{C}_s = \overline{\boldsymbol{c}\boldsymbol{c}^\dagger} \tag{2}$$

where the dagger indicates a Hermitian transpose operation, and the overbar the correlation product. For each noise wave c_i the correlation matrix yields values for the noise power deliverable to the terminations, given in terms of the statistical expectation of c_i, and written as $\overline{|c_i|^2}$, as well as a correlation product with each of the other ports, given by $\overline{c_i c_j^*}$. An n-port linear component's signal and noise properties are completely described by $n \times n$ scattering and noise wave correlation matrices.

In order to compute the noise properties of a network consisting of passive and active components, the scattering and correlation matrices of each component must be known. For an active device, these are usually found by measurement. The correlation matrix for a passive component is derived by examining how noise power is delivered to reflectionless terminations in thermodynamic equilibrium. The simplest case is that of a passive one-port where (1) simplifies to the scalar equation $b = sa + c$. The incident wave a emanates from the termination and is uncorrelated with the noise wave c, and so $\overline{|b|^2} = |s|^2\overline{|a|^2} + \overline{|c|^2}$. With a reflectionless termination, the incident noise power $\overline{|a|^2}$ in a 1-Hz bandwidth is known to be kT, where k is Boltzmann's constant, T is the temperature, and quantum effects have been neglected. Thermodynamic equilibrium requires a balance in power flow such that $\overline{|a|^2} = \overline{|b|^2}$ and therefore

$$\overline{|c|^2} = kT(1 - |s|^2). \tag{4}$$

For the one-port, the correlation matrix (2) is reduced to a single parameter: the deliverable noise wave power given by (4). The subject of this letter is the derivation of noise wave correlation matrices for passive *multiports*. Although previously discussed by Bosma [3], the derivation given here is based on simple physical arguments. In all noise calculations, a 1-Hz bandwidth will be assumed.

II. Two-Ports in Thermodynamic Equilibrium

The properties of the passive multiport are derived by first considering a two-port passive component $\boldsymbol{S}$ with reflectionless terminations, as shown in Fig. 1. Thermodynamic equilibrium is assumed; the terminations and two-port having reached a common temperature T. Since no external sources are present, the incident waves a_1 and a_2 are due solely to the thermal noise waves emanating from the two terminations. These waves will be uncorrelated, and from the derivation of (4) it is known that

$$\overline{|a_1|^2} = \overline{|a_2|^2} = kT \tag{5a}$$

$$\overline{a_2 a_1^*} = 0. \tag{5b}$$

Thermal noise waves generated by network $\boldsymbol{S}$, and the scattering of noise waves a_1 and a_2, contribute to output waves b_1 and b_2. Yet, in thermodynamic equilibrium the net power flow must be balanced, so these output waves must satisfy

$$\overline{|b_1|^2} = \overline{|b_2|^2} = kT. \tag{6}$$

As yet unknown is the value of the correlation product $\overline{b_2 b_1^*}$, but the properties of noise waves allow a means by which it may be determined. Given in Fig. 2 is the network of Fig. 1 with a directional coupler inserted between the two-port and its terminations. The primed wave variables are those that should be effected by the insertion of the coupler which is assumed to be an ideal (lossless and matched) 3-dB 180° hybrid. The effects on b_1 and b_2 are found by applying

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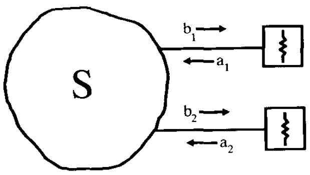

Fig. 1. Passive two-port S with reflectionless terminations, both assumed to be at temperature T. Incident and output waves are due to thermal noise.

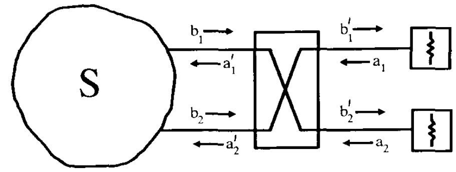

Fig. 2. Directional coupler inserted between the passive two-port S and its reflectionless terminations.

the coupler equations

$$b_1' = \frac{1}{\sqrt{2}}(b_1 + b_2) \quad (7)$$

$$b_2' = \frac{1}{\sqrt{2}}(b_1 - b_2) \quad (8)$$

resulting in

$$\overline{|b_1'|^2} = \tfrac{1}{2}\left(\overline{|b_1|^2} + \overline{|b_2|^2}\right) + \mathrm{Re}\left(\overline{b_2 b_1^*}\right) \quad (9)$$

$$\overline{|b_2'|^2} = \tfrac{1}{2}\left(\overline{|b_1|^2} + \overline{|b_2|^2}\right) - \mathrm{Re}\left(\overline{b_2 b_1^*}\right) \quad (10)$$

