The three-dimensional instability of strained vortices in a viscous fluid

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The recent theory describing 3-D exact solutions of the Navier–Stokes equations is applied to the problem of stability of 2-D viscous flow with elliptical streamlines. An intrinsically inviscid instability mechanism persists in all such flows provided the length scale of the disturbance is sufficiently large. Evidence is presented that this mechanism may be responsible for 3-D instabilities in high Reynolds number flows whose vortex structures can be locally described by elliptical streamlines.

I. INTRODUCTION

Craik and Criminale[1] recently presented a new class of 3-D exact solutions of the incompressible Navier–Stokes equations, describing spatially periodic "disturbances" superimposed on an unbounded basic flow of uniform shear. An important case covered is where the basic flow is one of 2-D strain and rotation and has uniform constant vorticity. In this case the components of the basic flow, which has elliptical streamlines, can be described by

$$\mathbf{U} = A\mathbf{x}, \quad A = \begin{pmatrix} 0 & -\gamma-\epsilon & 0 \\ \gamma-\epsilon & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (1)$$

where ϵ is the strain defined by the maximum rate of extension, 2γ is the vorticity, and $|\epsilon| < |\gamma|$. The exact solutions take the form

$$\mathbf{u} = \mathbf{U} + \mathbf{u}',$$

where

$$\mathbf{u}' = \mathrm{Re}\{\mathbf{v}(t)\exp[i\mathbf{k}(t)\cdot\mathbf{x}]\}. \qquad (2)$$

In their paper, Craik and Criminale[1] point out the possibility of the unbounded growth of $\mathbf{v}(t)$ despite viscous dissipation and explain this mechanism via vorticity dynamics for stagnation point flow ($\gamma = 0$) which has hyperbolic streamlines and can be solved exactly for $\mathbf{v}(t)$. In the general elliptical case, however, the solution must be performed numerically by solving a Floquet problem to determine whether $\mathbf{u}'$ grows or decays in time.

Quite independently, Bayly[2] considered the instability of the basic elliptical flow (1) in an inviscid fluid and also noted that the disturbances of form (2) are exact solutions of the Euler equations. Bayly's study was prompted by the recent work of Pierrehumbert,[3] who found a short-wave instability in a numerical stability calculation for the Euler equations linearized about a locally elliptical flow. This instability led Pierrehumbert to propose that this mechanism is responsible for the 3-D instability observed in many shear flows containing 2-D coherent structures.

Bayly[2] solves the Floquet problem for $\mathbf{v}(t)$ and finds that for $\epsilon \neq 0$ there is always a spatial mode with initial wavevector $\mathbf{k}(0)$ that exhibits exponential growth in time. Bayly's results agree very well with Pierrehumbert's growth rates. Bayly does not mention that his calculations can be simply modified to include the effects of viscosity as described by Craik and Criminale; it is the purpose of this paper to repeat Bayly's calculations, taking into account viscous dissipation.

The results that follow confirm that the inviscid instability mechanism persists when the viscosity is nonzero. We find a stability boundary in the Ekman number versus streamline eccentricity plane, with the Ekman number a measure of the ratio of viscous dissipation to vorticity in the basic flow. At a given finite eccentricity the flow is found to exhibit a 3-D instability, with a short wavelength cutoff caused by the action of viscosity.

In Sec. IV we apply this result quantitatively to wake and shear layer experiments and the numerical study of plane channel flow. We find that typical vortex structures appear to be large enough to support unstable disturbances even when viscosity is present, thus suggesting that the inviscid mechanism leads to real vortex instability, as proposed by Pierrehumbert.[3]

II. FORMULATION

We consider the basic elliptical flow (1) with uniform vorticity 2γ and streamfunction

$$\psi = -\tfrac{1}{2}\gamma(y^2 + x^2) - \tfrac{1}{2}\epsilon(y^2 - x^2),$$

where without loss of generality $0 \leqslant \epsilon < \gamma$. This is the same basic flow considered by the investigators mentioned previously up to a rotation of coordinates. This flow is an exact solution of the Navier–Stokes equations. The perturbation velocity $\mathbf{u}'$, which is at present not assumed to be small with respect to the basic flow, satisfies

$$\frac{\partial \mathbf{u}'}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}' + \mathbf{u}'\cdot\nabla\mathbf{u} + \mathbf{u}'\cdot\nabla\mathbf{u}' = -\frac{1}{\rho}\nabla p' + \nu\nabla^2\mathbf{u}',$$

