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**A KINETIC THEORY
DESCRIPTION OF RAREFIED GAS FLOWS**

by
Lester Lees



GUGGENHEIM AERONAUTICAL LABORATORY
CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

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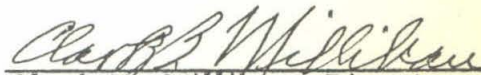
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ABSTRACT

An approach to the kinetic theory of gas flows is developed which starts with Maxwell's original integral equations of transfer, rather than with the Maxwell-Boltzmann equation for the velocity distribution function itself. In this procedure the Maxwell-Boltzmann equation is satisfied in a certain average sense, rather than at every point. The advantage of this method is that relatively simple distribution functions are utilized which contain a small number of unknown functions to be determined by applying the conservation laws, plus several additional higher moments. For simplicity a "two-stream Maxwellian" is employed, which is a natural extension and generalization of Mott-Smith's function for a normal shock, but differs from it in certain essential respects. As an illustration, the method is applied to linearized plane Couette flow and Rayleigh's problem. Reasonable results are obtained for macroscopic quantities such as mean velocity and shear stress over the whole range of densities from free-molecule flow to the Navier-Stokes regime. This technique is now being applied to some typical non-linear rarefied gas flows.

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LIST OF SYMBOLS

a_I	isothermal speed of sound, \sqrt{RT}
b	impact parameter, or perpendicular distance from particle "i" to initial trajectory of particle "j"; also distance between two parallel plates
\vec{c}	relative particle velocity, $\vec{u}_j - \vec{u}_i$
\bar{c}	mean molecular speed, $\sqrt{(8/\pi) RT}$
C_D, c_f	drag coefficient or skin friction coefficient, $p_{xy}/\frac{1}{2}\rho U^2$
d	characteristic length
f, f_1	velocity distribution functions for "probe" and colliding particles, respectively
f_i	$m_i N_i \bar{f}_i^{(1)}$, single particle velocity distribution function ($f_i = f$ for a pure gas)
$f_i^{(1)}, \bar{f}_i^{(1)}$	single particle probability functions
$f_{ij}^{(2)}$	joint probability density of two interacting particles
$f^{(n)}, \bar{f}^{(n)}$	simultaneous probability density for n particles ($n < N$)
$f^{(N)}(P, Q; t)$	simultaneous probability density for all N particles
$\bar{f}^{(N)}(P, Q; t)$	time-smoothed probability density
f_0	local full-range Maxwellian
f_{F-M}	free molecule flow distribution function
f_1, f_2	components of two-stream Maxwellian
\vec{F}	vector external force acting on a single particle
F_i	component of \vec{F} in the i^{th} direction
F_a	interparticle force exerted on particle a by all other
$F_{a\beta}$	force exerted on particle a by particle β
\vec{F}	vector external force per unit volume

h_i	stretching factor (Appendix A)
k	Boltzmann constant
K	constant in expression for central force field, $F = (m_1 m_2 K)/r^s$
K_{ji}	collision integral (Section II. 2.)
L	characteristic length
m	mass of a particle
M	Mach number, $U/\sqrt{\gamma R T}$
n	particle number density, per unit volume
n_1, n_2	number density functions in two stream Maxwellian
N	number of particles in system
N_j	number of particles of the j^{th} species
p	$nkT \equiv pRT$
P_{ii}	defined by the relation $P_{ii} = -p + P_{ii}$
$p_{ij} \equiv P_{ij}$	shear stress, $i \neq j$, $p_{ij} = -m \int f c_i c_j d\vec{\xi}$
P_k	momentum of a particle in k - direction
P	vector in momentum space
P_{ii}	normal stress, $P_{ii} = -m \int f c_i^2 d\vec{\xi}$
P_a	vector momentum of particle a
\vec{q}	heat flux vector, $\vec{q} = m \int \vec{c} (c^2/2) f d\vec{\xi}$
\dot{q}_j	heat flux in j - direction
q_k	generalized coordinate in k - direction
Q	vector in configuration space; also arbitrary function of particle velocity
ΔQ	change in Q produced by collisions

r_1, r_2	radii of inner and outer cylinders, respectively
R	radius vector
Re	Reynolds number, $\rho U b / \mu$ $\left[\text{or } \frac{\rho U (r_2 - r_1)}{\mu} \text{ in cylindrical Couette flow} \right]$
R_o	sensible range of interparticle force field, $R_o = 2 - 3 \sigma_o$
R_a	radius vector of particle a (Section II)
\mathcal{R}	gas constant, k/m
s	time variable; also $(u - u_w) / \sqrt{2 \mathcal{R} T}$, non-dimensional relative velocity (Section III. 4)
t	time
\bar{t}	characteristic flow time
T	absolute temperature, $3/2 n k T = m \int (c^2/2) f d\vec{\xi}$; also t/τ_f
T_1, T_2	temperature functions in two stream Maxwellian
\vec{u}	mean velocity vector, $\rho \vec{u} = m \int f \vec{\xi} d\vec{\xi}$
u, v	components of mean velocity parallel to x- and y- axes, respectively
u_i	component of mean velocity in i^{th} direction
\vec{u}_1, \vec{u}_2	vector velocity functions in two-stream Maxwellian
\vec{u}_∞	free stream velocity vector
U	plate velocity
V	relative velocity between two interacting particles = $ \vec{\xi}_1 - \vec{\xi} $
w	u/U
x, y	coordinates along and normal to mean flow direction
x_i	coordinate in i^{th} direction
X_a	external force acting on particle a
Y	$y/a_I \tau_f$

α	scattering angle parameter (Section III. 3.); also energy accommodation coefficient
α_i	curvilinear coordinate
β	$(b/2\lambda)^{-1} = \left[(1/\sqrt{2\pi\sigma}) (Re/M) \right]^{-1}$
γ	ratio of specific heats, c_p/c_v
ε	small parameter; also angle between plane of the orbit and plane containing the original relative velocity and the x- axis in a binary collision
θ'	scattering angle, or angle between relative velocity before and after collision
λ	Maxwell mean free path
μ	viscosity
ν	kinetic viscosity, μ/ρ
$\vec{\xi}$	vector particle velocity, $\xi^2 = \vec{\xi} ^2$
$d\vec{\xi}$	$d\xi_i d\xi_j d\xi_k$
ξ_j	component of particle velocity in j- direction
ξ_1	velocity of colliding particle
ρ	nm, mass density, $\rho = m \int f d\vec{\xi}$
σ, σ'	tangential and normal momentum exchange coefficients, respectively
σ_0	separation distance between two interacting particles at which potential energy is a minimum
τ	time interval
τ_c	average duration of a binary collision
τ_f	mean free time between two successive collisions
ϕ	perturbation on local Maxwellian; also property $\phi(P, Q)$ (Section II.)
ω_{kx}	angle between plane containing OG and the k- axis, and the plane containing OG and the x- axis, where $k = x, y, z$ (Section III. 3. and Appendix B)

$\vec{\Omega}$

instantaneous local angular velocity vector associated with coordinate curvature

 $\vec{\nabla}_R, \vec{\nabla}_P$

gradient operators with respect to coordinates and momenta, respectively

The subscripts "1" and "2" generally denote the two components of the two-stream Maxwellian. A prime denotes quantities evaluated after a collision, while unprimed quantities refer to conditions before a collision. A dot denotes differentiation with respect to time. The subscript "o" denotes ambient conditions, while the subscript "w" denotes surface values, and the subscript " ∞ " denotes free stream quantities far ahead of the body.

I. INTRODUCTION

We are so accustomed to the universality and success of the Navier-Stokes stress-rate of strain relations and Fourier's "law" of heat conduction for Newtonian liquids, and for gases at normal density, that the purely empirical character of these relations is often forgotten. The conditions of zero velocity "slip" and temperature "jump" at a solid surface are equally empirical. They were finally accepted only after the practical success of the no-slip condition had been demonstrated, and experimental investigations in liquids had shown that the velocity slip at a solid boundary must be extremely small, if not zero.* On the macroscopic level the coefficients of viscosity and heat conduction must be determined experimentally. Here one performs an exercise in circular logic: The Navier-Stokes relations and the no-slip boundary conditions are accepted as correct, so that the viscosity (for example) is regarded as a local property of the fluid, and not of the flow. Then analytical solutions based on these same empirical relations are utilized to obtain the viscosity from measurements of mass flow and pressure drop in a pipe flow, or from the measured torque on a rotating cylinder. The values of viscosity so determined are then utilized in all other flows. The justification for this procedure is purely operational.

This same empirical procedure could also be followed for rarefied gas flows. The viscosity would then depend not only on the temperature of the gas, but also on the density and some characteristic length. For example, Jeans^{2a} points out that for highly rarefied gas flow $\mu \sim \rho \bar{c} d$, because the average path length traversed by the molecules between successive collisions is of the order of some relevant dimension of the apparatus, d , rather than the mean free path.** Historically, an alternative approach to highly rarefied gas flows was developed, based on the kinetic theory rather than macroscopic concepts. In this "free-molecule" flow regime collisions between particles in the gas are regarded as secondary compared with collisions between gas particles and the solid boundaries.

* Goldstein¹ has given a short account of the interesting history of this question. (Superscripts denote references at the end of the text.)

** See also Liepmann³ and Roshko, Chapter 14.

The probability that a molecule emitted from a body surface has not collided with another molecule is roughly proportional to $e^{-r/\lambda}$, where r is the distance traversed and λ is the mean free path. Thus, if the mean free path is much larger than a characteristic body dimension ($\lambda \gg R$) few collisions occur in the vicinity of the body. Of course far from the body ($r \geq \lambda$) collisions in the gas can never be neglected. But the sensible influence of the body is not expected to extend outward more than 10 - 20 times the relevant body dimension. Therefore this approach should provide a good first approximation to the drag, heat transfer and flow field near the body when $\lambda \gg R$, even though the basic question of a proper "matching" to the "far field" is ignored.

At present there is no general agreement concerning the connection between gas kinetics ($\lambda \gg R$) and gas dynamics as described by the Navier-Stokes equations.³ Yet the kinetic theory ought to provide a framework that includes both of these limiting cases, and the transitional regime in between. The difficulties involved in constructing such a kinetic theory for liquids and dense gases are well known. However, for dilute, unionized, monatomic gases the statistical mechanical approach is greatly simplified, because the sensible range of intermolecular forces is orders of magnitude shorter than either the average distance between molecules, or the mean free path. For example, the average range of intermolecular forces is of the order of $3 \text{ \AA} \approx 3 \times 10^{-8} \text{ cm.}$, and the duration of a representative collision is of the order of 10^{-12} seconds at room temperature. Both values are virtually independent of gas density. Now the mean free path in air at one atmosphere is about $5 \times 10^{-6} \text{ cm.}$, and the average time interval τ_f between successive collisions is about 10^{-10} seconds. At 10^{-4} atm , $\lambda \approx 5 \times 10^{-2} \text{ cm.}$, and $\tau_f \approx 10^{-6} \text{ sec.}$; at 10^{-10} atm , $\lambda = 5 \times 10^4 \text{ cm.}$, or 500 meters, and $\tau_f \approx 1 \text{ second.}$ * Thus, the duration of a collision is very short compared with the mean free time, and the probability of a ternary collision is negligibly small compared with the probability of a binary collision. By introducing a time average over an interval long compared with the duration of a collision, but short compared with the mean free time, Liouville's equation for the

* At this pressure the statistical fluctuation in density in a volume of 1 mm.³ is still only 0.1 per cent.

simultaneous probability density of all particles is reduced to the Maxwell-Boltzmann equation for the single particle velocity distribution function (Section II). By this procedure statistical fluctuations (such as Brownian motion) are eliminated and a dissipative mechanism is introduced.

In spite of this apparent generality of the Maxwell-Boltzmann equation for dilute gases, we approach the one-hundredth anniversary of Maxwell's fundamental paper^{4a} without a kinetic theory of gas flows. In part, at least, such a deficiency must be attributed to the fact that the kinetic theory of gases has been dominated for over fifty years by the specialized approach of Chapman and Enskog, as expounded in the famous treatise of Chapman⁵ and Cowling. This approach is concerned primarily with the transport properties of gases and gas mixtures for intermolecular force fields somewhat more realistic than Maxwell's simple inverse fifth power "law". A considerable amount of effort is devoted to the evaluation of the collision integral in the Maxwell-Boltzmann equation. Departures from the Navier-Stokes regime are studied by means of an expansion procedure that is by no means obvious, and is certainly of little value in answering the basic question of the relation between gas kinetics and gas dynamics.