$$\overline{b_1' b_2'^*} = \tfrac{1}{2}\left(\overline{|b_1|^2} - \overline{|b_2|^2}\right) + j\,\mathrm{Im}\left(\overline{b_2 b_1^*}\right) \quad (11)$$

and a measure of $\overline{b_2 b_1^*}$ is possible by comparing these values. Similar expressions are obtained for a_1' and a_2', but since (5) remains valid for Fig. 2 as well as for Fig. 1, application of (9)–(11) results in

$$\overline{|a_1'|^2} = \overline{|a_2'|^2} = kT \quad (12a)$$

$$\overline{a_2' a_1'^*} = 0, \quad (12b)$$

demonstrating that insertion of the directional coupler has had no effect on the statistics of the noise waves originating from the terminations. Thermodynamic equilibrium then requires that (6) remain valid, as well. This allows (9) and (10) to be simplified to

$$\overline{|b_1'|^2} = kT + \mathrm{Re}\left(\overline{b_2 b_1^*}\right) \quad (13)$$

$$\overline{|b_2'|^2} = kT - \mathrm{Re}\left(\overline{b_2 b_1^*}\right). \quad (14)$$

Correlation between output waves from the two-port would cause power levels other than kT to be delivered to the terminations. This cannot be the case. The continued validity of (5) and thermodynamics require that $\overline{|b_1'|^2} = \overline{|b_2'|^2} = kT$, and indeed the power in all waves must be kT. The only conclusion is that $\mathrm{Re}\,(\overline{b_2 b_1^*}) = 0$.

The same reasoning may be repeated with a 3-dB 90° coupler substituted for the 180° hybrid. The result is then that $\mathrm{Im}\,(\overline{b_2 b_1^*}) = 0$. The statistics of all waves considered are unaffected by the insertion of the directional couplers, leading to the conclusion that

$$\overline{b_2 b_1^*} = 0. \quad (15)$$

This result is true for all passive two-ports since no restrictions other than passivity have been placed on the network.

III. Noise from Passive Multiports

The results of Section II can be generalized for the case of a passive multiport with reflectionless terminations as described by matrix equation (1). The terminations will produce uncorrelated waves with thermal noise power kT, written in vector form as

$$\overline{\boldsymbol{a}\boldsymbol{a}^\dagger} = kT\boldsymbol{I}, \quad (16)$$

where $\boldsymbol{I}$ is the identity matrix. For the multiport, the directional coupler test can be applied to any two ports at a time. The result is the same. For any i and j, $\overline{|b_i|^2} = \overline{|b_j|^2} = kT$ and $\overline{b_i b_j^*} = 0$. As before, correlation between b_i and b_j would result in power delivered to the loads differing from kT and is not possible. This result can be written in vector form as

$$\overline{\boldsymbol{b}\boldsymbol{b}^\dagger} = kT\boldsymbol{I}. \quad (17)$$

This lack of correlation is a rather remarkable outcome. Although input noise waves $\boldsymbol{a}$ will be scattered, generating correlated waves $\boldsymbol{Sa}$ that contribute to the output wave $\boldsymbol{b}$, the net correlation in the output waves must vanish. This puzzle is solved upon considering the noise wave correlation matrix for the multiport. Equation (17) allows a simple derivation. From (1)

$$\begin{aligned}\overline{\boldsymbol{b}\boldsymbol{b}^\dagger} &= \overline{(\boldsymbol{Sa} + \boldsymbol{c})(\boldsymbol{a}^\dagger\boldsymbol{S}^\dagger + \boldsymbol{c}^\dagger)} \\ &= \boldsymbol{S}\overline{\boldsymbol{a}\boldsymbol{a}^\dagger}\boldsymbol{S}^\dagger + \overline{\boldsymbol{c}\boldsymbol{c}^\dagger}\end{aligned} \quad (18)$$

where $\overline{\boldsymbol{a}\boldsymbol{c}^\dagger} = \overline{\boldsymbol{c}\boldsymbol{a}^\dagger} = 0$ since the noise waves produced by a multiport and its terminations are uncorrelated. Substitutions from (2), (16), and (17) yield the expression

$$\boldsymbol{C}_s = kT(\boldsymbol{I} - \boldsymbol{S}\boldsymbol{S}^\dagger) \quad (19)$$

known as Bosma's theorem [3]. The noise waves contributed by the network are simply those necessary to cancel the effects of correlation present in the scattered waves $\boldsymbol{Sa}$ in order to maintain the lack of correlation in output waves, and to achieve balance in power flow, both required for thermodynamic equilibrium.

IV. Conclusion

The noise wave correlation matrix for a passive multiport is found by assuming thermodynamic equilibrium with reflectionless terminations. Under these conditions, it has been shown that a passive multiport produces uncorrelated output waves. This result allows a simple derivation of Bosma's theorem: the relation between a passive multiport's scattering and noise wave correlation matrices.

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