$$\nabla\cdot\mathbf{u}' = 0.$$

Seeking solutions of the form

$$(\mathbf{u}',p) = [\hat{\mathbf{v}}(t),\hat{p}(t)]\exp[i\mathbf{k}(t)\cdot\mathbf{x}], \qquad (3)$$

the continuity equation ensures that the nonlinear term vanishes, and on projecting out the pressure term we find

$$\dot{\hat{\mathbf{v}}} = (2\mathbf{k}\mathbf{k}^T/|\mathbf{k}|^2 - I)A\hat{\mathbf{v}} - \nu|\mathbf{k}|^2\hat{\mathbf{v}}, \qquad (4)$$

where $\mathbf{k} = \mathbf{k}(t)$ satisfies

$$\dot{\mathbf{k}} = -A^T\mathbf{k}. \qquad (5)$$

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Details leading to these equations are given in Craik and Criminale[1] and in the inviscid form by Bayly.[2]

Equation (5) can be solved in closed form. Letting

$$\Omega=\sqrt{\gamma^2-\epsilon^2},\quad \alpha=\sqrt{(\gamma+\epsilon)/(\gamma-\epsilon)},$$

where $\alpha \geqslant 1$ is the aspect ratio of the elliptical streamlines, the solution may be written as

$$\mathbf{k}=k_0[\sin\theta\cos\Omega(t-t_0),\alpha\sin\theta\sin\Omega(t-t_0),\cos\theta], \tag{6}$$

where the constants of integration are k_0, θ, and an arbitrary shift in time t_0. The wave vector $\mathbf{k}$ is thus seen to precess elliptically with the period $2\pi/\Omega$, with minimum inclination angle to the z axis of θ and arbitrary length scale determined by k_0.

Following Craik and Criminale,[1] the viscous decay term in (4) is removed by the transformation

$$\hat{\mathbf{v}}=\exp\left(-\nu\int_0^t|\mathbf{k}|^2\,dt\right)\mathbf{v}, \tag{7}$$

leaving the equation for the growth of the perturbation as

$$\dot{\mathbf{v}}=Q(t)\mathbf{v},\quad Q(t)=(2\mathbf{k}\mathbf{k}^T/|\mathbf{k}|^2-I)A. \tag{8}$$

This is the inviscid equation solved by Bayly.[2] To find the general solution of (8), we consider the Floquet problem

$$\dot{M}=Q(t)M,\quad M(0)=I,$$

where M is the solution matrix associated with (8). The eigenvalues of $M(2\pi/\Omega)$ are the Floquet multipliers μ_i which determine the growth rate of the solution. This is because the general solution for $\mathbf{v}$ is a superposition of modes of the form

$$\mathbf{v}(t)=e^{\sigma_i t}\mathbf{f}_i(t),\quad \sigma_i=(\Omega/2\pi)\log\mu_i(\alpha,\theta),$$

where $\mathbf{f}_i$ has period $2\pi/\Omega$. The structure of the matrix Q shows that one multiplier $\mu_3=1$ and that we need only solve problem (8) for the x and y components. From conservation and symmetry properties of the system one can show that $\mu_1\mu_2=1$, with the μ_i being complex conjugates of each other (all modes stable) or real positive reciprocals with $\mu_2<1<\mu_1$ (there exists an unstable mode).[2] When the latter occurs, we define the inviscid growth rate as

$$\sigma_I=(\Omega/2\pi)\log\mu_1. \tag{9}$$

To summarize Bayly's[2] results for the Floquet problem for fixed Ω, there is a region of θ-α space for which unstable modes exist, which emanates from the point of linear instability of perturbed solid body rotation at $\alpha=1$, $\theta=\pi/3$. As α increases, a larger range of wave vectors of angle θ are unstable, with the maximum inviscid growth rate over θ a monotonically increasing function of α provided Ω is fixed.

We now consider the effect of viscosity on this inviscid instability. From Eqs. (6) and (7), we can calculate the time-averaged viscous decay rate as

$$\sigma_V=-\nu\frac{\Omega}{2\pi}\int_0^{2\pi/\Omega}|\mathbf{k}|^2\,dt$$
$$=-\nu k_0^2[1+\tfrac{1}{2}(\alpha^2-1)\sin^2\theta].$$