Maxwell himself was well aware of the fact that one is not particularly interested in the velocity distribution function itself, but in certain lower moments of this function. Accordingly he constructed^{4b} integral equations of transfer of any quantity Q that is a function only of the components of the particle velocity. By taking Q to be successively the collisional invariants of mass, linear momentum and translational kinetic energy of a particle, Maxwell obtained the usual gasdynamic conservation laws, independently of the form of the distribution function. By taking Q to be the cross-products of particle velocity, and the flux of kinetic energy, he derived relations for the stresses and heat flux for a special choice of the distribution function. Clearly this procedure amounts to satisfying the Maxwell-Boltzmann equation in a certain average sense, rather than point-by-point. In this respect Maxwell's approach is analogous to integral methods in fluid mechanics, such as the Karman⁶-Pohlhausen method for boundary layer flows, and its recent modification by Tani.⁷

Recently there have been encouraging signs of a return to Maxwell's transport integral method. However, Grad⁸, and Gross⁹, et al, utilize

polynomials of the Chapman-Enskog type in the distribution function, which introduces an undesirable rigidity into the procedure. In the present report we try to make a clean break with the Chapman-Enskog tradition. The distribution function is regarded merely as a convenient weighting function, and is expressed in terms of a certain number of arbitrary, initially unknown functions of space and time. A sufficient number of Maxwell moments are taken to determine these functions. For simplicity the problem is formulated in terms of a "two-stream Maxwellian", which is a natural extension and generalization of Mott-Smith's¹⁰ approach for the normal shock wave, but differs from it in certain essential respects. Of course no integral method is "unique", and in fact, one of the purposes of this report is to stimulate discussion of other possible integral techniques for rarefied gas flows.*

Because of its importance for all our later work the question of the range of validity and basic approximations underlying the Maxwell-Boltzmann equation is reexamined in Section II. After a brief survey of methods of attack based on the Maxwell-Boltzmann equation itself (Section III. 1), the present integral method is formulated and its connection with Mott-Smith's work indicated (Section III. 2). The important simplification introduced by Maxwell's inverse fifth power law of force is discussed in Section III. 3, and the boundary conditions for the present method are treated in Section III. 4. As an illustration of the method, linearized plane Couette flow and Rayleigh's problem are analyzed in Sections IV and V. In later papers it is hoped to apply the present technique to non-linear aspects of rarefied gas flows, including dissipation and large temperature differences.

* A paper by Max Krook⁴⁴ has just appeared in which a series of Maxwellians modified by polynomials is employed. No examples of this method are available as yet.

II. STATISTICAL MECHANICAL THEORY OF TRANSPORT PROCESSES IN GASES

II. 1. Phase Space, Liouville's Equation and the Averaging Process

In classical mechanics the dynamical state of a system of N particles at any time t is represented by a single point in $6N$ -dimensional phase space containing the $3N$ momentum coordinates $(p_k)_1, (p_k)_2, \dots, (p_k)_N$, and the $3N$ configuration coordinates $(q_k)_1, \dots, (q_k)_N$, where $k = 1, 2, 3$. Actually it is more convenient to deal with a vector P in momentum space, consisting of the N vectors P_1, P_2, \dots, P_N , and a vector Q in configuration space, composed of the vectors R_1, \dots, R_N specifying the positions of the centers of gravity of the particles. Now it is clear that a specification of the macroscopic properties ρ, T, u_i, p_{ij} , and \dot{q}_j of the system at a given time t certainly does not specify the dynamical state of the system. We are led to Gibb's conception of a statistical ensemble, consisting of many different dynamical states all corresponding to the same macroscopic state at a given time t , but all starting from different (and arbitrary) initial conditions. This ensemble is characterized by a probability density $f^{(N)}(P, Q; t)$, such that $f^{(N)}(P, Q; t) dP dQ$ is the probability of finding N distinct particles at time t in the region $R_1 \rightarrow R_1 + dR_1, R_2 \rightarrow R_2 + dR_2, \dots, R_N \rightarrow R_N + dR_N$, with momenta in the range $P_1 \rightarrow P_1 + dP_1, \dots, P_N \rightarrow P_N + dP_N$.

By analogy with ordinary gas dynamics, the "equation of continuity" in phase space takes the form

$$\frac{\partial f^{(N)}}{\partial t} + \text{div} (f^{(N)} V) = 0, \quad (1)$$

where V is the "velocity vector" along the trajectory of the dynamical system in phase space, with components \dot{P}, \dot{Q} . According to Hamilton's equations*

$$(\dot{p}_k)_\alpha = - \frac{\partial H}{\partial (q_k)_\alpha} \quad \text{and} \quad \dot{q}_k = \frac{\partial H}{\partial (p_k)_\alpha}, \quad (2)$$

* The Hamiltonian H represents the total energy of the system, including intermolecular and external forces.

so that

$$\operatorname{div} V = \sum_{a=1}^N \sum_{k=1}^3 \left[\frac{\partial \dot{p}_k}{\partial (p_k)_a} + \frac{\partial \dot{q}_k}{\partial (q_k)_a} \right] = 0. \quad (3)$$

In other words the volume occupied by a given swarm of points in phase space is unchanged along a trajectory. By Eq. (1) it follows that the probability density $f^{(N)}$ is also unchanged, or (Liouville's equation)

$$\frac{\partial f^{(N)}}{\partial t} + \sum_{a=1}^N \left[\frac{P_a}{m_a} \cdot \nabla_{R_a} f^{(N)} + (X_a + F_a) \cdot \nabla_{P_a} f^{(N)} \right] = 0, \quad (4)$$

where

$$P_a = m_a \dot{R}_a \quad \text{and} \quad X_a + F_a = \dot{P}_a. \quad (5)$$

Here X_a denotes the external force acting on a particle a , and is assumed to depend only on the coordinates of a ; F_a is the interparticle force exerted on particle a by all the other particles of the system. We consider only non-polar interparticle force fields, for which

$$F_a = \sum_{\substack{\beta=1 \\ \beta \neq a}}^N F_{a\beta} \quad \text{and} \quad F_{a\beta} = F_{a\beta} (|R_\beta - R_a|).$$

In Lagrangian form

$$\left. \begin{aligned} f^{(N)}(P, Q; t+s) &= f^{(N)}(P_0, Q_0; t), \\ \text{and} \\ (dP dQ)_{t+s} &= (dP dQ)_t \equiv dP_0 dQ_0, \\ \text{or} \\ \frac{\partial(P, Q)}{\partial(P_0, Q_0)} &= 1, \end{aligned} \right\} \quad (6)$$

where P_0, Q_0 are the values of P and Q at some arbitrary initial instant.

As in most statistical problems the probability density $f^{(N)}$ is utilized to obtain certain interesting average properties of the system that can be identified with observable quantities. Kirkwood¹¹ points out that three distinct averaging processes are involved in the macroscopic measurement of a property $\phi(P, Q)$ not explicitly dependent on the time. The ensemble average is defined by the operation

$$[\phi]_e = \iint \phi(P_o, Q_o) f^{(N)}(P_o, Q_o; t) dP_o dQ_o, \quad (7)$$

at time t . In order to smooth out microscopic fluctuations in particle distribution we define a time average as follows:

$$[\phi]_\tau = (1/\tau) \int_0^\tau \phi(P; Q) ds, \quad (8)$$

where τ is a time interval that is long compared with the duration of a representative collision, but is still very short on the macroscopic time scale. (In Section II.2 we show that $\tau_c \ll \tau \ll \tau_f$.) The "observed value" of ϕ is defined by the combined operation

$$(\phi)_{\text{obs}} = [(\phi)_\tau]_e = (1/\tau) \iiint \phi(P, Q) f^{(N)}(P_o, Q_o; t) ds dP_o dQ_o. \quad (9)$$

If these averaging operations are commuted, one obtains

$$[(\phi)_e]_\tau = \iint \phi(P_o, Q_o) \bar{f}^{(N)}(P_o, Q_o; t) dP_o dQ_o, \quad (10)$$

where the "time-smoothed" probability density $\bar{f}^{(N)}$ is given by

$$\bar{f}^{(N)}(P_o, Q_o; t) = (1/\tau) \int_0^\tau f^{(N)}(P_o, Q_o; t+s) ds. \quad (11)$$

According to Eq. (6),

$$f^{(N)}(P_o, Q_o; t) dP_o dQ_o = f^{(N)}(P, Q; t+s) dP dQ,$$

so that Eq. (9)

$$[(\phi)_\tau]_e = \iint \phi(P, Q) \bar{f}^{(N)}(P, Q; t) dP dQ, \quad (12)$$

which is clearly identical with $[(\phi)_e]_\tau$.

The commutability of the time and ensemble averaging processes is evidently closely connected with the reversibility of dynamical states expressed by Liouville's equation. At this stage there is no evidence of the dissipative mechanism required to describe macroscopic transport processes correctly. Such a mechanism is introduced only when one considers the time history of the "lower-order" probability densities,

$\bar{f}^{(1)}(P_1, Q_1; t)$, $\bar{f}^{(2)}(P_1, Q_1, P_2, Q_2; t)$ etc., where $\bar{f}^{(1)}$ and $\bar{f}^{(2)}$ are generally sufficient to specify most of the interesting macroscopic

quantities in a gas. These functions are defined by the relations

$$f^{(n)}(P_1, Q_1, \dots, P_n, R_n; t) = \iint f^{(N)}(P, Q) dP_{n+1} dR_{n+1} \dots dP_N dR_N, \quad (13)$$

where the integration extends over the residual phase space of the other $N-n$ particles, and

$$\bar{f}^{(n)}(P_1, R_1, \dots, P_n, R_n; t) = 1/\tau \int_0^\tau f^{(n)}(P_1, R_1, \dots, P_n, R_n; t+s) ds. \quad (14)$$

When the function $\phi(P, Q)$ does not depend on all the coordinates in the $6N$ dimensional phase space, but only on a subset of n particles, then according to Eqs. (12), (13), and (14), the average value of $\phi(P, Q)$ is given by

$$\begin{aligned} \phi_{\text{obs}} &= \iint \left[\phi(P_1, R_1, \dots, P_n, R_n) dP_1 dR_1 \dots dP_n dR_n \right] \bar{f}^{(N)} dP_{n+1} dR_{n+1} \dots dP_N dR_N \\ &= \iint \phi(P_1 \dots R_N) \bar{f}^{(n)}(P_1 \dots R_n; t) dP_1 \dots dR_N. \end{aligned} \quad (15)$$

For example, for a system of one component consisting of N particles the mass flux is given by

$$\rho \bar{u} = N \int P_1 \bar{f}^{(1)}(P_1, Q_1; t) dP_1.$$

In defining macroscopic quantities in a gas it is often necessary to perform still another averaging operation over a macroscopically small region in configuration space having a representative linear dimension large relative to the sensible range of interparticle forces, but small compared to the mean free path (Section II. 2).

Our task now is to develop the equations governing the behavior of the sequence of particle distribution functions $\bar{f}^{(n)}$ by starting from Liouville's equation [Eq. (4)] and the definitions given by Eqs. (13) and (14), utilizing essential information about the structure of the dynamical system. For a dilute gas this procedure leads to the Maxwell-Boltzmann equation for the important single-particle distribution function $\bar{f}^{(1)}$.

