Resonance Oscillations in a Hot Nonuniform Plasma

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(Received 20 April 1964)

A quantitative theory of resonance oscillations, such as observed by Dattner and others, is given. The first two moments of the collisionless Boltzmann equation assuming a scalar pressure are used in conjunction with a physically reasonable radial electron density profile to describe the oscillations of a hot nonuniform plasma cylinder. These equations coupled with Maxwell's equations assuming a scalar potential are solved numerically to yield the frequency spectrum of the plasma wave resonances. It is found that the frequency spectrum depends on the parameter $r_w^2/(\lambda_d^2)$ where r_w is the radius of the plasma column and $\langle \lambda_d^2 \rangle$ is the mean square Debye length of the electron plasma. New experimental observations of dipole and quadrupole spectra for two plasma columns of differing radii are reported and the results of these observations are in good agreement with the theory. The physical mechanism of the resonances is described.

I. INTRODUCTION

IN his early work¹ on plasma oscillations Tonks showed that a bounded uniform plasma exhibits resonance oscillations at a frequency

$$\omega = R\omega_p \quad (1)$$

where $\omega_p = (ne^2/\epsilon_0 m)^{1/2}$ is the electron plasma frequency, dependent upon the electron density n , and R is a dimensionless constant whose value is less than unity, which depends upon the *shape* of the bounded plasma, upon the exact nature of the mode of oscillation, and upon the dielectric constant of the material surrounding the plasma (generally glass in laboratory experiments). This result, obtained by simply regarding the plasma as a uniform dielectric with frequency dependent permittivity $\epsilon = \epsilon_0(1 - \omega_p^2/\omega^2)$, was approximately verified in discharge tubes by Tonks and Langmuir and is generally accepted. Twenty years later this phenomenon assumed new importance in connection with resonant scattering from ionized meteor trails.² Scattering experiments by Romell,³ again using a discharge tube, showed that there was not just one resonance in the scattering cross section, but in addition a *series* of somewhat weaker resonances. Subsequently Dattner⁴ has studied the resonances extensively in a waveguide and they have been shown to be all of a dipole nature⁵ and closely related to the plasma frequency.^{4,6}

Attempts to explain the additional resonances using a dielectric theory of the cold plasma were

unsuccessful. Kaiser and Closs⁷ discussed the resonances of a non-uniform cylindrical plasma by approximating the non-uniform plasma by a series of uniform annular dielectric shells. One new resonance was found for each new discontinuity in density. Keitel⁸ later showed that the scattering cross section of a plasma column with a smooth Gaussian electron density profile has no resonance of appreciable amplitude, but exhibits some scattering at all frequencies. This was in general agreement with the earlier result of Herlofson² who showed that when a uniform plasma column has a diffuse boundary, the resonance predicted by Eq. (1) should be damped by an amount which increases with the diffuseness. These results are all consistent with the idea that the nonuniform dielectric model with a smooth density profile falling monotonically to zero exhibits a *continuous* spectrum of resonances with an accumulation of resonances whenever there is a sharp gradient in electron density. Furthermore, Astrom⁹ has shown that at the point in the plasma where the plasma frequency is equal to the incident wave frequency, the electron motions become very large unless limited by collisions or nonlinearities. Closer examination of the behavior of the plasma at this point brings out more clearly the inadequacy of the dielectric model.

The dielectric model neglects the random thermal motions of the electrons and it is implicit that each electron oscillates back and forth about a fixed point. In fact, the thermal motion of each electron is much greater than that induced by the wave and it is only because the latter is coherent from electron to electron that the resulting current is significant. The introduction of the random thermal motions can be shown to

¹ L. Tonks, Phys. Rev. **37**, 1458 (1931); Phys. Rev. **38**, 1219 (1931).

² N. Herlofson, Arkiv Fysik **3**, 247 (1951).

³ D. Romell, Nature **167**, 243 (1951).

⁴ A. Dattner, Ericsson Technics **2**, 309 (1957); Ericsson Technics **8**, 1 (1963).

⁵ F. I. Boley, Nature **182**, 790 (1958).

⁶ R. W. Gould, Proceedings of the Linde Conference on Plasma Oscillations, Indianapolis (1959); see also, Bull. Am. Phys. Soc. **5**, 322 (1960).

⁷ T. R. Kaiser and R. L. Closs, Phil. Mag. **43**, 1 (1952).

⁸ G. Keitel, Proc. IRE **43**, 1481 (1955).

⁹ E. Astrom, Arkiv Fysik **19**, 163 (1961).

eliminate the singularity at the local plasma frequency which was crucial to the previous results. Gould first applied these ideas to find the resonant frequencies of a uniform hot plasma column.⁶ The plasma electrons were treated as a single fluid with a scalar pressure obeying an adiabatic law. In addition to the resonance similar to that given by the dielectric theory, a spectrum of resonances with frequencies ω_m near the plasma frequency is obtained (see Sec. IV where the uniform plasma is treated as a special case)

$$\omega_m^2 = \omega_p^2 + (3\kappa T/m)k_m^2 \quad m = 1, 2, 3, \dots, \quad (2)$$

where T is the electron temperature and k_m is a wave vector dependent upon the radius of the plasma column r_w . Equation (2) is just the approximate dispersion equation of Bohm and Gross¹⁰ for longitudinal plasma waves which have been multiply reflected back and forth across the plasma column. The resonances in the scattering cross section predicted in this model are thus associated with the existence of "standing" density waves in the plasma column. This model too has its shortcomings for the resonances given by Eq. (2) are clustered very near to the plasma frequency. The spacing between resonances, which is related to the tube diameter, is about an order of magnitude less than is experimentally observed.

The assumption of a uniform density plasma is, of course, not warranted in laboratory plasmas, and that the plasma wave resonances should be seriously affected by a density gradient may be seen as follows.^{11,12} When the density depends on position we may solve (approximately) the plasma wave dispersion for the *local* wavenumber

$$k^2(r) = [\omega^2 - \omega_p^2(r)]/[3\kappa T/m]. \quad (3)$$

In those regions where the density is low enough [$\omega_p(r) < \omega$] the plasma waves propagate (k real) and in those regions where the density is high [$\omega_p(r) > \omega$] the plasma waves are evanescent (k imaginary). Since the electron density (and hence the plasma frequency) decreases monotonically from its value at the center of a cylindrical plasma column, in general, the longitudinal plasma wave does not penetrate to the center of the column, but is instead reflected at the radius r_c at which $\omega_p(r_c) = \omega$. One might therefore expect resonance conditions to be expressible approximately as

$$\int_{r_c}^{\infty} k(r) dr = \theta_m, \quad m = 1, 2, 3, \dots, \quad (4)$$

where the various θ_m 's differ approximately by π . Such an approach does indeed lead to a resonance spectrum in qualitative agreement with experimental observations¹¹ since it gives a greater separation between resonances due to the smaller region over which the plasma waves propagate. However, it is clear from the phase integral (4) that not only is the electron temperature important, but the resonance spectrum is greatly influenced by the electron density profile. The electron density profile in a low density cylindrical plasma has recently been calculated by Parker¹³ and later we summarize those aspects of his solutions which are important here.