We could now proceed to calculate the total growth rate $\hat{\sigma}_T=\sigma_I+\sigma_V$ of the perturbation $\hat{\mathbf{v}}$. However, it is more instructive to write the growth rate as a function of the original parameters γ, proportional to the vorticity, and ϵ, the strain rate. This is because in the $\alpha-\Omega$ formulation, Ω tends to zero as the strain rate is increased ($\alpha\to\infty$) provided the vorticity remains finite. In so doing, it is convenient to define the dimensionless parameters

$$E_\gamma\equiv 2\pi\nu k_0^2/\gamma,\quad \beta\equiv\epsilon/\gamma=(\alpha^2-1)/(\alpha^2+1)$$

as the Ekman number based on vorticity and the streamline eccentricity parameter, respectively. The dimensionless growth rate based on γ (half the vorticity), $\sigma_T=\hat{\sigma}_T/\gamma$ is thus

$$\sigma_T=(1/2\pi)\{\sqrt{1-\beta^2}\log\mu_1(\beta,\theta)$$
$$-E_\gamma[(1-\beta\cos^2\theta)/(1-\beta)]\}.$$

To find the curve of marginal stability in the β-E_γ plane, we solve

$$G(\beta,E_\gamma,\sigma;\theta)\equiv\sigma-\sigma_T(\beta,E_\gamma;\theta)=0 \tag{10a}$$

for $\sigma=0$, subject to the constraint

$$G_\theta(\beta,E_\gamma,\sigma;\theta)=0, \tag{10b}$$

as we seek to maximize the growth rate over the inclination angle θ. More generally, any one of the parameters $\{\beta,E_\gamma,\sigma\}$ may be held fixed so that Eqs. (10) define a two-parameter continuation problem.

III. NUMERICAL RESULTS

In the solution of Eqs. (10), the matrix $M(2\pi/\Omega)$ was computed using a standard ODE solving package, giving μ_1 and thus σ_T. The continuation was performed using the software package AUTO,[4] which is able to detect and follow the fold in G with respect to θ as required by the constraint (10b).

First, we display the inviscid ($E_\gamma=0$) growth rate maximized over all inclination angles θ as the eccentricity is varied, as in Fig. 1. This was calculated by Bayly[2]; however, Bayly took Ω to be fixed, which means that as $\alpha\to\infty$ ($\beta\to 1$), the vorticity and the strain tend to infinity. In this way Bayly's results are misleading and support Pierrehumbert's conjecture[3] that the growth rate continues to increase as the plane Couette limit is reached ($\epsilon\to\gamma$), which would make the limit strongly singular. This is because inviscid Couette flow is known to be stable, although a disturbance can undergo an initial stage of growth before the ultimate algebraic decay, as was shown by Lord Kelvin.[5] We believe our calculation keeping the vorticity finite to be more physically relevant and shows that there is a maximum in the growth rate at an aspect ratio of 3.1 ($\beta=0.81$). The growth rate then decreases to zero as the Couette limit is reached because of the rotation rate Ω in (9) tending to zero. Thus the limit is essentially regular, but this conclusion is not inconsistent with Pierrehumbert's[3] numerical stability results showing a monotonically increasing growth rate for $\beta\leqslant 0.8$.

On the introduction of viscosity, the stability boundary in the β-E_γ plane is calculated as described above and gives the results shown in Fig. 2. Contours of constant growth rate are also shown. The main result is that the introduction of viscosity does not eliminate the instability, but produces a

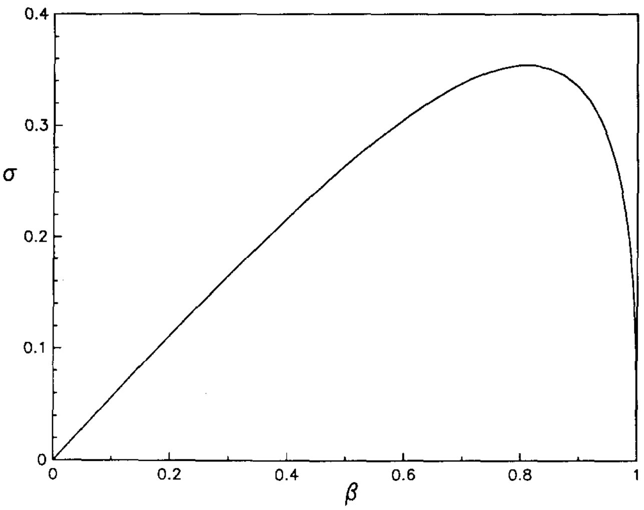

FIG. 1. Maximum inviscid growth rate $\sigma = \sigma_I$ as a function of the eccentricity parameter β.