II.2. Review of Kirkwood's Derivation of the Maxwell-Boltzmann Equation from Liouville's Equation¹¹⁻¹²

Suppose the dynamical system consists of N particles, where N_j is the number of particles of the j^{th} species, and $j = 1, 2, \dots, \mathcal{J}$. In deriving the basic equation for $\bar{f}^{(1)}$ from Liouville's Equation it is helpful to rewrite that equation as follows:

$$\begin{aligned} \frac{\partial f^{(N)}}{\partial t} + \left[\frac{P_i}{m_i} \cdot \nabla_{R_i} f^{(N)} + (X_i + F_i) \cdot \nabla_{P_i} f^{(N)} \right] \\ + \sum_{\substack{a=1 \\ a \neq i}}^N \left[\nabla_{R_a} \cdot (f^{(N)} \dot{R}_a) + \nabla_{P_a} \cdot (f^{(N)} \dot{P}_a) \right] = 0 \end{aligned} \quad (16)$$

By integrating Eq. (16) over the residual phase space of all particles except "i", taking the time average specified by Eqs. (11) and (12), and making use of Eqs. (13) and (14), one obtains*

$$\begin{aligned} \frac{\partial \bar{f}_i^{(1)}}{\partial t} + \frac{P_i}{m_i} \cdot \nabla_{R_i} \bar{f}_i^{(1)} + X_i \cdot \nabla_{P_i} \bar{f}_i^{(1)} \\ + \sum_{\substack{a=1 \\ a \neq i}}^N \iint \left[\nabla_{R_a} \cdot (f^{(N)} \dot{R}_a) + \nabla_{P_a} \cdot (f^{(N)} \dot{P}_a) \right] dP_1 \dots dR_N \\ = - \sum_{j=1}^{\mathcal{J}} N_j \left[\frac{1}{\tau} \int_0^\tau \iint F_{ji} \cdot \nabla_{P_i} f^{(N)}(P, Q; t+s) dP_1 dR_N ds \right] \\ = - \sum_{j=1}^{\mathcal{J}} N_j K_{ji} \end{aligned} \quad (17)$$

where K_{ji} is the quantity in brackets. Here F_{ji} represents the force between any particle of the j^{th} species and particle "i". The summation extends over all species, including the species to which particle "i" belongs, but of course excludes the particle "i" itself. By Green's

* The extension of this procedure to $\bar{f}^{(2)} \dots \bar{f}^{(n)}$ is straightforward. See also Reference 13.

theorem the integrals on the left-hand side of Eq. (17) are converted into surface integrals of velocity and acceleration "currents" over the boundaries of phase space containing the dynamical system. Presumably the boundaries can always be chosen so that these currents vanish. Then the left-hand side of Eq. (17) is identical with the left-hand side of the Maxwell-Boltzmann Equation, and the right-hand side contains the effect of the interparticle forces ("collisions") on the distribution function.

In order to extend the integrations over all of phase space we introduce the delta function, as follows:

$$f^{(N)}(P_i, R_i; P_1 \dots R_N; t+s) = \iint f^{(N)}(\tilde{P}_i, \tilde{R}_i, P_1 \dots R_N; t+s) \delta(P_i - \tilde{P}_i) \delta(R_i - \tilde{R}_i) d\tilde{P}_i d\tilde{R}_i.$$

But according to Eq. (6),

$$f^{(N)}(\tilde{P}_i, \tilde{R}_i, P_1 \dots R_N; t+s) = f^{(N)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}; P_{1_0} \dots R_{N_0}; t)$$

and

$$d\tilde{P}_i d\tilde{R}_i dP_1 \dots dR_N = d\tilde{P}_{i_0} d\tilde{R}_{i_0} dP_{1_0} \dots dR_{N_0}.$$

Also, the Lagrangian coordinates P_{i_0} and R_{i_0} are connected with the "local" coordinates by the relations $P_{i_0} = P_i - \Delta P_i$ and $R_{i_0} = R_i - \Delta R_i$.

By introducing all of these relations on the right-hand side of Eq. (17) and changing the variables of integration from $\tilde{P}_i, \tilde{R}_i, P_1 \dots R_N$ to $\tilde{P}_{i_0}, \tilde{R}_{i_0}; P_{1_0} \dots R_{N_0}$, one obtains the following expression for K_{ji}

$$K_{ji} = (1/\tau) \iiint_0^\tau F_{ji}(t+s) \cdot \nabla_{P_i} f^{(N)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}, P_1 \dots, R_{N_0}; t) \delta(\tilde{P}_{i_0} + \Delta P_i - P_i) \delta(\tilde{R}_{i_0} + \Delta R_i - R_i) d\tilde{P}_{i_0} d\tilde{R}_{i_0} dP_{j_0} dR_{j_0} dP_1 \dots dR_{N_1} ds. \quad (18)$$

In order to make any further progress one must introduce simplifying approximations based on the properties of the dynamical system. For dilute gases the most important of these is the approximation of "molecular chaos". In 3-configuration space this approximation states that the probability of finding three or more particles in a region of linear

dimensions of the order of two or three times the sensible range of intermolecular forces is negligibly small compared to the probability of finding two particles in such a region. The validity of this approximation clearly depends on the fact that the time interval between two successive collisions of a particle is larger than the duration of a binary collision by at least several orders of magnitude (See Introduction.)*. According to the statement of molecular chaos the coordinates R_i and R_j of the interacting particles "i" and "j" depend only on the "initial" coordinates and momenta of these particles R_{i_0} , R_{j_0} , P_{i_0} , P_{j_0} , and not on those of any other particles in the system. Thus $F_{ji} = F_{ji}(|R_j - R_i|)$ also depends only on these quantities, and $(\partial/\partial s)(\Delta P_i) = F_{ji}(R_i, R_j; t + s)$. This last relation, which is usually called the binary collision approximation, evidently follows directly from the approximation of molecular chaos in 3-configuration space.

When the molecular chaos approximation is employed, all of the quantities except $\bar{F}^{(N)}$ in the integrand of K_{ji} [Eq. (18)] depend only on R_{i_0} , R_{j_0} , P_{i_0} , P_{j_0} . Now

$$f^{(N)}(P_0, Q_0; t) = f_{ij}^{(2/N)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}, P_{j_0}, R_{j_0}, P_1 \dots Q_N; t) f_{ij}^{(2)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}, P_{j_0}, R_{j_0}; t),$$

where $f_{ij}^{(2/N)}$ is the relative probability density in the residual phase space of the other $N-2$ particles, when the particles "i" and "j" have specified coordinates P_{i_0} , R_{i_0} , P_{j_0} , R_{j_0} in the 12-dimensional phase space of that pair. The integration of $f_{ij}^{(2/N)}$ over this residual phase space can now be carried out to give unity, and the expression for K_{ji} becomes

$$K_{ji} = \frac{1}{\tau} \int_0^\tau \iiint F_{ji} \cdot \nabla_{P_i} f_{ij}^{(2)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}, P_{j_0}, R_{j_0}; t) \delta(\tilde{P}_{i_0} + \Delta P_i - P_i) \delta(\tilde{R}_{i_0} + \Delta R_i - R_i) d\tilde{P}_{i_0} d\tilde{R}_{i_0} dP_{j_0} dR_{j_0} ds. \quad (19)$$

* Actually Kirkwood^{11, 12} introduces a "cut-off" distance R_0 such that F_{ji} vanishes for all $R > R_0$. For neutral particles the interparticle force field decays so rapidly with increasing separation distance that this limitation is not an essential one. For charged particles the situation is not quite so simple because of the long-range Coulomb forces.

With this last result for K_{ji} the differential equation for $\bar{f}_i^{(1)}$ Eq. (17) is identical (except for the time averages) with that found by Born and Green by a somewhat different approach [Eq. (6.1), page 7, Reference 13, Part I].

In velocity space the approximation of molecular chaos states that

$$f_{ij}^{(2)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}, P_{j_0}, R_{j_0}; t) = f_i^{(1)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}; t) \cdot f_j^{(1)}(P_{j_0}, R_{j_0}; t), \quad (20)$$

i. e., the joint probability density of two interacting particles is equal to the product of the a priori independent probabilities of the two particles considered separately. Evidently this statement is closely connected with the approximation of binary encounters. During such an encounter $f_{ij}^{(2)}$ is unchanged, since no third particle intervenes. Each particle experiences a certain finite change in momentum and then goes its separate way, with no "memory" of the collision process itself. Thus after a collision, $f_{ij}^{(2)} = f_i^{(1)} \cdot f_j^{(2)}$. But $f_{ij}^{(2)}$ is constant during the collision, so that Eq. (20) holds throughout the duration of the binary encounter. Clearly this approximation plays a key role; without it the differential equation for $\bar{f}_i^{(1)}$ involves $f_{ij}^{(2)}$, and one must deal with the sequence of equations for $f_{ij}^{(2)}$ in terms of $f_{ijk}^{(3)}$, $f_{ijk}^{(3)}$ in terms of $f_{ijkl}^{(4)}$, etc. (See for example, Eq. (6.2), page 7, Reference 13, Part I.)

Our list of approximations is completed by requiring that $f_i^{(1)}$ and $f_j^{(1)}$ do not vary appreciably over a distance of the order of the range of interparticle forces, or in a time interval of the order of the duration of a representative collision. In other words this analysis cannot deal with bodies that are themselves of molecular size, but the linear dimensions of the objects involved can certainly be small compared with the mean free path. By employing this approximation, one can replace $\delta(\tilde{R}_{i_1} + \Delta R_{i_1} - R_{i_1})$ by $\delta(\tilde{R}_{i_1} - R_{i_1})$, and R_{j_0} by \tilde{R}_{i_0} in Eq. (19). By changing the variables of integration from \tilde{R}_{i_0}, R_{j_0} to \tilde{R}_{i_0} and $(R_{i_j})_0 = R_{j_0} - \tilde{R}_{i_0}$, employing Eq. (20) and making use of the properties of the delta function one obtains

$$K_{ji} = (1/\tau) \iiint f_i^{(1)}(\tilde{P}_{i_0}, \tilde{R}_{i_0}; t) f_j^{(1)}(P_{j_0}, \tilde{R}_{i_0}; t) \cdot I_1 \cdot d\tilde{P}_{i_0} \cdot dP_{j_0} (dR_{i_j})_0, \quad (21)$$

where

$$I_1 = \int_0^{\tau} \frac{\partial}{\partial s} (\Delta P_i) \cdot \nabla_{P_i} \left[\delta(\tilde{P}_{i_0} + \Delta P_i - P_i) \right] ds \quad (22)$$

Carrying out the integration in Eq. (22), one obtains

$$I_1 = \delta(\tilde{P}_{i_0} + \Delta P_i(\tau) - P_i) - \delta(\tilde{P}_{i_0} - P_i) ; \quad (23)$$

therefore,

$$K_{ji} = \frac{1}{\tau} \iiint \left[\delta(P_i' - P_i) - \delta(\tilde{P}_{i_0} - P_i) \right] f_i^{(1)}(\tilde{P}_{i_0}, \tilde{R}_i; t) f_j^{(1)}(P_{j_0}, \tilde{R}_j; t) d\tilde{P}_{i_0} dP_{j_0} (dR_{ij})_0, \quad (24)$$

where $P_i' = \tilde{P}_{i_0} + \Delta P_i(\tau)$.

The final question to be decided is the magnitude of the time interval τ . Of course one would like to make this time interval long enough so that the value of K_{ji} [Eq. (24)] is independent of τ . That this question is not a trivial one can be seen by considering the limiting case $\tau \rightarrow \infty$. According to Poincaré's theorem, for a system of particles confined to a finite region of phase space, in which the forces depend only on the spatial coordinates, a given initial state must recur at least once to any desired degree of accuracy, if one waits long enough. Thus there exist Poincaré cycle periods within which each orbit is traversed in the reverse sense, thereby cancelling its initial contribution to the integral K_{ji} . Now for dilute gases the Poincaré cycle periods are extremely long even on the macroscopic time scale. Therefore, this apparent paradox between dynamic reversibility and thermodynamic or macroscopic irreversibility is resolved by choosing τ to be large relative to the duration of a representative collision, but short relative to the Poincaré cycle period.

In fact it seems to the present writer that the magnitude of τ is clearly dictated by the dynamical structure of the system, and particularly by the molecular chaos — binary encounter approximation. Suppose we consider a binary encounter between a "j" particle and an "i" particle in coordinates relative to the "i" particle (Figure 1). Suppose that F_{ji} vanishes outside of a sphere of radius $R_0 = 2 - 3 \sigma_0$, where σ_0 is the minimum energy distance. Let b equal the perpendicular distance between "i" and the initial trajectory of "j" before the encounter, and let

$$P_{ij} = m_{ij} \left[\frac{P_j}{m_j} - \frac{P_i}{m_i} \right] ;$$

$\frac{(p_{ij})_0}{m_{ij}}$ is the scalar magnitude of the initial relative velocity and m_{ij} is the usual reduced mass given by the relation $1/m_{ij} = 1/m_i + 1/m_j$.

The duration of a representative collision is $\tau_c \approx \frac{2R_0}{(p_{ij})_0 / m_{ij}}$.

By referring to Figure 1 one sees that the regions of relative configuration space occupied by the particle "j" at a given initial instant can be classified as follows (Reference 12):

Region I

$$z > \frac{(p_{ij})_0}{m_{ij}} \tau + \sqrt{R_0^2 - b^2}.$$

For particles lying in this region the sphere of radius R_0 is not penetrated in the interval τ , so that $\Delta P_1 = 0$ and the integrand in Eq. (24) vanishes.