Although the phase integral method, Eq. (4), gives a qualitative understanding of how a larger spacing between resonances can arise, it is not possible to rigorously justify the use of Eq. (3) in this manner, nor is it possible to calculate the values of θ_m . In this paper we therefore return to the basic equations of a collisionless, nonuniform, electron gas and obtain solutions appropriate to this problem. Rather than attempt to solve the collisionless Boltzmann equation in a nonuniform electron gas, we employ only the first two moments¹⁴ (continuity and momentum equations) and replace the tensor pressure by a scalar pressure obeying an adiabatic law. No dissipation mechanism is included in our theory, since the moment method does not describe Landau damping¹⁵ and we have not included collisional damping. A similar approach was outlined by Vandenplas and Gould¹⁶ with two important differences: (a) the use of a tensor electron pressure, and (b) the use of a simplified parabolic electron density profile. Since numerical results have only recently been obtained,¹⁷ a comparison with experiment was not possible. Other treatments of a hot nonuniform plasma slab^{18,19} and cylinder²⁰ have recently appeared, all employing different approximations to the electron density profile.

There have been a number of experimental

¹³ J. V. Parker, *Phys. Fluids* **6**, 1657 (1963); see also, California Institute of Technology Report No. 19, Contract Nonr 220(13) (1962).

¹⁴ L. Spitzer, *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956).

¹⁵ L. Landau, *J. Phys. USSR* **10**, 25 (1946).

¹⁶ P. Vandenplas and R. W. Gould, *Nucl. Fusion Suppl.* Pt. 3, 1115 (1962).

¹⁷ P. E. M. Vandenplas and A. M. Messiaen, to be published.

¹⁸ P. Weissglas, *Phys. Rev. Letters* **10**, 206 (1963).

¹⁹ F. C. Hoh, *Phys. Rev.* **133**, A1016 (1964).

²⁰ J. A. Fejer, *Phys. Fluids* **7**, 439 (1964).

¹⁰ D. Bohm and E. P. Gross, *Phys. Rev.* **75**, 1851 (1949).

¹¹ R. W. Gould, California Institute of Technology Report No. 1, Contract DA36-039 SC-85317 (1960).

¹² F. W. Crawford, *Phys. Letters* **5**, 244 (1963).

measurements of the dipole resonances reported in recent years,^{3,4,6,11,12,21,22} the most comprehensive by Dattner.⁴ In this paper we report new measurements on the dipole as well as *quadrupole* resonance spectra of several mercury discharge tubes of different radii. We show that the observed spectra are in excellent agreement with the theory outlined above when the electron temperature of about 3 eV

(a)

(b)

FIG. 1. Block diagram of experimental arrangement for observing resonances: (a) in waveguide systems; (b) with multipole devices.

²¹ W. D. Hershberger, J. Appl. Phys. **31**, 417 (1960).

²² A. M. Messiaen and P. E. M. Vandenplas, Physica **28**, 537 (1962).

FIG. 2. Multipole devices. (a) Split cylinder dipole; (b) wire dipole; (c) split cylinder quadrupole; (d) wire quadrupole.

is assumed.²³ In the presentation below we first outline the observational techniques (Sec. II) and report our experimental results (Sec. III). A discussion of the theoretical model and the plasma equations is given in Section IV. The uniform density plasma⁶ is examined first as a special case, and before considering the nonuniform case, the electron density profile calculations of Parker¹³ are briefly reviewed.

II. OBSERVATIONAL TECHNIQUE

The various radial plasma wave resonances in the cylindrical plasma column can be experimentally observed in several different ways. One method is to observe the frequency-dependence of the scattering of plane electromagnetic waves from the column. Provided that the incident electric field E has a component perpendicular to the cylinder axis, the scattered radiation displays a maximum when the incident wave frequency is equal to one of the resonant frequencies. Such scattering experiments have previously been carried out in free space^{3,5} or in a waveguide system.^{4,6,11} A waveguide system such as employed by Dattner and in part of our experiments is shown in Fig. 1(a).

Another method of observation is based upon the fact that at resonance the plasma column absorbs a more than average amount of energy. Figure 1(b) shows the general experimental arrangement, including a directional coupler used to monitor the signal reflected from the plasma. Figs. 2(a)-(d) show the specific multipole electrode configurations referred to in Fig. 1(b) and used in our experiments. The power of the reflected signal will be that of the incident signal minus any power dissipated in the

²³ A brief account of the results appearing here is to be found in J. C. Nickel, J. V. Parker, and R. W. Gould, Phys. Rev. Letters **11**, 183 (1963).

FIG. 3. Typical absorption spectrum. The frequency is held constant and the electron density is varied by varying the discharge current.

plasma column, radiated from the multipole device, or dissipated in the transmission lines. In the off-resonance condition the column will absorb relatively little energy. However, when the electron density in the column is adjusted (by adjusting the discharge current) to give a resonance condition, the plasma column will absorb more energy and a decrease will be noted in the reflected signal. Thus by monitoring the reflected signal, we can obtain an absorption spectrum of the positive column. A typical absorption spectrum is shown in Fig. 3. The multipole electrode devices shown in Fig. 2 can be constructed so that they preferentially excite the dipole, quadrupole, or higher order multipole modes of the positive column. We have used these devices in the 200–1300 Mc/sec range where their dimensions could be made small compared to a free space wavelength so as to minimize radiation and resonances in the devices themselves.