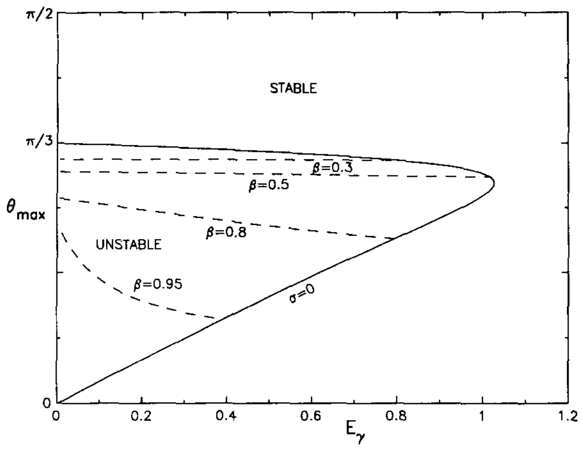

FIG. 3. Most unstable wavenumber inclination versus Ekman number for various streamline eccentricities β.

short wavelength cutoff at all nonzero Ekman numbers. Because the inviscid instability is scale independent, the maximum growth rate occurs at the largest length scales possible in the viscous problem (i.e., at the smallest Ekman number). Note that whenever the flow is unstable, there is a range of wavenumber inclinations θ for which the disturbance wave vector (6) grows. There is, however, an inclination angle $\theta_{\max}$ at which the growth rate is a maximum, and Fig. 3 plots $\theta_{\max}$ versus Ekman number for various aspect ratios.

The curve of marginal stability defines the relationship $E_\gamma = E_\gamma^*(\beta)$ for the critical Ekman number as a function of eccentricity. At a fixed viscosity ν and vorticity 2γ, a given straining field determines β and thus from Fig. 2 and Eq. (6) determines the length scale $l = 2\pi/k_0\sqrt{1+\alpha^2}\sin\theta$ at which the instability mechanism operates. Moreover, from Fig. 3 we can deduce that $1 < \sqrt{1+\alpha^2}\sin\theta_{\max} \lesssim 2$ for the range of aspect ratios $1 < \alpha \lesssim 3$ $(0 < \beta \lesssim 0.8)$, so that $l = 2\pi/k_0$ is a good measure of the length scale of the unstable modes in the x-y plane of the basic flow. Note, also, that the corresponding length scale of the instability in the z direction is $2\pi/k_0\cos\theta \approx 2l$ for moderate straining fields.

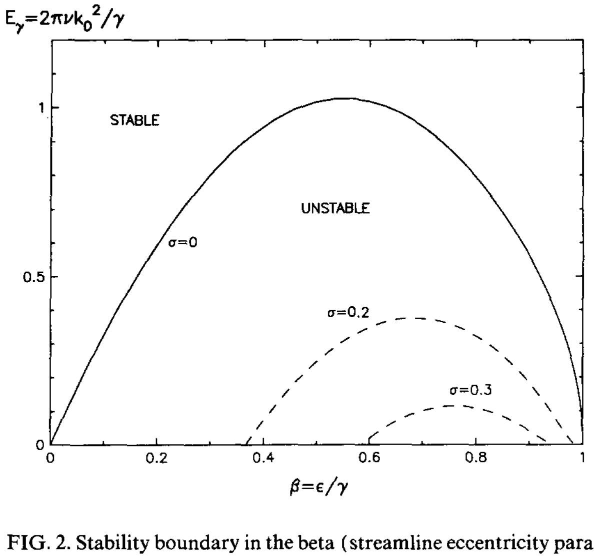

FIG. 2. Stability boundary in the beta (streamline eccentricity parameter)-Ekman number plane; contours of the growth rate $\sigma = \sigma_T$ are also shown.

IV. COMPARISON WITH EXPERIMENTS

In applying the above results to the stability of a flow whose local structure can be approximated by elliptical regions of uniform vorticity, a disturbance will not in general consist of a single mode (3), which is an exact solution of the Navier–Stokes equations. It will, however, be a Fourier-type superposition of such modes over k_0, θ, and t_0, as described by Bayly.[2] The disturbance can therefore be localized in space and will satisfy the linearized perturbation equations, so that the above analysis acts as a linear stability theory. An unstable disturbance will have a minimum length scale which is of the order of $l^* = 2\pi/k_0^*$, where for a given flow field k_0^* is determined by the critical Ekman number E_γ^*. In this way the length scale l of the instability is limited to lie between that given by the critical Ekman number and that determined by the macroscopic size L below which the flow is well approximated by a uniform shear, i.e., $l^* < l \ll L$, where

$$l^* = (2\pi)^{3/2}(\nu/\gamma E_\gamma^*)^{1/2}.$$

We thus speculate that this instability will act within isolated vortices which can be approximated locally by a region of constant vorticity of size l, where the eccentricity of the closed elliptical streamlines is determined by the external strain caused by neighboring vortices. To test this hypothesis, we apply it first to experimental data of a wake and shear layer.