Region II

$$\frac{(p_{ij})_0}{m_{ij}} (\tau - \tau_c) + \sqrt{R_0^2 - b^2} < z \leq \frac{(p_{ij})_0}{m_{ij}} \tau + \sqrt{R_0^2 - b^2}.$$

For particles lying in this region, occupying a volume

$$\frac{(p_{ij})_0}{m_{ij}} \tau_c \, b \, db \, d\epsilon, \text{ only partially completed collisions occur in the interval } \tau.$$

Region III

$$\sqrt{R_0^2 - b^2} \leq z \leq \frac{(p_{ij})_0}{m_{ij}} (\tau - \tau_c) + \sqrt{R_0^2 - b^2}.$$

These particles, occupying a volume $\frac{(p_{ij})_0}{m_{ij}} (\tau - \tau_c) \, b \, db \, d\epsilon$, experience complete collisions in the interval τ .

Region IV

$$-\sqrt{R_0^2 - b^2} \leq z < \sqrt{R_0^2 - b^2}.$$

For these particles, occupying a volume $2 \sqrt{R_0^2 - b^2} \, b \, db \, d\epsilon$, partially completed collisions are completed, or multiply periodic orbits may be traversed in the interval τ .

Now the contribution to I [Eq. (24)] of the incomplete collisions and multiply periodic orbits can be made as small as desired by choosing $\tau \gg \tau_c$, because the contribution of the completed collisions is proportional to τ . Thus

$$K_{ji} = \iint \dots \int \frac{(P_{ij})_0}{m_{ij}} \left[\delta(P_i' - P_i) - \delta(\tilde{P}_{i0} - P_i) \right] f_i^{(1)}(\tilde{P}_{i0}, \tilde{R}_i; t) f_j^{(2)}(P_{j0}, \tilde{R}_i; t) \\ \cdot dP_{i0} dP_{j0} dbdbd\varepsilon + O(\tau_c/\tau) \quad (25)$$

On the other hand, it seems clear (although Kirkwood does not say so) that in a gas τ must be short compared with the average time between two successive collisions; otherwise, the molecular chaos-binary encounter approximation makes no sense. Thus $\tau_c \ll \tau \ll \tau_f$.

By conservation of total linear momentum,

$$\Delta P_i = -\Delta P_j = -\Delta P^*; \text{ so that } P_i' - P_i = \tilde{P}_{i0} - (\Delta P^* + P_i),$$

and $P_{j0} = P_j' - \Delta P^*$, where the prime denotes quantities evaluated after

collision. By utilizing the properties of the delta function and changing variables, K_{ij} is evaluated as follows:

$$K_{ji} = \int_0^\infty \int_0^{2\pi} \int \frac{(P_{ij})_0}{m_{ij}} \left[f_i^{(1)}(P_i + \Delta P^*, \tilde{R}_i; t) f_j^{(1)}(P_j - \Delta P^*, \tilde{R}_i; t) \right. \\ \left. - f_i^{(1)}(P_i, \tilde{R}_i; t) f_j^{(1)}(P_j, \tilde{R}_i; t) \right] dbdbd\varepsilon dP_j \quad (26)$$

Substituting this result for K_{ji} into Eq. (17), and taking time averages of both sides of the equation over the time interval τ , one obtains

$$\left[\overline{f_i^{(1)'} f_j^{(1)'}} - \overline{f_i^{(1)} f_j^{(1)}} \right] \text{ for the bracket in the integrand of Eq. (26).}$$

If $f_i^{(1)}$ and $f_j^{(1)}$ do not vary appreciably over this time interval, the bracket

can be replaced by $\left[(\bar{f}_i^{(1)} \bar{f}_j^{(1)})' - \bar{f}_i^{(1)} \bar{f}_j^{(1)} \right]$; when $m_i N_i \bar{f}_i^{(1)}$ is replaced by f_i , Eq. (17) becomes identical with the Maxwell-Boltzmann integro-differential equation:

$$\frac{\partial f_i}{\partial t} + \frac{P_i}{m_i} \cdot \nabla_{R_i} f_i + X_i \cdot \nabla_{P_i} f_i = \sum_{j=1}^j \int_0^\infty \int_0^{2\pi} \int \frac{(P_{ij})_0}{m_{ij}} [f_i' f_j' - f_i f_j] b db d\epsilon d\vec{\xi}_j, \quad (27)$$

where $d\vec{\xi}_j = dp_j/m_j$.

To summarize, the Maxwell-Boltzmann equation for a dilute gas is derived from Liouville's equation for an arbitrary dynamical system by utilizing the approximation of molecular chaos, or independent a priori probabilities, in both configuration and velocity space. The essential macroscopic dissipative mechanism is introduced by taking averages over a suitable time interval τ , which eliminates the contributions made by incomplete collisions and multiply periodic particle orbits. This time interval must be long compared with the duration of a representative collision, but short compared with the average time between two successive collisions of a particle. In addition we require that the single particle distribution function f_i must not vary appreciably over a distance of the order of the sensible range of interparticle forces, or in a time interval of the order of the duration of a representative collision.

Clearly these approximations are intimately related to the structure of the dynamical system, but there is no rigorous proof of their validity; they must be regarded as working hypotheses to be subjected to the test of experience. These approximations would be quite inappropriate for liquids, for example. In that case the equation for $f_i^{(1)}$ involves $f_{ij}^{(2)}$ [Eq. (19)], and this equation must be supplemented by the equation for $f_{ij}^{(2)}$ in terms of $f_{ijk}^{(3)}$, etc. Kirkwood¹¹, and Born¹³ and Green, break this ascending sequence by employing the approximation

$$\frac{f_{ijk}^{(3)}}{f_k^{(1)} f_{ij}^{(2)}} = \left(\frac{f_{ik}^{(2)}}{f_i^{(1)} f_k^{(1)}} \right) \cdot \left(\frac{f_{jk}^{(2)}}{f_j^{(1)} f_k^{(1)}} \right),$$

i. e., the relative probability of the occurrence of a third molecule in conjunction with two others is equal to the product of the relative probabilities of the occurrence of this same molecule in conjunction with each of the others separately. It is remarkable that in spite of the basic differences between the two systems the statistical mechanical treatment yields the same Navier-Stokes relations between the stresses and rates of strain for liquids and for gases at normal densities. From this point of view one should not be surprised to find that the behavior of rarefied gases is quite different, at least according to the kinetic theory (Sections III and IV).

III. MAXWELL'S EQUATIONS OF TRANSFER AND THEIR APPROXIMATE SOLUTION

III. 1. Previous Investigations of the Maxwell-Boltzmann Equation*

Because of the non-linear collision integral in the Maxwell-Boltzmann equation [Eq. (27)] most investigators are forced to resort to various linearization procedures. The most famous of these is the Chapman-Enskog⁵ method, in which f is expressed as a series expansion of the form**

$$f = f_0 + \epsilon f_1 + \epsilon^2 f_2 + \dots,$$

where f_0 is the "local Maxwellian", and the small parameter ϵ is of the order of τ_f/\bar{t} . [Here \bar{t} is a characteristic flow time; for steady flow $\epsilon \sim \lambda/L$, where L is a characteristic length and λ is the mean free path.] The motivation for this scheme can be appreciated by writing Eq. (27) in non-dimensional form. A factor $(\bar{c} \sigma_0^2 n) \bar{t}$ appears on the right-hand side, and this factor is proportional to $\bar{t}/\tau_f \sim 1/\epsilon$. When the Chapman-Enskog series expansion for f is substituted into the Maxwell-Boltzmann equation, the terms in the collision integral containing f_0 alone drop out, by definition, and the leading terms contain f_1 . Thus, in first approximation f is replaced by f_0 on the left-hand side, and f_1 is determined so as to satisfy the reduced equation. No proof of convergence has ever been given, and one strongly suspects that this method, which leads to the Navier-Stokes relations for a monatomic gas, is applicable only when ϵ is "sufficiently small" compared with unity.

At the opposite end of the scale Jaffé¹⁷ suggested that the distribution function for nearly-free molecule flow could be represented by a series of the form $f_{F.-M.} + (L/\lambda)f_1 + (L/\lambda)^2 f_2 + \dots$, where (L/λ) is now the small parameter. Wang Chang¹⁸ and Uhlenbeck have applied this method, which they call "Knudsen iteration", to the shear flow and

* Excellent reviews of previous work in this field have been given by Gross¹⁴, Willis¹⁵, and Grad¹⁶, so that only some of the main points need to be discussed here.

** For a pure gas $f_1 = f$.

heat transfer between two infinite, parallel plates. This geometry turns out to be an unfortunate choice for Jaffé's method, because a linearization procedure of this type leads to distribution functions containing a term like $\exp \left[-\frac{L}{\lambda} \frac{1}{\xi_y} \right]$ where ξ_y is the particle velocity in the direction normal to the plates. In this case the mean velocity distribution contains a term like $(L/\lambda) \log_e (L/\lambda)$ (see References 14 and 15), and Jaffé's procedure is clearly invalid. Szymanski¹⁹ has applied a similar method to low Mach number flow over a sphere. Again no estimate of the range of validity of this method is available.

In an effort to generalize the Chapman-Enskog procedure, Mott-Smith²⁰ and Wang Chang¹⁸ and Uhlenbeck utilized series expansions of the form

$$f = \sum_{n=0}^{\infty} A_n(\vec{R}, t; \theta) \phi_n(c),$$

where θ is the angle between c and some prescribed direction, and the ϕ_n 's are suitable orthogonal polynomials, e.g., Sonine or Hermite. But the slow convergence of such series expansions is only another indication of their inappropriateness for rarefied gas flow, where the discontinuity, or "two-sidedness" in the distribution function is essential.

In order to circumvent some of the difficulties experienced with the non-linear collision integral, Max Krook^{21, 22} proposed that this integral, which is of the form $(\partial f / \partial t)_{\text{coll.}}$, be replaced by the linear approximation $\left[- \left(\frac{f - f_{\text{eq.}}}{a} \right) \right]$, where a is some characteristic time. This approximation amounts to the choice of a single characteristic time for all particle velocities. The resulting equation for f , called the "Krooked Boltzmann equation", or the "ersatz" equation, has been studied by Bhatnager, Gross, and Krook²¹.

Gross⁹, Jackson, and Ziering recognized some of the unsatisfactory features of the Krook equation and were well aware of the essential requirement of "two sidedness" in the distribution function for rarefied gas flows. For steady plane Couette flow they employed a distribution function of the form

$$f = f_0 \left[1 + \phi(\vec{\xi}, y) \right], \text{ where } \phi(\vec{\xi}, y) = \phi^+ \text{ for } \xi_y > 0,$$

and

$$\phi(\vec{\xi}, y) = \phi^- \text{ for } \xi_y < 0, \text{ with } \phi^\pm = a_0^\pm(y) \xi_x + a_1^\pm(y) \xi_x \xi_y.$$

[Here x and y are directions parallel and normal to the plates, respectively.]

They obtained approximate solutions by satisfying the boundary conditions exactly, and utilizing the first few full-range (Method A) or half-range (Method B) velocity moment equations obtained from the Krook equation to determine $a_0^\pm(y)$ and $a_1^\pm(y)$. In a later paper Gross²³ and Ziering studied plane Couette flow by applying this procedure to the linearized Maxwell-Boltzmann equation for f , and Gross²⁴ and Jackson analyzed the linearized Rayleigh problem by the same method. This procedure comes closer to the spirit of Maxwell's moment integral approach (Section III. 2), but is still unnecessarily restricted by the use of polynomials for ϕ . A comparison between this method and the present scheme for plane Couette flow is given in Section V. 1.