It is shown in the Appendix that the resonance condition for a plasma column in the split cylinder geometry [Figs. 2(a), (c)] may be written

$$L_n = -(n/r_w)K_{\text{eff}}, \quad (5)$$

where

$$K_{\text{eff}} = K_{\text{eff}}(K, r_w, b, c) \\ = K \frac{(b/r_w)^n(K+f) - (r_w/b)^n(K-f)}{(b/r_w)^n(K+f) + (r_w/b)^n(K-f)}$$

and where r_w , b , and c are the inner and outer radii of the glass tube and the radius of the device respectively, and the integer n denotes the angular dependence ($e^{in\theta}$). K is the relative dielectric constant of the surrounding glass, and

$$f = [1 + (b/c)^{2n}] / [1 - (b/c)^{2n}] \quad (6)$$

measures the influence of the split metal cylinder ($f \rightarrow 1$ when the cylinder is removed). The left-hand side of expression (5) is dependent only on

the solution of the equations for the plasma and the electromagnetic field in the plasma region, while the right-hand side is dependent only on the dielectric and electrode configuration exterior to the plasma. L_n is the ratio of the normal electric field to the potential in the plasma region at the surface of the plasma column, and the method of calculating it is the subject of Sec. IV. For the dipole split cylinder configuration [Fig. 2(a)], $n = 1, 3, 5, \dots$ harmonics are present, while for the quadrupole configuration [Fig. 2(c)] $n = 2, 6, 10$ harmonics are present. However, when $c \geq 3r_w$ only the lowest nonzero multipole component ($n = 1$ and $n = 2$, respectively) has a significant amplitude at the plasma.

In a similar manner the resonance condition for the ideal wire multipoles may be written (Appendix):

$$L_n = -(n/r_w)K_{\text{eff}} \quad (7)$$

where

$$K_{\text{eff}} = K_{\text{eff}}(K, r_w, b, c) \\ = K \frac{(b/r_w)^n(K+1) - (r_w/b)^n(K-1)}{(b/r_w)^n(K+1) + (r_w/b)^n(K-1)}$$

and again the $n = 1, 3, 5, \dots$ harmonics are present for the dipole configuration [Fig. 2(b)] and the $n = 2, 6, 10, \dots$ harmonics are present for the quadrupole configuration [Fig. 2(d)]. For $c \geq 3r_w$ only the $n = 1$ and the $n = 2$ harmonics, respectively, are of importance. In the experiments reported here, we have used the devices of Fig. 2(a), (b) for studying dipole resonances and the devices of Fig. 2(c), (d) for studying quadrupole resonances by keeping the device radius several times larger than the column radius.

Some care must be given, however, to the feeding of the quadrupole and higher order multipole devices. For proper operation alternate elements must be driven 180° out of phase with each other. For the quadrupole devices we ensure this phase condition by driving opposite elements with two sections of coaxial cable of equal length which are fed from a "tee." When the coaxial cable sections are of equal length, a pure quadrupole mode is excited. However, when the sections are of unequal length the dipole harmonic is introduced. In this way it may be clearly seen that the lowest quadrupole resonance occurs for a slightly higher current than the lowest dipole resonance.

It is also seen from Eq. (7) that the resonance condition for the ideal wire multipoles should be independent of the wire position c . Experimentally it is found that the resonance condition is somewhat

dependent upon c , especially for the main resonance. This may be due to the finite size of the wires, an effect which would be difficult to include in the calculation.

In our experiments we have kept the incident frequency constant and varied the tube current to obtain the resonance conditions. After the current is adjusted for resonance, the average electron density $\langle n_e \rangle$ corresponding to that current, is measured by observing the frequency shift of a right circular cavity operating in a TM_{010} mode. According to perturbation theory,²⁴ the average density is related to the frequency shift of the cavity by

$$\langle n_e \rangle = \frac{8\pi^2 m \epsilon_0 J_i^2 (2.405)}{e^2} \frac{r_w^2}{d^2} f_0 \Delta f \quad (8)$$

where d is the cavity radius, r_w is the column radius, f_0 is the resonant frequency of the cavity with the glass but no plasma, and $\Delta f = f - f_0$, where f is the resonant frequency with the plasma. For these experiments f_0 was about 3000 Mc/sec which is well above the plasma frequency for the densities used. For the range of densities explored, the frequency shifts varied from about 0.5 to 30 Mc/sec and could be determined to within 0.2 Mc/sec.

In presenting our experimental results we have normalized the squares of all resonance frequencies to the mean square plasma frequency $\langle \omega_p^2 \rangle = \langle n_e \rangle e^2 / \epsilon_0 m$, which depends only upon the measured mean density $\langle n_e \rangle$. A single dimensionless parameter of the discharge also enters: the ratio of the column radius r_w to the Debye length. We have chosen to use $r_w^2 / \langle \lambda_d^2 \rangle$, where $\langle \lambda_d^2 \rangle = \epsilon_0 k T / \langle n_e \rangle e^2$, since it depends on the measured *mean* electron density (note especially the type of average Debye length implied by this definition) and an electron temperature T .

III. EXPERIMENTAL RESULTS

Experiments have been conducted on the positive columns of two separate mercury discharge tubes of different radii (r_w). The column of Tube No. 1 was contained in a Corning type 3320 glass tube of internal radius 0.51 cm and wall thickness 0.11 cm, while the column of Tube No. 2 was contained in a Corning type 7740 glass tube of internal radius 0.3 cm and wall thickness 0.1 cm. For $r_w^2 / \langle \lambda_d^2 \rangle < 10^4$ the split cylinder and wire multipole resonance detecting devices were used. The split cylinder device was used for the main resonance in all cases since this resonance is the most sensitive to external conditions and a well defined K_{eff} could be calculated.

²⁴ See, for example, S. J. Buchsbaum, L. Mower, and S. C. Brown, Phys. Fluids 3, 806 (1960).

FIG. 4. Dipole resonance spectrum for Tube No. 1.

ted. In the lower range of $r_w^2 / \langle \lambda_d^2 \rangle$ the tubes were operated in a steady state with a vacuum tube current controller. A 500Ω carbon resistor in series with the tubes and located directly at their anode was used to eliminate oscillations associated with the tubes and their circuitry. These oscillations were found predominantly at tube currents less than 30 or 40 Ma and otherwise masked any resonances occurring in this range. For $r_w^2 / \langle \lambda_d^2 \rangle > 10^4$ the resonances were examined using the (S-band) waveguide method. In this range the tubes were pulsed using a thyatron controlled RC discharge of time constant 1.65×10^{-4} sec. When interpreting the waveguide data it was assumed that the waveguide walls had a negligible effect on the resonance condition, i.e., that $K_{eff} = K_{eff}(K, r_w, b, c = \infty)$.

For the discharge tubes used it was found that the electron density gradients along the tube were negligible over a 6 or 7-in. length near the center of the column. Therefore the plasma resonance device and the cavity could be placed side by side as in Fig. 1.