Davies[6] describes the array of vortices in the wake of a bluff body in a wind tunnel. Approximating these as elliptical regions of uniform vorticity at a Reynolds number $\mathrm{Re} \approx 4\times10^4$, we find the vortex aspect ratio α is about 1.13 and $\gamma \approx 30\ \mathrm{sec}^{-1}$, thus giving the values $\beta \approx 0.12$, $E_\gamma^* \approx 0.4$, so that the critical length scale $l^* \approx 2$ cm. This compares with the macroscopic size of the regions of vorticity $L \approx 15$ cm, suggesting that the vortices are large enough to support unstable disturbances which may be the cause of 3-D instabilities observed in the far wake. This mechanism would be distinct from the 3-D resonance instability of elliptical vortices, where the axial wavelength is of the same order as the core diameter, which has been analyzed for finite strain by Robinson and Saffman.[7]

In the planar mixing layer, vortices are observed to be quite elliptical in shape. Examining the data of Browand and Weidman,[8] who look at coherent structures in water developing in the layer, vortices at $\mathrm{Re} \approx 300$ are found to have an aspect ratio of about 1.8, with a core which we can approximate as being uniform with $\gamma \approx 1.5\ \mathrm{sec}^{-1}$. These figures yield $\beta \approx 0.53$ and thus $E_\gamma^* \approx 1.0$, so $l^* \approx 1$ cm. In this case the scale of the vortices is $L \approx 4$ cm. This measurement was made on a vortex formed after a single pairing instability had taken place. Further downstream, we can argue that the influence of the 3-D instability becomes more important after the occurrence of multiple pairing instabilities in the shear layer (which are essentially 2-D in nature). This is because in a given shear layer $\omega A/d =$ const, where ω is the (uniform) vorticity of a single elliptical vortex, A is its area, and d is the spacing between adjacent vortices. If we assume that the eccentricity of the vortices remains essentially constant and A scales as d^2, then ω and thus γ must decrease by a factor of 2 after each pairing. As the critical Ekman number is fixed for a given eccentricity, the smallest unstable length scale l^* of the disturbance scales as $l^* \sim \gamma^{-1/2}$, so that l^* increases by a factor of only $\sqrt{2}$. Hence the ratio l^*/L decreases after each pairing, allowing for a greater range of unstable disturbances. This scenario is consistent with experimental observations, where 3-D streaking instabilities only become dominant after a succession of pairings have taken place.

Finally, we consider the secondary instability of 2-D finite amplitude waves in plane Poiseuille flow, as studied numerically by Orszag and Patera.[9] In a moving frame the basic flow consists of roughly elliptical vortices, which undergo a 3-D core instability. From Fig. 1 in their paper showing a plot of the streamlines, at $\mathrm{Re} = 4000$ the aspect ratio of a basic 2-D vortex is approximately 2.2, so that $\beta \approx 0.7$. In units nondimensionalized with respect to the channel half-width and centerline velocity, the vortex diameter is approximately 1.0. The vorticity in this region is approximately 0.5, where the latter is estimated from the contour plot of vorticity,[9] and we suppose that the shear stress at the wall is similar to that of laminar flow. Assuming the form of the basic wave remains roughly the same over a wide range of Reynolds numbers and over the streamwise wavenumber, Orszag and Patera[9] find that the growth rate (which we double as our growth rate is nondimensionalized by γ, half the vorticity) approaches $\sigma \approx 0.3$ as the Reynolds number increases, which is to be compared with the inviscid growth rate from Fig. 1 of about 0.32. At finite Reynolds number, from Fig. 2 we find a critical Ekman number of $E_\gamma^* \approx 1.0$ for $\beta \approx 0.7$. Given the above vortex properties, this corresponds to a minimum unstable Reynolds number of about 1000. This compares favorably with the findings of Orszag and Patera[9] that weakly decaying 2-D waves exist exhibiting a 3-D instability down to Reynolds numbers near 1000, which is roughly the threshold found in experiments. Finally, note that the length scale of the instability in the cross-stream direction is about two to three times that in the plane of the vortex core, which is consistent with the magnitude of the most unstable wavenumber found in the spanwise direction in numerical simulation. The above evidence therefore supports the conjecture that the 3-D core instability of strained vortices is responsible for secondary instability in shear flows.

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