Recently Willis¹⁵ developed an iterative method primarily for nearly-free molecular flows, based upon a transformation of Eq. (27) into an integral equation. For steady flow in the absence of external forces Eq. (27) for a pure gas can be written as follows:

$$\vec{\xi} \cdot \nabla_{\vec{R}} f = -f \cdot D(f, \vec{R}, \vec{\xi}) + P(f, \vec{R}, \vec{\xi}), \quad (28)$$

where

$$D(f, \vec{R}, \vec{\xi}) = \iiint f_1 |\vec{\xi}_1 - \vec{\xi}| b d b d \epsilon d \vec{\xi}_1, \quad (29a)$$

and

$$P(f, \vec{R}, \vec{\xi}) = \iiint f' f_1' |\vec{\xi}_1 - \vec{\xi}| b d b d \epsilon d \vec{\xi}_1. \quad (29b)$$

The superscript "1" denotes the colliding particle and $\vec{\xi}$ is the vector particle velocity. This equation is integrated formally to give

$$f(\vec{R}_0 + \frac{|\vec{R} - \vec{R}_0|}{|\vec{\xi}|} \vec{\xi}, \vec{\xi}) = f(\vec{R}_0, \vec{\xi}) \exp - \left\{ \int_{\vec{R}_0}^{\vec{R}} \frac{D}{|\vec{\xi}|} d\vec{R}'' \right\} + \int_{\vec{R}_0}^{\vec{R}} d\vec{R} \frac{P}{|\vec{\xi}|} \exp \left\{ - \int_{\vec{R}}^{\vec{R}} \frac{D}{|\vec{\xi}|} d\vec{R}'' \right\}, \quad (30)$$

where $f = f(\vec{R}_0, \vec{\xi})$ at $\vec{R} = \vec{R}_0$, and the integration is carried out along straight lines parallel to $\vec{\xi}$. For nearly free-molecular flow the iteration is begun by evaluating D and P from the free-molecule distribution function. The first iteration is equivalent under certain conditions¹⁵ to the "first

collision" method, in which the only collisions considered are the first ones experienced by a particle as it leaves the body.²⁵ Willis utilizes Krook's model in applying his method to linearized plane Couette flow, and to the flow over a sphere, disc, and two-dimensional strip. The results obtained are quite instructive, and it would be interesting to apply this promising method to the Boltzman equation itself rather than to Krook's equation, in order to clarify the role of the non-analytic behavior of "f" near $\xi = 0$ in Krook's model. (See Section V. 1.)

To summarize the situation, none of the methods proposed for attacking Eq. (27) directly appears to be generally satisfactory. In fact some of the so-called "improvements" on the Navier-Stokes relations only make matters worse. A thorough exploitation of the Maxwell integral equations of transfer would seem to be long overdue.

III. 2. Maxwell's Equation of Transfer: Present Method

Suppose that Q is any function of the velocity components of a particle, (momentum, energy, etc.) and we want to obtain the general equation for the rate of transport Q . This equation can be derived from Eq. (27), or it can be obtained more directly by considering the various sources of change in the amount of Q contained in a fixed volume. For simplicity, consider a pure gas.* The total amount of Q per unit volume is given by

$$n(\vec{R}, t) \cdot \bar{Q}(\vec{R}, t) = \int Q(\vec{\xi}) f(\vec{R}, \vec{\xi}, t) d\vec{\xi}, \quad (31)$$

where \bar{Q} is the mean value of Q . The rate of increase of this amount per unit time is $\partial/\partial t (n \bar{Q})$, and this quantity must be equal to the sum of three terms: (1) the net rate of change in total amount of Q per unit volume caused by the flux of particles across the bounding surface of the fixed volume; (2) the effect of external forces and curvature of the coordinate system on particle acceleration; (3) the effect of collisions.

Now in vector notation the first effect is given by

* There is no conceptual difficulty in extending this treatment to mixtures.

$$- \vec{\nabla}_R \cdot \left[\int f \vec{\xi} Q d\vec{\xi} \right], \quad *$$

while the second is given by

$$\int f \left\{ \vec{F}/m - (\vec{\Omega} \times \vec{\xi}) \right\} \cdot \vec{\nabla}_{\vec{\xi}} Q d\vec{\xi},$$

where $\vec{\Omega}$ is the local instantaneous angular velocity vector introduced by the curvature of the coordinate system. ** The number of collisions per particle per unit time is equal to

$$V f_1(\vec{\xi}_1, \vec{R}, t) b db d\epsilon d\vec{\xi}.$$

[Figure 1] where $V = |\vec{\xi}_1 - \vec{\xi}|$, and the number of particles per unit volume is $f(\vec{\xi}, \vec{R}, t) d\vec{\xi}$, so that the rate of change of total amount of Q from this source, ΔQ , is given by

$$\Delta Q = \iiint \int (Q' - Q) f f_1 V d\vec{\xi} d\vec{\xi}_1 b db d\epsilon, \quad (32)$$

where $Q' - Q$ is the change in Q experienced in a collision. Finally, the Maxwell integral equation of transfer is obtained as follows:

$$(\partial/\partial t) \left(\int f Q d\vec{\xi} \right) + \vec{\nabla}_R \cdot \left[\int f \vec{\xi} Q d\vec{\xi} \right] = \int f \left\{ \vec{F}/m - (\vec{\Omega} \times \vec{\xi}) \right\} \cdot \vec{\nabla}_{\vec{\xi}} Q d\vec{\xi} + \Delta Q, \quad (33)$$

where ΔQ is given by Eq. (32).

This transport equation has been applied in studies of the motion of electrons and ions²⁶, and it was utilized by Maxwell^{4b} and by Chapman⁵ in developing the Chapman-Enskog procedure (Section III.1). However, it does not seem to have been employed in later fluid mechanical studies until the work of H. Grad⁸. In Grad's thirteen-moment method the distribution function is a linear function of the stresses and heat fluxes, which are now regarded as separate dependent variables not explicitly related to ρ , u , T and their derivatives. Thus, in a rectangular Cartesian coordinate system,

$$f = f_0 \left[1 + \frac{p_{ij}}{2pRT} c_i c_j - \frac{q_i c_i}{pRT} \left(1 - \frac{c^2}{5RT} \right) \right],$$

where f_0 is the local Maxwellian. By substituting this expression for f into Eq. (33), and by taking Q to be equal successively to m , $m \xi_i$,

* The components of $\vec{\xi}$ are independent of \vec{R} , but the unit vectors are not.

** In a Cartesian coordinate system the second term inside the braces vanishes (Appendix A).

$m(\xi^2/2)$, $m\xi_i\xi_j$, and $m\xi_i(\xi^2/2)$, thirteen partial differential equations (including the conservation relations) are obtained for the thirteen independent moments ρ , \vec{u} , T , p_{ij} and \dot{q}_i . Grad's method yields a qualitatively correct transition from free-molecule flows to the Navier-Stokes regime when it is applied to relatively simple problems such as plane Couette flow²⁷ and Rayleigh's problem²⁸ at low Mach number. But the expression for f does not have the bimodal character that is an essential feature of rarefied gas flows, and it is still quite restrictive in its use of polynomials in the particle velocity. This formulation introduces undesirable cross-couplings between the normal stresses and heat fluxes.* These deficiencies probably account for the unsatisfactory results obtained in the case of a normal shock wave²⁹.

In order to make any further progress it seems necessary to drop the search for "higher order" macroscopic equations such as Burnett's, Grad's, etc. An important advantage of Eq. (33) over Eq. (27) is just the fact that it permits a large amount of flexibility in the choice of f , which is obscured by the Chapman-Enskog⁵, Grad⁸, and Gross⁹ procedures. The distribution function can be expressed in terms of a number of arbitrary functions of \vec{R} and t , selected in such a way that the boundary conditions of the problem are satisfied. Physical reasoning can be introduced at an early stage of the analysis. The essential non-linear character of the collision term can be preserved and questionable expansion and linearization procedures avoided. Of course the proper number of moments must be taken to insure that a complete set of first-order partial differential equations is obtained for the undetermined functions of \vec{R} and t . The number of these functions (or moments) employed will depend on the "level" of information or the degree of detail desired. The value of such an integral method depends upon the fact that for many problems the moments are not particularly sensitive to the assumed weighting function, provided only that certain lower-order moment equations are satisfied.

The present approach was stimulated by Mott-Smith's¹⁰ treatment of the normal shock wave. In that analysis he employs a distribution function that is the sum of two full-range Maxwellian terms, corresponding to the subsonic and supersonic streams "far" downstream and "far" upstream.

* Paper by D. K. Ai to appear shortly.

The velocity and temperature functions appearing in these Maxwellians are taken as constants, and are therefore related by the usual Rankine-Hugoniot conditions, but the number densities of these Maxwellians are initially undetermined. The number density distributions through the shock region are found by solving the transport equation [Eq. (33)] with $Q = m\xi_x^2$, or alternatively, with $Q = m\xi_x^3$. As Mott-Smith points out, his analysis is designed especially for strong shocks, where the bimodal character of f is expected to predominate. One cannot expect this treatment to be valid for weak shocks, and in fact the recent experiments of Sherman³⁰ and Talbot show that it is not.

Mott-Smith's formulation must be modified and generalized in such a way that the following basic requirements are satisfied by the distribution function: (1) It must have the "two-sided" character that is an essential feature of rarefied gas flow; (2) It must be capable of providing a smooth transition from rarefied flows to the Navier-Stokes regime; (3) It should lead to the simplest possible set of differential equations and boundary conditions consistent with requirements (1) and (2).

A simple distribution function that satisfies the requirement of "two-sidedness" is suggested by the situation for free-molecule flow. In that limiting case the collision term in the Maxwell-Boltzmann equation is negligibly small, and in the absence of external forces the "characteristics" of Eq. (27) are very nearly straight lines, representing the paths of particles moving with unchanged velocity both toward and away from solid surfaces. The distribution function at a point P (Figure 2) is governed by the "line of sight" principle; the influence of the body at P is confined to the conical surface formed by the limiting tangent rays drawn from P to the surface. Consider the simple case of diffuse reflection at the surface. For all outwardly directed particle velocity vectors lying within this cone (Figure 2) f is the Maxwellian corresponding to the velocity and temperature of the surface. For all other velocity vectors f is the free stream Maxwellian. A natural generalization of this distribution function is provided by the following representation:

For $\vec{\xi}$ lying in Region I (Figure 2)

$$f = f_1 = n_1(\vec{R}, t) \left[\frac{1}{2\pi \mathcal{R} T_1(\vec{R}, t)} \right]^{3/2} \exp \left\{ - \frac{[\vec{\xi} - \vec{u}_1(\vec{R}, t)]^2}{2 \mathcal{R} T_1(\vec{R}, t)} \right\}. \quad (34a)$$

In Region II (all other $\vec{\xi}$)

$$f = f_2 = n_2(\vec{R}, t) \left[\frac{1}{2\pi R T_2(\vec{R}, t)} \right]^{3/2} \exp \left\{ - \frac{(\vec{\xi} - \vec{u}_2(\vec{R}, t))^2}{2 R T_2(\vec{R}, t)} \right\}, \quad (34b)$$

where $n_1(\vec{R}, t)$, $n_2(\vec{R}, t)$, $\vec{u}_1(\vec{R}, t)$, $\vec{u}_2(\vec{R}, t)$, $T_1(\vec{R}, t)$, and $T_2(\vec{R}, t)$ are ten initially undetermined functions of \vec{R} and t . In a sense, the functions n_2 , \vec{u}_2 , T_2 represent the "screening effect" on the oncoming stream of the particles reflected from the surface, while the functions n_1 , \vec{u}_1 , T_1 contain the effect of the "free stream" particles on those emanating from the surface. All macroscopic quantities of interest are uniquely determined by these arbitrary functions.

Evidently a similar representation can be utilized for flows confined between two solid surfaces. A simple example is provided by the problem of steady, plane Couette flow between two infinite, parallel plates in relative motion (Figure 3). In that case f is given by the following representation, involving six undetermined functions:

For $\xi_y > 0$,

$$f = f_1 = n_1(y) \left[\frac{1}{2\pi R T_1(y)} \right]^{3/2} \exp \left\{ - \frac{[\xi_x - u_{x_1}(y)]^2 + \xi_y^2 + \xi_z^2}{2 R T_1(y)} \right\}. \quad (35a)$$

For $\xi_y < 0$,

$$f = f_2, \quad (35b)$$

where f_2 is a similar generalized Maxwellian containing the functions $n_2(y)$, $T_2(y)$, $u_{x_2}(y)$. A simple example of a non-steady flow (Rayleigh's problem) is given in Sections IV. 3. and V. 2.

One important difference between the present formulation and Mott-Smith's¹⁰ is that the distribution functions given by Eqs. (34) - (35) are discontinuous in the particle velocity, whereas Mott-Smith's function is continuous. This discontinuous behavior is dictated by the requirements of "two-sidedness" and a non-vanishing shear stress, but it also turns out to be quite useful in satisfying the requirement of a smooth transition to the Navier-Stokes regime. Because of the success of the Chapman-Enskog procedure for $t/\tau_f \gg 1$, one is accustomed to associating shear stress in this region with an anti-symmetric distribution function containing terms

like $c_i c_j$. Yet, in the "simplified" kinetic theory derivation of transport properties^{2b} the shear stress is equated to the momentum flux carried by particles crossing a given plane from above with the mean velocity $u + a\lambda(\partial u/\partial y)$, minus the momentum flux carried by particles crossing from below, with the mean velocity $u - a\lambda(\partial u/\partial y)$. Clearly the present two-stream Maxwellian model is fully capable of representing this situation, provided the functions \vec{u}_1 and \vec{u}_2 approach $u \pm a\lambda(\partial u/\partial y)$, respectively, in the Navier-Stokes limit ($Re/M \rightarrow \infty$). In the two simple examples of plane Couette flow and Rayleigh's problem studied so far (Section IV), one finds that the solutions obtained do in fact join smoothly to the Navier-Stokes solutions when $b/\lambda \gg 1$, or when $t/\tau_f \gg 1$.