The dipole ($n = 1$) data for Tube No. 1 ($r_w = 0.51$ cm) are shown as circles in Fig. 4, while the quadrupole ($n = 2$) data are shown in Fig. 5. Here we label the resonances as Main, 1st, 2nd, etc. The main resonance refers to the lowest order plasma wave resonance which is sometimes called the Herlofson resonance, while the other resonances are the higher order plasma wave resonances. For Tube No. 1 we have assumed that $T_e = 3$ eV. The circles in Fig. 6 give the dipole data for Tube No. 2 ($r_w = 0.3$ cm).

Here we have assumed a slightly higher electron temperature, $T = 3.7$ eV. The theoretical results for the tubes (Sec. IV) are also shown in Figs. 4–6 as solid lines; the dotted lines indicate the results

FIG. 5. Quadrupole resonance spectrum for Tube No. 1.

predicted by the dielectric theory (Eq. 1) for the main resonance, assuming that ω_p^2 may be replaced by its *mean* value and that $R^2 = 1/(1 + K_{eff})$. It is interesting to note that by plotting $r_w^2/\langle\lambda_d^2\rangle$ on a logarithmic scale the effect of assuming a larger or smaller electron temperature in reducing the data is simply to shift all experimental points to the left or the right of the theoretical curves. This property is very useful in selecting an electron temperature which best fits the data. Furthermore, the temperatures obtained by this procedure are in good agreement with probe measurements of temperatures in mercury discharges.

IV. THEORETICAL TREATMENT

We assume that the plasma wave resonances are describable by the collisionless Boltzmann equation. This assumption seems justified in view of the fact that electron collision frequencies in a Hg discharge at 1μ pressure are approximately 100 times smaller than the corresponding plasma frequencies. We also make use of the quasistatic approximation (see Appendix) to replace Maxwell's equations by a simple scalar potential equation for the electric field since the dimensions are small compared to an electromagnetic wavelength.

The basic equations are therefore

$$\frac{\partial f}{\partial t} + \mathbf{U} \cdot \frac{\partial f}{\partial \mathbf{r}} - \frac{e}{m} \mathbf{E} \cdot \frac{\partial f}{\partial \mathbf{U}} = 0, \quad (9)$$

$$\mathbf{E} = -\nabla\phi, \quad (10)$$

$$\nabla^2\phi = -\rho/\epsilon_0, \quad (11)$$

where $f(\mathbf{r}, \mathbf{U})$ is the electron distribution function, $\mathbf{U}$ and $\mathbf{r}$ are the velocity and position respectively, and $\mathbf{E}$, ϕ , and ρ are the electric field, potential, and charge density.

The difficulty of solving equations (9)–(11) is very great when one considers spatial nonuniform plasmas. We have elected to use the alternate procedure of taking the velocity moments¹⁴ of Eq. (9). This replaces the single equation by an infinite number of coupled equations which deal only with the macroscopic quantities of the plasma, i.e.,

$$\text{density} \quad n_e = \int_{-\infty}^{\infty} f d^3u$$

$$\text{mean velocity} \quad v_i = \int_{-\infty}^{\infty} u_i f d^3u \quad (\text{vector})$$

$$\text{pressure tensor} \quad p_{ij} = \int_{-\infty}^{\infty} w_i w_j f d^3u \quad (\text{second rank tensor})$$

$$\text{heat flux tensor} \quad Q_{ijk} = \int_{-\infty}^{\infty} w_i w_j w_k f d^3u \quad (\text{third rank tensor})$$

where $w_i = u_i - v_i$.

One may arbitrarily terminate this infinite chain after the n th moment equation by setting

$$\int_{-\infty}^{\infty} w_i w_j \cdots w_{n+1} f d^3u = 0.$$

FIG. 6. Dipole resonance spectrum for Tube No. 2.

Whether this termination process is justified, and if so at what moment, can be partially answered by considering the problem of a uniform hot plasma. For propagation of longitudinal waves one can show^{16,25} that retaining higher and higher moments is equivalent to including higher terms in the expansion of $(\omega/\omega_p)^2$ in powers of (thermal speed/wave speed)² in the dispersion relation for longitudinal plasma waves. This expansion agrees, except for Landau damping, with that obtained if one integrates the Boltzmann equation directly and then expands the resulting plasma dispersion function.²⁶ This expansion is, however, an asymptotic series and by convention should be terminated when the higher terms begin to increase.

If we let $\xi^2 = (\text{thermal speed/wave speed})^2$ then this series may be written

$$\omega^2 = \omega_p^2 \sum_{m=0}^{\infty} \frac{1 \cdot 3 \cdot 5 \cdots (2m+1)}{2^m} \xi^{2m}, \quad \xi^2 = \left(\frac{kv_e}{\omega} \right)^2$$

and the terms decrease monotonically until $m = m_0$ where

$$m_0 = 1/\xi^2 - \frac{1}{2}.$$

To estimate ξ^2 we use for the wave speed the ratio of wave vector to frequency, where the wave vector squared is approximately

$$k^2 = \frac{2}{3}(\omega^2 - \omega_p^2)/v_e^2$$

and the thermal speed squared is

$$v_e^2 = 2kT/m_e.$$

Finally, for m_0 we obtain

$$m_0 = \frac{2}{3}\omega^2/(\omega^2 - \omega_p^2) - \frac{1}{2}.$$

Since we are concerned with the propagating region of the plasma where $0 < \omega_p^2 \leq \omega^2$ we find that the smallest value of m_0 is $m_0 = 1$. This corresponds to including terms as high as the pressure tensor and possibly the heat flux tensor, but no more. One can see then that inclusion of higher terms, besides increasing the difficulty of solution, should not contribute materially to the accuracy of the results.

In view of this we choose to set the heat flux tensor equal to zero. Even so, the resulting differential equation is sixth order and to facilitate solution we employ a common assumption which reduces the order to fourth. This assumption simply replaces the pressure tensor with a scalar pressure proportional to the density (this approximation is

exact for a uniform plasma). We should therefore consider the following as only a semiquantitative theory.

The moment equations, modified as described above, are

$$\partial n_e / \partial t + \nabla \cdot (n_e \mathbf{v}) = 0 \quad (12)$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = \frac{1}{m_e n_e} \{-n_e e \mathbf{E} - \nabla p\}. \quad (13)$$

Because we are concerned with a small perturbation induced in the plasma by an applied oscillating field we choose to represent all quantities as the sum of a steady state term (subscript zero) and a perturbation (subscript 1). These new variables are defined as follows,

$$n_e = n_{e0}f(r) + n_1(\mathbf{r})e^{-i\omega t}, \quad \mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1,$$

$$p = p_0(\mathbf{r}) + p_1(\mathbf{r})e^{-i\omega t}, \quad \mathbf{v} = \mathbf{v}_1(\mathbf{r})e^{-i\omega t}.$$

Here we have let n_{e0} be the electron density at the center of the column and $f(r)$ be a dimensionless function which describes the non-uniformity of the electron density; the ions are assumed unperturbed due to their higher mass, and in the steady state the electron velocity is zero. When these definitions are inserted in equations (10)–(13) the following equations result.

Zero order in the perturbation (equilibrium equations):

$$\nabla p_0 = -en_{e0}f(r)\mathbf{E}_0, \quad (14)$$

$$\nabla \cdot \mathbf{E}_0 = (e/\epsilon_0)[n_i - n_{e0}f(r)]. \quad (15)$$

First order in the perturbation:

$$i\omega n_1 = n_{e0}\nabla \cdot (f(r)\mathbf{v}_1), \quad (16)$$

$$i\omega m_e n_{e0}f(r)\mathbf{v}_1 = en_1\mathbf{E}_0 + en_{e0}f(r)\mathbf{E}_1 + \nabla p_1, \quad (17)$$

$$\nabla \cdot \mathbf{E}_1 = -(e/\epsilon_0)n_1. \quad (18)$$

As mentioned earlier, some assumption must be made relating p to the other variables in order to make these equations complete. We actually make different assumptions regarding p_0 and p_1 . First, that the steady state pressure p_0 is related to the steady state density n_0 by

$$p_0 = n_{e0}f(r)kT, \quad (19)$$

which follows from the assumption of a Maxwellian velocity distribution for the electron gas. Second, that the perturbed pressure p_1 is related to the perturbed density by

$$p_1 = \gamma kTn_1, \quad (20)$$

where γ is a constant depending on the type of compression involved. The collisionless plasma

²⁵ C. Oberman, Princeton University, Project Matterhorn, MATT-57 (1960).

²⁶ B. D. Fried and S. Conte, *The Plasma Dispersion Function* (Academic Press Inc., 1961).

undergoes a simple one-dimensional adiabatic compression, hence we take $\gamma = 3$. From Eqs. (14) and (19) one may find the static electric field in terms of the electron density

$$\mathbf{E}_0 = -\frac{\kappa T}{e} \frac{\nabla f(r)}{f(r)}. \quad (21)$$

Equations (16)–(18), (20) and (21) can now be combined to yield a single fourth order differential equation for the potential ($\mathbf{E}_1 = -\nabla\phi_1$):

$$\begin{aligned} \nabla^2 \nabla^2 \phi_1 - \frac{1}{\gamma} \left(\frac{\nabla f}{f} \cdot \nabla \right) \nabla^2 \phi_1 + \left[\frac{1}{\gamma \lambda_{d0}^2} \left(\frac{\omega^2}{\omega_{p0}^2} - f \right) \right. \\ \left. - \frac{1}{\gamma} \nabla \cdot \left(\frac{\nabla f}{f} \right) \right] \nabla^2 \phi_1 - \frac{1}{\gamma} \nabla f \cdot \nabla \phi_1 = 0. \end{aligned} \quad (22)$$

When it is assumed that $\phi_1(r, \theta) = \Phi_n(r) \exp(in\theta)$, Eq. (22) becomes a fourth-order ordinary differential equation in $\Phi_n(r)$. It is also convenient for numerical computation to have this equation in dimensionless form. When one introduces the dimensionless length $z = r/r_w$ where r_w is the radius of the plasma column, then the fourth order differential equation for $\Phi_n(z)$ is seen to depend only upon the two dimensionless quantities (r_w^2/λ_{d0}^2) and (ω^2/ω_{p0}^2) , where λ_{d0} and ω_{p0} are the Debye length and plasma frequency at the center of the plasma column.

Solution of Eq. (22) for a Uniform Plasma

In order to understand the general nature of the solutions to this equation one can examine the simpler problem of a uniform hot plasma.⁶ In this case $f(r) = 1$, $\nabla f(r) = 0$ and Eq. (22) reduces to

$$(\nabla^2 + k^2) \nabla^2 \phi_1 = 0, \quad (22a)$$

where

$$k^2 = \frac{1}{\gamma \lambda_{d0}^2} \left(\frac{\omega^2}{\omega_{p0}^2} - 1 \right). \quad (22b)$$

The solutions of Eq. (22a) are easily seen to be the solutions to

$$\nabla^2 \phi_1 = 0 \quad (22c)$$

and

$$(\nabla^2 + k^2) \phi_1 = 0. \quad (22d)$$

Equation (22c) describes the transverse waves (of infinite wavelength in the quasi-static approximation) whereas the second equation describes the longitudinal (plasma) waves. In a uniform plasma these two types of solutions are uncoupled, as this separation indicates. However, in a nonuniform plasma [as described by the more complicated equation (22)] the longitudinal and transverse waves are coupled by the electron density gradient. The scale

of the density gradient is of the order (a few Debye lengths) of the longitudinal wavelength and much shorter than the “infinite” wavelength transverse waves. Thus the coupling between longitudinal and transverse waves is strong.

Two of the four solutions to Eqs. (22c) and (22d) are singular at the origin and must be discarded. To uniquely determine the solution one must specify two additional conditions. One of these is simply an amplitude factor and we select it so as to make $\Phi_n(1) = 1$. The other condition is that the radial electron current at the wall should vanish. This is implicit in the assumption of a nonconducting boundary around the plasma.

Applying these boundary conditions to the solutions of Eq. (22a) and using the previously derived condition for resonance, $L_n = [(d\Phi_n/dr)/\Phi_n]_{z=1} = -nK_{eff}$, we obtain the following transcendental equation for the n -pole resonances ($\exp[in\theta]$ angular dependence)

$$\frac{nJ_n(kr_w)}{kr_w J'_n(kr_w)} = \frac{\omega^2}{\omega_{p0}^2} + \frac{1}{K_{eff}} \left(\frac{\omega^2}{\omega_{p0}^2} - 1 \right).$$

This equation is to be solved simultaneously with expression (22b) for k . Eliminating k between these equations results in a single implicit equation for (ω/ω_{p0}) with (r_w/λ_{d0}) as a parameter. One solution of this equation is possible for $\omega < \omega_{p0}$ (and k imaginary, hence an evanescent plasma wave) and infinitely many solutions are possible for $\omega > \omega_{p0}$ (k real, hence plasma waves propagate across plasma). When (r_w/λ_{d0}) is large, as is the case for all experiments so far, the frequency of the lowest resonance agrees quite closely with that obtained from the dielectric theory [$\omega \approx \omega_{p0}/(1 + K_{eff})^{1/2}$]. The first few plasma wave resonances ($\omega > \omega_{p0}$) have frequencies only slightly greater than ω_{p0} . This is not in agreement with experiment, where large separations between the plasma wave resonances are found. By means of the qualitative arguments given in the introduction we conclude that approximate agreement with experiment might be obtained if the proper electron density profile were taken into account.