Of course these remarks apply equally well to the normal stresses, so that (for example) Mott-Smith's¹⁰ treatment of the normal shock wave must be modified by taking a distribution function that is discontinuous at $\xi_x = 0$.

The question naturally arises as to the minimum number of arbitrary functions that can be employed in this representation of the distribution function. The minimum number is imposed by the requirement that at least the conservation laws should be satisfied. As shown by Maxwell^{4b} the ordinary gas dynamic conservation equations are obtained regardless of f by taking Q in Eq. (33) to be the collisional invariants of mass, momentum, and energy, successively. If $Q = m$, the particle mass, then the right-hand side vanishes. By definition $\rho = \int m f d\vec{\xi}$ and $\rho \vec{u} = \int m f \vec{\xi} d\vec{\xi}$, so that Eq. (33) yields the equation of continuity:

$$\partial \rho / \partial t + \nabla_R \cdot (\rho \vec{u}) = 0, \quad (36)$$

When $Q = m \vec{\xi}$ the right-hand side of Eq. (33) becomes $\int f \vec{F} d\vec{\xi}$, provided that the force is not a function of the particle velocity, because the vector momentum is not affected by curvature of the coordinate system. But $\int f \vec{F} d\vec{\xi} = n \vec{F} = \mathcal{F}_1$, the vector force per unit volume. By recognizing that $\vec{\xi} = \vec{c} + \vec{u}$, where \vec{c} is the intrinsic, or relative particle velocity, the second integral on the left-hand side of Eq. (33) is evaluated as follows:

$$\int m f \vec{\xi} \vec{\xi} d\vec{\xi} = \int m (\vec{c} + \vec{u})(\vec{c} + \vec{u}) f d\vec{\xi} = m \int \vec{c} \vec{c} f d\vec{\xi} + \rho \vec{u} \vec{u}. \quad (37)$$

But $m \int \vec{c} \vec{c} f d\vec{\xi} = -\underline{\sigma}$, where $\underline{\sigma}$ is the stress tensor. In this case, Eq. (33) becomes

$$(\partial/\partial t) (\rho \vec{u}) + \vec{\nabla}_R \cdot (\rho \vec{u} \vec{u}) = \vec{\mathcal{F}} + \vec{\nabla}_R \cdot \underline{\sigma}, \quad (38a)$$

or

$$\rho (\partial \vec{u} / \partial t) + \rho \vec{u} \cdot \vec{\nabla}_R \vec{u} = \vec{\mathcal{F}} + \vec{\nabla}_R \cdot \underline{\sigma}, \quad (38b)$$

which is identical with the usual gas dynamic momentum equation.

Similarly, by taking $Q = m \xi^2/2$, utilizing the kinetic theory definitions of the temperature and the heat flux vector,

$$(3/2) n k T = (3/2) \rho R T = m \int (c^2/2) f d\vec{\xi}, \quad (39)$$

and

$$\vec{q} = m \int \vec{c} (c^2/2) f d\vec{\xi}, \quad (40)$$

and making use of the continuity equation [Eq. (36)], one obtains the energy equation:

$$\rho \frac{\partial}{\partial t} \left(\frac{3}{2} RT + \frac{u^2}{2} \right) + \rho \vec{u} \cdot \vec{\nabla}_R \left(\frac{3}{2} RT + \frac{u^2}{2} \right) = - \vec{\nabla}_R \cdot \vec{q} + \vec{\nabla}_R \cdot (\underline{\sigma} \vec{u}) + \vec{\mathcal{F}} \cdot \vec{u}. \quad (41)$$

By utilizing the momentum equation [Eq. (38)], the energy equation [Eq. (41)] is transformed to the following relation

$$(\partial p / \partial t) + \vec{\nabla}_R \cdot (p \vec{u}) = - 2/3 (\vec{\nabla}_R \cdot \vec{q}) + 2/3 (\underline{\sigma} \cdot \vec{\nabla}_R \vec{u}), \quad (42)$$

where

$$p = \rho R T. \quad (43)$$

We may also write

$$\underline{\sigma} = - p \underline{I} + \underline{\mathcal{Z}}, \quad (44a)$$

where \underline{I} is the identity tensor, and $\underline{\mathcal{Z}} = - m \int (\vec{c} \vec{c} - 1/3 c^2 \underline{I}) f d\vec{\xi}$; (44b)
e.g., in any locally orthogonal coordinate system

$$P_{ii} = - p + P_{ii}, \quad (45a)$$

where

$$P_{ii} = - (m/3) \int (2 c_i^2 - c_j^2 - c_k^2) f d\vec{\xi}, \quad (45b)$$

and

$$P_{ij} = P_{ji} = - m \int c_i c_j f d\vec{\xi}, \quad i \neq j. \quad (45c)$$

Thus the minimum number of moments is equal to the number of collisional invariants. In an axially-symmetric or two-dimensional flow, for example, a minimum of four moments is required. Now the two-stream Maxwellian distribution function represented by Eqs. (34) and (35) contains eight undetermined functions in this case, so that a maximum of four of these functions can be chosen in advance. Of course one will generally utilize one or two moments in addition to the collisional invariants, and the moments corresponding to the stresses and heat fluxes are the logical

choices (Sections III. 3. and IV.). Evidently the approximation is systematically improved by taking additional moments in Eq. (33), but it is very difficult to make any useful statements in advance regarding the convergence of this process. Experience with this important question must be obtained by solving certain typical flow problems (Section IV.).

III. 3. Collision Integral for Maxwell Particles

Once the two-stream Maxwellian is selected as the particle velocity distribution function, the collision integral appearing in Maxwell's integral transport equation [Eqs. (32) and (33), Section III. 2.] can be evaluated for any arbitrary law of force between the particles. At this stage of the analysis it seems desirable to select the simplest inter-particle force field that contains the main short-range features of the binary collision "model" [Section II. 2]. This approach will be particularly useful for non-linear problems, where we want to preserve the essential non-linear character of the collision integral. More sophisticated inter-particle potentials can be employed later if a more refined description of real gas flows is found to be desirable.

Maxwell's famous inverse fifth-power force law provides just the simplification required, because with the proper choice of variables the relative velocity $V = |\xi_1 - \xi|$ is eliminated from the collision integral ΔQ . By considering the classical two-body problem in the plane of the orbit one finds^{2c, 4b} that the motion of the colliding particle relative to the "probe" particle (Figure 1) is equivalent to the motion of a particle of unit mass about a fixed center of force. For a central force field of the type

$F = \frac{m_1 m_2 K}{r^s}$ the potential energy at a distance r for this particle of unit mass is given by $\frac{(m_1 + m_2) K}{(s-1) r^{s-1}}$. The important parameter appearing in the equations of motion is the ratio of the kinetic energy $V^2/2$ to the potential energy at the distance $r = b$, or $\frac{V^2 (s-1) b^{s-1}}{2(m_1 + m_2) K}$. * This

* As in Sections II. and III. 2., b is the impact parameter, or perpendicular distance between the probe particle and the initial trajectory of the colliding particle.

parameter determines the total deflection or scattering angle during a collision. Suppose that b is replaced by a new variable a , defined by the relation

$$a = b \left[\frac{v^2}{(m_1 + m_2) K} \right]^{1/s-1},$$

which is just the scattering angle parameter raised to the $[1/(s-1)]$ power, except for a numerical factor. Then the quantity $V b d b$ appearing in the collision integral ΔQ [Eq. (33)] is replaced by

$$[(m_1 + m_2) K]^{2/(s-1)} V^{1-4/s-1} a d a.$$

For $s = 5$, V disappears, and

$$\Delta Q = \sqrt{(m_1 + m_2) K} \iint f f_1 J d\vec{\xi} d\vec{\xi}_1, \quad (46)$$

where

$$J = \int_0^\infty \int_0^{2\pi} (Q' - Q) a d a d \xi. \quad (47)$$

In other words for $s = 5$ the collision integral ΔQ is given by the value of J averaged over the velocity space of both the probe and colliding particles, where J depends only on the binary collision process and not on the velocity distribution function itself. The value of ΔQ is proportional to the average value of Q itself, and for the lower moments ΔQ contains the components of the shear stress tensor and heat flux vector.

Since Q is always some function of the particle velocity and its components the change in the velocity components of the particles during a collision must be evaluated before the quantity J (and ΔQ) can be computed. Following Maxwell^{4b} and Jeans^{2c} we construct a plane parallel to NP , the original direction of the colliding particle relative to the probe particle at O (Figure 4), and containing the (arbitrary) x -axis. The intersection of this plane with the plane through $O \perp NP$ gives the direction OR , from which the angle ξ is now measured. The vectors OG and OG' represent the relative velocity between the particle before and after the encounter, respectively; of course, if OG is directed inward then OG' is directed outward, and vice versa. The angle θ' between these two vectors denotes the scattering angle or total angle of deflection in the encounter.

By means of vector geometry (Appendix B) one finds that

$$(\xi' - \xi_1')_k = (\xi_1 - \xi)_k \cos \theta' + \sqrt{V^2 - (\xi_1 - \xi)_k^2} \sin \theta' \cos (\xi + \omega_{kx}), \quad (48)$$

where ξ_k and $(\xi_1)_k$ are components of the particle velocities in the k -direction before collision, and ξ'_k and $(\xi_1')_k$ are these components after collision. Here ω_{kx} is the angle between the plane containing OG and the k -axis, and the plane containing OG and the x -axis; thus,

$$\omega_{xx} = 0; \cos \omega_{kx} = - \frac{[(\xi_1')_k - \xi_k] [\xi_{1x} - \xi_x]}{\sqrt{V^2 - (\xi_1 - \xi)_k^2} \sqrt{V^2 - (\xi_1 - \xi)_x^2}}, \quad (49)$$

where $k = y, z$. By combining these geometric relations with the statement expressing conservation of linear momentum, i. e.,

$$m_1 \vec{\xi}' + m_2 \vec{\xi}_1' = m_1 \vec{\xi} + m_2 \vec{\xi}_1,$$

the quantity ξ_{1k}' can be eliminated, and one obtains

$$\xi_k' = \xi_k + \frac{m_2}{m_1 + m_2} \left[2(\xi_1 - \xi)_k \cos^2 (\theta'/2) + \sqrt{V^2 - (\xi_1 - \xi)_k^2} \sin \theta' \cos (\xi + \omega_{kx}) \right].$$

For a pure gas ($m_1 = m_2$) ΔQ vanishes when Q is one of the collisional invariants of mass, momentum, or energy (Section III. 2.). The next higher moment of interest is the momentum flux, $Q = m \xi_j \xi_k$. In evaluating J [Eq. (47)], it is helpful to recognize that all terms in the expression for $(Q' - Q)$ that are proportional to $\cos m\xi$ or $\sin m\xi$ ($m \neq 0$) integrate out to zero. With these terms discarded,

$$\frac{Q' - Q}{m} = \xi_j' \xi_k' - \xi_j \xi_k = [\xi_j (\xi_{k_1} - \xi_k) + \xi_k (\xi_{j_1} - \xi_j)] \cos^2 (\theta'/2)$$

$$+ (\xi_{j_1} - \xi_j)(\xi_{k_1} - \xi_k) \cos^4 (\theta'/2) - \frac{1}{4} (\xi_{j_1} - \xi_j)(\xi_{k_1} - \xi_k) \sin^2 \theta' \cos^2 \xi,$$

and

$$J = 2\pi (\xi_{j_1} \xi_{k_1} - \xi_j \xi_k) \int_0^\infty \cos^2 (\theta'/2) a da - [(3\pi)/4] [(\xi_{j_1} - \xi_j)(\xi_{k_1} - \xi_k)] \int_0^\infty \sin^2 \theta' da. \quad (50)$$

The first term in the expression for J will give no net contribution to ΔQ , because the average value of $\xi_{j_1} \xi_{k_1}$ is identical with the average value of

$\xi_j \xi_k$. In the second term clearly $(\xi_{j_1} - \xi_j)$ and $(\xi_{k_1} - \xi_k)$ can be replaced by $(c_{j_1} - c_j)$ and $(c_{k_1} - c_k)$, where $\vec{c} = \vec{\xi} - \vec{u}$, the intrinsic velocity. Finally,

$$\Delta Q = -3/4 A_2 \sqrt{2mK} m \iiint [(c_j c_k)_1 + c_j c_k] f f_1 d\vec{\xi} d\vec{\xi}_1, \quad (51)$$

where $A_2 = \pi \int_0^\infty \sin^2 \theta' a da$. [Maxwell^{4b} found that $A_2 = 1.3682$.]