Static Electron Density Profile

Before discussing the solution of Eq. (22) for a nonuniform plasma, a few remarks are necessary about the function $f(r)$ which describes the steady state electron density profile. The function $f(r)$ has been calculated by Parker¹³ using the model of Tonks and Langmuir.²⁷ This calculation involves solving Eq. (15) with the following model of the

²⁷ L. Tonks and I. Langmuir, Phys. Rev. **34**, 876 (1929).

plasma. A Maxwellian electron gas is contained by the radial electric field created when the high speed electrons attempt to flow to the walls more rapidly than the massive ions can follow. The ions are assumed to be generated with negligible velocity at a rate proportional to the local electron density and to fall directly to the walls under the influence of the radial electric field. This implies a negligible ion-neutral collision frequency so that the ions suffer negligible deflection as they move to the walls. The ion density at radius r is an integral from 0 to r of $S(r')$, the number of ions created per second at radius r' , divided by the velocity they have attained in falling from r' to r .

$$n_i(r) = \int_0^r S(r') \frac{r'}{r} \left\{ \frac{2e}{m_i} [\phi_0(r') - \phi_0(r)] \right\}^{-1/2} dr'.$$

Since the electron gas is assumed Maxwellian, its density is given by

$$n_e(r) = n_{e0} \exp [e\phi_0(r)/kT]$$

where $\phi_0(r)$ is the static electric potential. When these two relations for the ion and electron densities are combined with Eq. (15) the resulting integro-differential equation is found to depend on a single parameter A (originally introduced by Langmuir²⁷). Aside from a slowly varying factor of order unity, A is the same as our parameter (r_w^2/λ_{d0}^2) . The boundary condition on this equation is simply that the current of ions and electrons to the insulating wall shall be equal. Details of this calculation appear elsewhere.^{13,28} The electron density profiles $f(r)$ which were required for this work are plotted in Fig. 7 for a mercury plasma. To facilitate comparison with experiment we have chosen to express all theoretical results in terms of the mean electron density $\langle n_e \rangle$ since it is the measured quantity and a straightforward conversion between $\langle n_e \rangle$ and n_{e0} is possible in the theory.¹³ Thus in Fig. 7 and in subsequent figures the parameter $r_w^2/\lambda_d^2 = r_w^2/\langle n_e \rangle / \epsilon_0 kT$ is used and resonant frequencies are expressed as $\omega^2/\langle \omega_p^2 \rangle$.

Solution of Eq. (22) for Nonuniform Plasma

Returning to the solution of Eq. (22), it remains only to remark that the general considerations such as separation of the angular dependence [let $\phi_1 = \Phi_n \exp(in\theta)$] and the boundary conditions discussed in connection with the uniform plasma apply here also. The boundary condition which requires the radial electron current to vanish at the boundary of

FIG. 7. Electron density profiles for a mercury plasma (from Ref. 13)

the plasma can be expressed mathematically as follows:

$$\left\{ \left[\frac{d}{dz} - \frac{1}{\gamma f} \frac{df}{dz} \right] \left[\frac{1}{z} \frac{d}{dz} \left(z \frac{d\Phi_n}{dz} \right) - \frac{n^2}{z^2} \Phi_n \right] - \frac{fr_w^2}{\lambda_{d0}^2} \frac{d\Phi_n}{dz} \right\}_{z=1} = 0. \quad (23)$$

The solutions to the dimensionless equation were obtained numerically on an IBM 7090 computer.²⁸ Power series were developed about the origin $r = 0$ for the two finite solutions and these were evaluated at approximately $z = 0.02$ to avoid the singular point at the origin. The solutions were continued from there to the wall $z = 1$ by a straightforward integration of the differential equation. After the two independent solutions were found in this manner the correct linear combination was taken so as to satisfy the boundary conditions (23) and $\Phi_1(r_w) = 1$. The value of $\Phi_1'(r_w) = (d\Phi_n/dz)_{z=1}$ was calculated and the final result was the value of $(\Phi_n'/\Phi_n)_{z=1}$, the logarithmic derivative of the potential at the edge of the plasma. This calculation was carried out as a function of $\omega^2/\langle \omega_p^2 \rangle$ in the range $0 < \omega^2/\langle \omega_p^2 \rangle < 2.0$ and for five values of $r_w^2/\langle \lambda_d^2 \rangle$. The highest value of $r_w^2/\langle \lambda_d^2 \rangle$ considered was $(r_w^2/\langle \lambda_d^2 \rangle) = 4500$. This limit is imposed by the exponential nature of the solutions in the cutoff region (plasma interior), since when $\exp [r_w/(\langle \lambda_d^2 \rangle)^{1/2}] > 10^{16}$ the machine is no longer capable of correct solutions even in double precision.

Theoretical Results and Discussion

The result of a calculation of the type just described is illustrated in Fig. 8. $\Phi_n'(1)/\Phi_n(1)$ is plotted versus $\omega^2/\langle \omega_p^2 \rangle$ for the dipole case ($n = 1$) and for the value $r_w^2/\langle \lambda_d^2 \rangle = 1600$. We have found that this curve, and similar curves for other values of $r_w^2/\langle \lambda_d^2 \rangle$ are all accurately represented by an expression of the form

$$\frac{\Phi_n'(1)}{\Phi_n(1)} = n \left[1 + \sum_m \frac{A_{nm} \langle \omega_p^2 \rangle}{\omega_{nm}^2 - \omega^2} \right] \quad (24)$$

²⁸ J. V. Parker, Ph.D. thesis, California Institute of Technology (1964).

FIG. 8. Logarithmic derivative of the potential as a function of the frequency $r_w^2/\langle\lambda_d^2\rangle = 1600$.

where in the frequency range of interest, $0 < \omega^2 < \langle 2\omega_p^2 \rangle$, several terms of the series suffice. The constants A_{nm} and $\omega_{nm}^2/\langle\omega_p^2\rangle$ are slowly varying functions of the parameter $r_w^2/\langle\lambda_d^2\rangle$ and also depend slightly on the multipole order (n). Tables I and II presents the results of the calculations in a convenient form.