But according to the definition of P_{jk} [Eq. (45c)],

$$\Delta Q = (3/2 A_2 \sqrt{2mK}) n P_{jk}.$$

Now Maxwell^{4b} had already shown that the ordinary coefficient of viscosity based on the local full-range Maxwellian velocity distribution is given by the expression

$$\mu = \frac{kT}{3/2 A_2 \sqrt{2mK}},$$

where k is the Boltzmann constant. Therefore, with $Q = m \xi_j \xi_k$ and $s = 5$, $\Delta Q = (p/\mu) P_{jk}$, regardless of the form of the velocity distribution function.* Maxwell^{4b} also showed that $p/\mu = \pi/4 (1/\tau_f)$, where τ_f is the mean free time, and this relation will be useful later (Section IV.1.).

Similarly, by taking $Q = m \xi_j (\xi^2/2)$ one finds that

$$\Delta Q = \frac{-p}{(k/c_p)} \dot{q}_j + \sum_k (p/\mu) P_{jk} u_k = (p/\mu) \left[-(2/3) \dot{q}_j + \sum_k P_{jk} u_k \right]. \quad ** \quad (52)$$

In a rectangular Cartesian coordinate system with no external forces the Maxwell integral equations of transfer for the first nine moments beyond the collisional invariants take the following form for Maxwell particles:***

* By definition $p = nkT$; note that $P_{ii} = -p + p_{ii} \neq p$, in general.

** The expression for $\Delta Q = \Delta (m \xi_j \cdot \xi^2/2)$ given by Jeans [Reference 2d, page 241, Eq. (667)] contains numerical errors. The correct expression reads as follows:

$$\Delta Q = \frac{1}{2} A_2 \sqrt{2mK} m \left[u_j \overline{c^2} - 3 \sum_k (u_j \overline{c_j c_k}) - \overline{c_j^2 c^2} \right],$$

from which one obtains the result given by Eq. (52). Here $(c_p \mu)/k = 2/3$ for a monatomic gas.

*** The equations for an orthogonal curvilinear coordinate system with external forces are given in Appendix A.

$$(\partial/\partial t)(\int m \xi_j \xi_k f d\vec{\xi}) + \sum_i (\partial/\partial x_i)(\int m \xi_i \xi_j \xi_k f d\vec{\xi}) = (p/\mu) p_{jk} \quad (53)$$

$$(\partial/\partial t)(\int m \xi_j \frac{\xi^2}{2} f d\vec{\xi}) + \sum_i (\partial/\partial x_i)(\int m \xi_i \xi_j \frac{\xi^2}{2} f d\vec{\xi}) = (p/\mu) \left[-\frac{2}{3} \dot{q}_j + \sum_k p_{jk} u_k \right]. \quad (54)$$

Thus the collision integrals are evaluated once and for all without the necessity for any further approximation, and p_{jk} and \dot{q}_j can always be expressed in terms of the arbitrary functions $\vec{u}_1, \vec{u}_2, n_{1,2}, T_{1,2}$ [Eqs. (34a) and (34b)].

III. 4. Boundary Conditions

When atoms or molecules impinge on a solid surface a complex interaction takes place which is only poorly understood at present. Particularly in the range of incident particle energies of 1/10 to 10 electron volts, we have not progressed very far beyond Maxwell's original rough suggestion³¹ that the emitted beam consists partly of specularly reflected and partly of diffusely reemitted particles. Following Maxwell, it is customary to define a tangential momentum exchange coefficient, σ , by the relation

$$\sigma = \frac{\tau_i - \tau_R}{\tau_i - \tau_w} = \frac{\tau_i - \tau_R}{\tau_i},$$

where τ_i and τ_R are the incident and reflected tangential momenta, respectively, and τ_w is the tangential momentum of diffusely reemitted particles having a Maxwellian velocity distribution corresponding to the temperature of the solid surface; by symmetry $\tau_w = 0$. Smoluchowski³² extended Maxwell's conception to include an energy accommodation coefficient, α , defined by the relation

$$\alpha = \frac{E_i - E_R}{E_i - E_w},$$

and Schaaf³³ and Bell introduced a normal momentum reflection coefficient σ' , where

$$\sigma' = \frac{P_i - P_R}{P_i - P_w}.$$

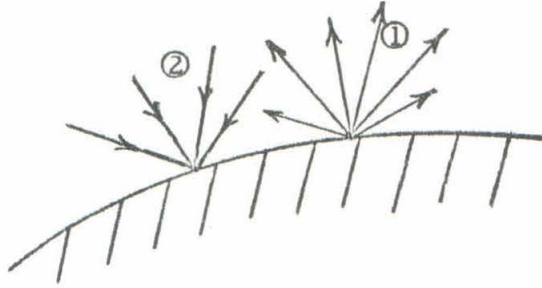
As suggested by Hurlbut³⁴, for most "engineering surfaces" the random orientation of the tilt planes of the surface material on the atomic scale insures that $\sigma \approx 1$, regardless of the details of the surface interaction

process itself. But experimental values of α (and presumably of σ') vary from a few hundredths to nearly unity, depending on the incident gas, the surface material, surface temperature, presence of an adsorbed gas layer, etc. Recently Schamberg³⁵ introduced a more rational description of the surface interaction process, in which the three arbitrary coefficients σ , σ' , and α are replaced by three physical interaction parameters: (1) the angular half-width of a reemitted conical beam of particles; (2) the inclination of the axis of the reemitted beam as a function of the angle of impingement of the incident beam; (3) the speed or kinetic temperature of the reflected particles.* On the basis of low energy molecular beam experiments³⁴ Schamberg assumes that the distribution of reemitted particle number density across the beam obeys a cosine law, which would be correct for purely diffuse scattering. He finds that the beam width plays only a minor role. In fact the most probable situation is diffuse reemission with α and $\sigma' \neq 1$, i. e., the axis of the reemitted beam is normal to the surface and the beam half-width is 90° , but the diffusely reemitted particles have a kinetic temperature lying between the surface temperature and the kinetic temperature of the incident particles.

Because of the scarcity of experimental data at the present time it does not seem worthwhile to enter into a detailed discussion of the possible values of α and σ' . The two-stream Maxwellian velocity distribution function introduced in Section III. 2. is well suited to a description of the surface interaction process, and the results of new experiments can be incorporated into the analysis as they become available. In fact this consideration was one important reason for selecting the two-stream Maxwellian in the first place. In order to illustrate the types of boundary conditions encountered in the present approach two special models are discussed here: (1) diffuse reemission ($\sigma = 1$) with $\alpha = \sigma' = 1$ (2) diffuse reemission according to the Maxwell-Smoluchowski model, with α and $\sigma' \neq 1$.

* Schamberg developed his analysis first for hyperthermal incident velocity, for which the thermal motion can be neglected, but he subsequently extended it to include lower speeds³⁶, at least approximately. Schamberg takes the speed (or kinetic temperature) of the reemitted particles to be independent of direction, but as we shall see later, (page 36) it seems more logical to associate this speed with the normal component of the velocity of the reemitted particles, at least for diffuse reemission ($\sigma = 1$; α , $\sigma' \neq 1$).

At a convex solid surface the incident particles all belong to group 2, and the reemitted particles belong to group 1 (Figure 2 and sketch below).*



For diffuse reemission with $\alpha = \sigma' = 1$ the reemitted particles have a Maxwellian velocity distribution corresponding to T_w , by definition, and the mean velocity of the reemitted particles is identical with the local surface velocity, i. e., $\vec{u}_1(\vec{R}, t) = \vec{u}_w$ and $T_1(\vec{R}, t) = T_w$ when $\vec{R} = \vec{R}_w$. When there is no net mass transfer at the surface an additional boundary condition must be satisfied which is similar to the usual free-molecule flow condition, except that now $\vec{u}_2 \neq \vec{u}_\infty$ in general:

$$n_1 \sqrt{\mathcal{R} T_1} = n_1 \sqrt{\mathcal{R} T_w} = n_2 \sqrt{\mathcal{R} T_2} C(-s_{2n}), \quad (55)$$

where

$$C(s_n) = e^{-s_n^2} + \sqrt{\pi} s_n (1 + \operatorname{erf} s_n). \quad (56a)$$

Here

$$s_{2n} = \frac{(u_{2n} - u_{wn})}{\sqrt{2 \mathcal{R} T_2}},$$

where u_{2n} and u_{wn} are the normal components of \vec{u}_2 and \vec{u}_w , respectively.

In the limiting case $s_{2n} \ll 1$, Eq. (56a) becomes

$$n_1 \sqrt{\mathcal{R} T_w} = n_2 \sqrt{\mathcal{R} T_2} [1 - \sqrt{\pi} s_{2n} + \dots], \quad (56b)$$

while in the case of hyperthermal velocity ($s_{2n} \gg 1$),

* For a concave surface the reemitted particles all belong to group 1, but the distribution of incident particles depends upon the particular geometry of the problem.

$$n_1 \sqrt{RT_w} = \sqrt{2\pi} n_2 (u_{wn} - u_{2n}) , \quad (56c)$$

or

$$n_1 = \sqrt{2\pi\gamma} n_2 \frac{(u_{wn} - u_{2n})}{\sqrt{\gamma RT_w}} . \quad (56d)$$

In considering the uniform rectilinear motion of a finite body in a fluid of infinite extent the following boundary conditions must also be imposed (in body coordinates):

$$\vec{u}_2 \rightarrow \vec{u}_\infty , \quad T_2 \rightarrow T_\infty , \quad \text{and} \quad n_2 \rightarrow n_\infty \quad \text{as} \quad x \rightarrow -\infty .$$

For axially-symmetric or two-dimensional flows (for example) the values of three of the unknown functions (\vec{u}_1 and T_1) are determined at the body surface, and Eq. (55) furnishes a fourth relation. If two functions are chosen in advance (say $\vec{u}_2 = \vec{u}_\infty$), then the values of n_2 and T_2 on the surface are the integration parameters, when the integration is carried outward from the body surface. The situation is somewhat simpler in the case of bounded flows. For plane (or cylindrical) Couette flow, for example, the boundary conditions are as follows (Figure 3):

$$\begin{aligned} u_1(y) &= -(U/2), \quad T_1(y) = T_\ell \quad \text{at} \quad y = -b/2 \\ u_2(y) &= +(U/2), \quad T_2(y) = T_u \quad \text{at} \quad y = +b/2 \\ v(y) &= 0 \quad \text{or} \quad n_1 \sqrt{T_1} = n_2 \sqrt{T_2} , \quad \text{all } y. \end{aligned} \quad (57)$$

A sixth relation connecting the six unknown functions is introduced by specifying the density level; for example, by selecting the value of n_1 at $y = -b/2$.

When $\sigma = 1$ but α and $\sigma' \neq 1$ the boundary conditions are more complicated. Of course the mean tangential velocity of the reemitted particles is equal to the tangential velocity of the surface, but in general the normal component of the mean velocity of the reemitted particles is not equal to the normal velocity of the surface. The condition of zero net mass transfer at the surface leads to the relation

$$n_1 \sqrt{RT_1} \quad C(s_{1n}) = n_2 \sqrt{RT_2} \quad C(-s_{2n}) \quad (58)$$

where $C(s_n)$ is defined by Eq. (56a), and $s_{1n} = \frac{u_{1n} - u_{wn}}{\sqrt{2RT_1}}$.