One could regard the bracketed quantity in Eq. (24) as an *apparent* dielectric constant of the plasma cylinder, although to be sure it depends also on multipole order (n). It is interesting to note, however, that this quantity satisfies the conditions for a lossless, causal dielectric and that, for dipole modes, a sum rule

$$\sum_m A_{1m} = 1, \quad (25)$$

seems to be obeyed. This result is not completely unexpected, since it leads to an apparent dielectric constant, $1 - \langle\omega_p^2\rangle/\omega^2$ at high frequencies ($\omega^2 \gg \langle\omega_p^2\rangle$). It also serves as a check on the accuracy of the calculations. A similar sum rule can be shown to hold for the quadrupole modes provided one uses a *different* definition for the mean square plasma frequency,

$$\langle\omega_p^2\rangle_{\text{quad}} = \int_0^{r_w} \omega_p^2(r) r^3 dr \left(\int_0^{r_w} r^3 dr \right)^{-1}, \quad (26)$$

TABLE I. Dipole resonance.

$r_w^2/\langle\lambda_d^2\rangle$	$\omega_{11}^2/\langle\omega_p^2\rangle$	A_{11}	$\omega_{12}^2/\langle\omega_p^2\rangle$	A_{12}	$\omega_{13}^2/\langle\omega_p^2\rangle$	A_{13}
72	0.6404	0.8521	2.656	0.1297	4.977	0.01859
194	0.3657	0.8412	1.618	0.1370	2.815	0.02177
515	0.2202	0.8507	1.106	0.1208	1.859	0.0254
1580	0.1270	0.8692	0.7983	0.0907	1.302	0.0242
4530	0.0767	0.8876	0.6431	0.0646	1.014	0.0204

TABLE II. Quadrupole resonance.

$r_w^2/\langle\lambda_d^2\rangle$	$\omega_{21}^2/\langle\omega_p^2\rangle$	A_{21}	$\omega_{22}^2/\langle\omega_p^2\rangle$	A_{22}	$\omega_{23}^2/\langle\omega_p^2\rangle$	A_{23}
72	0.9335	0.3431	3.204	0.00487	5.906	0.00564
194	0.5099	0.3998	1.839	0.03061	3.119	0.0065
515	0.3076	0.4631	1.219	0.0507	1.969	0.0099
1580	0.1825	0.5425	0.8593	0.0582	1.349	0.0135
4530	0.1141	0.6124	0.6797	0.0509	1.040	0.0142

where the average is weighted according to the square of the quadrupole electric field which would exist at high frequencies. In Tables I and II the usual definition of mean plasma frequency has been used so this sum rule is not immediately apparent.

In the appendix it was shown that resonance occurs when $\Phi_n'(1)/\Phi_n(1) = -nK_{eff}$, e.g., for those frequencies at which the dashed line ($K_{eff} = 2.1$) intersects the solid curve in Fig. 8. This procedure was employed in constructing the theoretical curves shown in Figs. 4–6. Several important conclusions can be drawn from Fig. 8 and similar curves for other values of $r_w^2/\langle\lambda_d^2\rangle$.

(1) Examination of Fig. 8 shows that the lowest resonance is the only one whose frequency is materially affected by the surrounding electrodes and glass wall, i.e., by a change in K_{eff} . This suggests that a relatively large fraction of the oscillation energy of the higher modes is “stored” within the plasma region.

(2) The value of A_{nm} is largest for the lowest resonance and decreases monotonically for the higher resonances. It is particularly small (but not zero) for resonances whose frequencies are higher than the maximum plasma frequency of the column. This is in qualitative agreement with the fact that higher resonances are weaker, but disagrees with Crawford’s prediction¹² of a series limit when $\omega = \omega_p(0)$.

(3) The value of A_{nm} for a quadrupole resonance is,

FIG. 9. Radio frequency electron density n_1 at resonance for the four lowest modes. The point r_c marks the radius where the plasma wave propagation cuts off [$\omega_p(r_c) = \omega$].

FIG. 10. Radio frequency electron density $n_1(z)$ at resonance for the next to lowest mode showing the influence of sheath thickness.

in general, smaller than the value for the corresponding dipole resonances.

(4) The average scattering power of the plasma column changes with $r_w^2/\langle\lambda_d^2\rangle$. When this parameter is small, for example, $r_w^2/\langle\lambda_d^2\rangle = 72$, then the ratio (scattered field/incident field) at the plasma column is about 0.05, while for $r_w^2/\langle\lambda_d^2\rangle = 4500$ this ratio is about 0.65, corresponding to almost metallic reflection.

The manner in which the nonuniform plasma causes such a large separation of the resonant frequencies is illustrated in Figs. 9 and 10, which are plots of the perturbation electron density in the plasma. Figure 9 shows progressively higher resonance modes for the value $r_w^2/\langle\lambda_d^2\rangle = 1600$. It can be seen that each higher resonance contains one more half-wavelength of the plasma wave. Since it requires greater radial distance in which to fit this extra half-wave, one must raise the incident frequency until more of the plasma can propagate waves (the point r_c indicates the radius where wave propagation ceases, i.e., $\omega^2 = f(r_c)n_0e^2/\epsilon_0m_e$). This is in qualitative agreement with the idea, presented in the introduction, that longitudinal waves propagate in an annular region between the boundary ($r = r_w$) and the critical point ($r = r_c$). Figure 10 illustrates the influence of sheath thickness on a resonance. The location of critical points for various resonances are indicated by circles in Fig. 7. Here the density perturbations are shown for the next to lowest resonance for various values of the parameter $r_w^2/\langle\lambda_d^2\rangle$. As this parameter increases, the thickness of the propagating region (the sheath region) decreases; this does not, however, result in a compensating rise in the frequency $\omega_2/\langle\omega_p^2\rangle$, since the wavelength of oscillation also decreases as $r_w^2/\langle\lambda_d^2\rangle$ increases. These two effects combine to hold the normalized frequency of oscillation nearly constant as the oscillation is restricted to an increasingly narrow region near the surface.

V. CONCLUSIONS

The good agreement between theory and experiment supports the validity of using the truncated moment equations with a scalar pressure. The uniform agreement for all values of $(r_w^2/\langle\lambda_d^2\rangle)$ increases confidence in the model used for calculating the static electron density profile.

It appears that for low density cylindrical plasmas this technique might be useful as a diagnostic tool. The quantities T , $\langle n \rangle$ can be obtained by observing a resonance spectrum. From these quantities the radial density profiles can be inferred.

ACKNOWLEDGMENTS

The authors wish to express their sincere appreciation to the Office of Naval Research for the support of this investigation and to the National Science Foundation for the fellowship held by Mr. Parker during this period.

APPENDIX

Here we derive the resonance conditions for a plasma column in the multipole devices. As the dimensions of these devices are small compared to the free space wavelength of the exciting signal, we assume that the fields in their neighborhood are quasistatic or are derivable from a scalar potential, or that

$$\mathbf{E}(r, \theta) = -\nabla\phi.$$

We also assume an $e^{-i\omega t}$ time dependence throughout. Regions I, II, and III will refer to the plasma, glass tube, and space between the glass tube and electrodes, respectively (see Fig. 2).

According to the theory presented in Sec. IV, the potential in Region I, $r < r_w$ has the form

$$\phi^I(r, \theta) = \sum_n \Phi_n \left(r, \frac{\omega^2}{\langle\omega_p^2\rangle} \frac{r_w^2}{\langle\lambda_d^2\rangle} \right) \sin n\theta.$$

The results of Sec. IV allow us to calculate the logarithmic derivative

$$L_n = \left[\frac{r}{\Phi_n} \frac{d\Phi_n}{dr} \right]_{r=r_w}$$

as a function of $\omega^2/\langle\omega_p^2\rangle$ and $r_w^2/\langle\lambda_d^2\rangle$.

In Region II, $r_w < r < b$, the potential has the form

$$\phi_n^{II}(r, \theta) = \sum_n V_n \left\{ B_n \left(\frac{r}{b} \right)^n + F_n \left(\frac{r}{b} \right)^{-n} \right\} \sin n\theta, \quad (A1)$$

while in Region III, $b < r < c$, the potential has the form

$$\phi^{\text{III}}(r, \theta) = \sum_n V_0 \left\{ D_n \left(\frac{r}{b} \right)^n + S_n \left(\frac{r}{b} \right)^{-n} \right\} \sin n\theta, \quad (\text{A2})$$

where V_0 is the potential of the electrode. The boundary conditions at $r = r_w$ and $r = b$ demand the

$$g_n = \frac{\frac{L_n}{nK} \left[(K+1) \left(\frac{b}{r_w} \right)^n + (K-1) \left(\frac{r_w}{b} \right)^n \right] + \left[(K+1) \left(\frac{b}{r_w} \right)^n - (K-1) \left(\frac{r_w}{b} \right)^n \right]}{\left[(K+1) \left(\frac{r_w}{b} \right)^n - (K-1) \left(\frac{b}{r_w} \right)^n \right] - \frac{L_n}{nK} \left[(K+1) \left(\frac{r_w}{b} \right)^n + (K-1) \left(\frac{b}{r_w} \right)^n \right]} \quad (\text{A4})$$

and K is the relative dielectric constant of the glass. Thus, in Region III, the potential becomes

$$\phi^{\text{III}}(r, \theta) = V_0 \sum_n \left[g_n \left(\frac{r}{b} \right)^n + \left(\frac{r}{b} \right)^{-n} \right] S_n \sin n\theta. \quad (\text{A5})$$

The coefficient S_n is determined by the known potential at $r = c$. For the split cylinder multipoles [Figs. 2(a), (c)] we find

$$S_n = \left(\frac{b}{c} \right)^n \frac{4}{n\pi \{g_n + (b/c)^{2n}\}}, \quad n = 1, 3, 5, \dots \text{dipole device}, \quad (\text{A6a})$$

$$= \left(\frac{b}{c} \right)^n \frac{8}{n\pi \{g_n + (b/c)^{2n}\}}, \quad n = 2, 6, 10, \dots \text{quadrupole device}, \quad (\text{A6b})$$

and considering the wires to be ideal line charges of charge q per unit length (replace V_0 by q in the general expressions A1, A2, and A5), we find for the wire multipoles [Figs. 2(b), (c)]

$$S_n = \pi\epsilon_0 n \left(\frac{b}{c} \right)^n \frac{1}{g_n}, \quad n = 1, 3, 5, \dots \text{dipole device},$$

$$= \frac{2}{\pi\epsilon_0 n} \left(\frac{b}{c} \right)^n \frac{1}{g_n}, \quad n = 2, 6, 10, \dots \text{quadrupole device}.$$

These expressions completely determine the potential in Regions II and III for any driving frequency, given the logarithmic derivative L_n from the solution for the interior region. In this paper we are primarily interested in the resonant frequencies of the system, i.e., those frequencies for which an applied voltage V_0 , or an applied current ($-i\omega q$) produces a very large response (field) in the system. For a system with no dissipation, such as described by our theoretical model, the resonant response is *infinite*. Thus the resonant frequencies are deter-

mined by the singularities of S_n . For the split cylinder multipoles, the poles in S_n occur when

$$D_n = g_n S_n \quad (\text{A3})$$

where

$$g_n + (b/c)^{2n} = 0$$

or when

$$L_n = -nK \frac{\left(\frac{b}{r_w} \right)^n (K+f) - \left(\frac{r_w}{b} \right)^n (K-f)}{\left(\frac{b}{r_w} \right)^n (K+f) + \left(\frac{r_w}{b} \right)^n (K-f)}$$

$$= -nK_{\text{eff}}(K, r_w, b, c)$$

where K is the relative dielectric constant of the glass and

$$f = \frac{1 + (b/c)^{2n}}{1 - (b/c)^{2n}}.$$

For the ideal wire multipoles, the poles of S_n occur when

$$g_n = 0$$

or when

$$L_n = -nK \frac{(K+1) \left(\frac{b}{r_w} \right)^n - (K-1) \left(\frac{r_w}{b} \right)^n}{(K+1) \left(\frac{b}{r_w} \right)^n + (K-1) \left(\frac{r_w}{b} \right)^n}$$

$$= -nK_{\text{eff}}(K, r_w, b, c = \infty).$$

Thus we see that for a plasma column in one of these multipole devices a resonance occurs each time the conditions in the plasma are adjusted such that

$$L_n = -nK_{\text{eff}}(K, r_w, b, c)$$

where K_{eff} is an *effective* dielectric constant for the region exterior to the plasma, i.e., if a plasma were surrounded by a homogeneous medium whose dielectric constant is K_{eff} it would exhibit exactly the same resonance spectrum as the more complicated system employed here.