By computing the incident and reemitted translational kinetic energies and normal momenta with the aid of the two-stream Maxwellian, and utilizing the definitions of α and σ' , two additional boundary conditions

are obtained, as follows:

Energy Accommodation

$$(1 - \alpha) T_2 \mathcal{E}(-s_{2n}, |s_{2n}|) + 2\alpha T_w = T_1 \mathcal{E}(s_{1n}), \quad (59a)$$

where

$$\mathcal{E}(s_n, |s|) = \frac{E(s_n, |s|)}{C(s_n)}, \quad (59b)$$

and

$$E(s_n, |s|) = e^{-s_n^2} (2 + s^2) + \sqrt{\pi} s_n (5/2 + s^2)(1 + \operatorname{erf} s_n). \quad (60)$$

Normal Momentum Accommodation

$$(1 - \sigma') \sqrt{T_2} \mathcal{P}(-s_{2n}) + \frac{\sigma' \sqrt{\pi}}{2} \sqrt{T_w} = \sqrt{T_1} \mathcal{P}(s_{1n}), \quad (61a)$$

where

$$\mathcal{P}(s_n) = \frac{P(s_n)}{C(s_n)} \quad (61b)$$

and

$$P(s_n) = \left[s_n e^{-s_n^2} + \sqrt{\pi} \left(\frac{1}{2} + s_n^2 \right) (1 + \operatorname{erf} s_n) \right]. \quad (62)$$

The functions $E(s_n, |s|)$ and $P(s_n)$ are similar to the usual free molecule expressions for energy and normal momentum flux³⁷, except that

$$s \neq \frac{u_\infty}{\sqrt{2RT_\infty}},$$

in general. In addition to the conditions given by Eqs. (58), (59a), and (61a), we have $u_{1t} = u_{wt}$ (for $\sigma = 1$), where "t" is the direction tangential to the surface.

Two limiting cases are of particular interest: (1) low Mach number flow with small temperature differences, i.e., $s_2 \ll 1$ and $T_w = T_2(1 + \mathcal{E})$, with $\mathcal{E} \ll 1$; (2) hyperthermal flow, or $s_2 < 0$ and $|s_2| \gg 1$. In the first case, Eq. (59a) takes the limiting form

$$2T_w [1 + \mathcal{E}(1 - \alpha) + \dots] = T_1 \mathcal{E}(s_{1n}), \quad (63)$$

while Eq. (61a) becomes

$$\sqrt{T_w} [1 + \mathcal{E}(1 - \sigma') + \dots] = \sqrt{T_1} \cdot 2/\sqrt{\pi} \mathcal{P}(s_{1n}). \quad (64)$$

If $\sigma' = \alpha$ then $s_1^n = 0$, but not otherwise. In other words, specifying the surface interaction process by means of a single parameter $\alpha = \sigma'$ is consistent in this limiting case with the statement that $u_{1n} = u_{wn}$. The boundary conditions here are

$$(1 - \alpha) T_2 + \alpha T_w = T_1 \quad (65)$$

$$\vec{u}_1 = \vec{u}_w \quad (66)$$

In the opposite limiting case when $s_2 < 0$ and $|s_2| \gg 1$, Eqs. (58), (59a), and (61a) yield the relation

$$\sigma' = \frac{\sin \theta_2 - \sqrt{(1 - \alpha) + \alpha \frac{4 R T_w}{(u_w - u_2)^2}}}{\sin \theta_2 - \frac{\sqrt{\pi}}{4} \sqrt{\frac{4 R T_w}{(u_w - u_2)^2}}}, \quad (67)$$

where θ_2 is the angle of inclination of the velocity vector \vec{u}_2 with respect to the surface.* In particular, if

$$\frac{4 R T_w}{(u_w - u_2)^2} \ll 1, \text{ then } \sigma' = 1 - \frac{\sqrt{1 - \alpha}}{\sin \theta_2}.$$

Thus σ' depends upon α , the ratio of the local surface temperature to the kinetic energy $\frac{(u_w - u_2)^2}{2}$, and the local inclination of \vec{u}_2 with respect to the surface. The relation between σ' and α given by Eq. (67) is similar to the result obtained by Schamberg³⁵ for hyperthermal free-molecule flow, except for certain numerical factors in the second terms of both numerator and denominator. These differences arise because Schamberg takes the relative reemission speed V_R as constant for all angles of reemission, whereas in the present analysis this speed is associated with the normal component of the relative velocity.

* Actually this limiting case is specified more precisely by the conditions $(1 - \alpha) s_2^2 \gg 1$ and $(1 - \sigma') s_{2n} \gg 1$.

IV. ILLUSTRATIVE EXAMPLE: PLANE PARALLEL FLOW AT LOW MACH NUMBER WITH SMALL TEMPERATURE DIFFERENCES

IV. 1. General Considerations, Differential Equations, and Boundary Conditions

Although the present integral method is designed mainly to deal with the non-linear aspects of rarefied gas flows, it seems desirable to explore the method first for some linear flow problems. For this purpose plane parallel flow at low Mach number with small temperature differences provides the simplest example. Solutions obtained by the integral method can be compared with studies of this class of flow problems carried out by means of some of the other techniques reviewed in Section III. 1. Most of the simplifications in this limiting case apply equally well to steady, plane Couette flow and to Rayleigh's problem of the non-steady flow generated by the impulsive start of an infinite flat plate set into uniform motion in its own plane (Figure 3). For this reason the differential equations and boundary conditions for the two problems are formulated jointly.

The requirement that $M^2 \ll 1$ is equivalent to the statement that the kinetic energy of mean motion is negligibly small compared to the static enthalpy. Even if all of this kinetic energy is converted into random particle motion, the increase in gas temperature thus produced is insignificant. If there are no internal heat sources, and if the temperature of the plate (or plates) differs only slightly from ambient temperatures, then the plane shear flow and the temperature field are completely independent of one another. This "splitting" between the velocity and temperature fields is well-known for the Navier-Stokes equations. It occurs also for the Grad equations^{27, 28}, and is easily shown to apply also to the present transport integral method.* In this paper, only the isothermal shear flow is considered; the treatment of the temperature field is entirely similar.*

Since $T = \text{const.}$, the only self-consistent choice of the temperature functions appearing in the two stream Maxwellian equations [Eqs. (35a) and (35b)] is that

* For details see the forthcoming paper on steady, compressible plane Couette flow by C. Liu and the present author.

$$T_1(y, t) = T_2(y, t) = T_0 \quad (68)$$

where T_0 is the ambient temperature. All physical quantities are independent of x , and to this approximation the equation of continuity [Eq. (36)] states that

$$\partial v / \partial y = 0 \quad , \quad (69)$$

for both steady and non-steady flow. But $v(y, t) = 0$ at a solid surface, so that

$$v(y, t) \equiv 0 \quad . \quad (70)$$

In this case the y -momentum equation [Eq. (38b)] yields

$$0 = \partial P_{yy} / \partial y \quad , \quad \text{or} \quad P_{yy} = \text{const.} \quad (71)$$

By definition, $P_{yy} = -m \int f (\xi_y - v)^2 d\vec{\xi}$, and $\rho = nm = \int f d\vec{\xi}$.

By carrying out the integrations for the two-stream Maxwellian [Eqs. (35a) and (35b)] one obtains

$$n(y, t) = \frac{1}{2} [n_1(y, t) + n_2(y, t)] \quad , \quad (72)$$

and

$$P_{yy} = -k/2 (n_1 T_1 + n_2 T_2) = -n k T = -p \quad . \quad (73)$$

In other words,

$$p = \text{const.} = p_0 \quad , \quad \text{and} \quad n = \text{const.} \quad , \quad \text{or} \quad \rho = \text{const.} = \rho_0 \quad , \quad (74)$$

as expected. Now

$$\rho v = m \left[\int_{-\infty}^{+\infty} \int_0^{\infty} \int_{-\infty}^{+\infty} f_1 \xi_y d\xi_x d\xi_y d\xi_z + \int_{-\infty}^{+\infty} \int_{-\infty}^0 \int_{-\infty}^{+\infty} f_2 \xi_y d\xi_x d\xi_y d\xi_z \right] ,$$

or

$$\rho v = \sqrt{mk/2\pi} \left[n_1 \sqrt{T_1} - n_2 \sqrt{T_2} \right] . \quad (75)$$

Therefore, Eqs. (68) and (70) through (75) lead to the conclusion that

$$n_1 = n_2 = n (\text{const.}) .$$

The two remaining unknown functions $u_1(y, t)$ and $u_2(y, t)$ determine the shear stress and x -component of flow velocity completely. By definition

$$\rho u = m \int f \xi_x d\vec{\xi} = \frac{1}{2} m [n_1 u_1(y, t) + n_2 u_2(y, t)] \quad , \quad (76a)$$

for the two-stream Maxwellian. In the present limiting case

$$u(y, t) = \frac{1}{2} [u_1(y, t) + u_2(y, t)] \quad . \quad (76b)$$

Also,

$$p_{xy} = -m \int f (\xi_x - u) (\xi_y - v) d\vec{\xi} = -m \int f \xi_x \xi_y d\vec{\xi} + \rho u v, \quad (77)$$

and the two-stream Maxwellian yields the following relation ($v = 0$):

$$p_{xy}(y, t) = \rho_0 \sqrt{RT_0/2\pi} [u_2(y, t) - u_1(y, t)] \quad (78)$$

In order to determine the shear flow we must now apply the x-momentum equation and one additional moment beyond the collisional invariants (Section III. 2.). The most natural choice is $Q = m \xi_x \xi_y$, which leads directly to an expression for the shear stress p_{xy} . The x-momentum equation is

$$\rho_0 (\partial u / \partial t) = (\partial p_{xy} / \partial y), \quad (79)$$

while the appropriate transport integral equation is given by Eq. (53) (Section III. 3.) :

$$\partial / \partial t \left(\int m \xi_x \xi_y f d\vec{\xi} \right) + \partial / \partial y \left(\int m \xi_x \xi_y^2 f d\vec{\xi} \right) = (p_0 / \mu_0) p_{xy} \quad (80)$$

According to Eq. (77), when $v \equiv 0$ the first term on the left-hand side of Eq. (80) is simply $(- \partial p_{xy} / \partial t)$. The second integral is evaluated as follows:

$$\begin{aligned} \int f \xi_x \xi_y^2 d\vec{\xi} &= \int_{-\infty}^{+\infty} \int_0^{\infty} \int_{-\infty}^{+\infty} f_1 [(\xi_x - u_1) + u_1] \xi_y^2 d\xi_x d\xi_y d\xi_z \\ &+ \int_{-\infty}^{+\infty} \int_{-\infty}^0 \int_{-\infty}^{+\infty} f_2 [(\xi_x - u_2) + u_2] \xi_y^2 d\xi_x d\xi_y d\xi_z. \end{aligned} \quad (81)$$

Since f_1 and f_2 are even functions of $(\xi_x - u_1)$ and $(\xi_x - u_2)$, respectively, the integral is reduced to the form

$$m \int f \xi_x \xi_y^2 d\vec{\xi} = u_1 \int_{-\infty}^{+\infty} \int_0^{\infty} \int_{-\infty}^{+\infty} m f_1 \xi_y^2 d\xi_x d\xi_y d\xi_z + u_2 \int_{-\infty}^{+\infty} \int_{-\infty}^0 \int_{-\infty}^{+\infty} m f_2 \xi_y^2 d\xi_x d\xi_y d\xi_z \quad (82)$$

But f_1 and f_2 are even functions of ξ_y , and each integral on the right-hand side of Eq. (82) contributes exactly $p_0/2$ in this case. Thus

$$m \int f \xi_x \xi_y^2 d\vec{\xi} = p_0/2 (u_1 + u_2) = p_0 u(y),$$

and Eq. (80) becomes

$$- \partial p_{xy} / \partial t + p_0 (\partial u / \partial y) = (p_0 / \mu_0) p_{xy} \quad (83)$$

In this limiting case the functions u_1 and u_2 can be eliminated in favor of the macroscopic quantities p_{xy} and u ; in general the situation is not quite so simple.

Several interesting conclusions can be drawn from Eq. (83). If a uniform shear stress distribution is applied at time $t = 0$ to a fluid at rest (or in uniform motion) then

$$p_{xy} \sim e^{-(p_0/\mu_0)t} = e^{-(\pi/4)(t/\tau_f)},$$

where τ_f is the mean free time between collisions. In other words, τ_f plays the role of a relaxation time, as shown by Maxwell^{4b}. In a non-steady flow such as Rayleigh's problem the shear stress depends not only on the local velocity gradient, but also on the time history of the shear stress itself through the term $-(\partial p_{xy}/\partial t)$. Roughly, the region of influence extends over a time interval of order τ_f . Thus when $t \gg \tau_f$, $p_{xy} \approx \mu(\partial u/\partial y)$, and the gas behaves like a Newtonian fluid, but when $t/\tau_f = O(1)$ the situation is entirely different. In the Chapman-Enskog⁵ expansion procedure (Section III. 1.) f is replaced by the local full-range single Maxwellian f_0 on the left-hand side of Eq. (80) in first approximation, and the term $-(\partial p_{xy}/\partial t)$ is lost. In the more general case spatial as well as time derivatives of various moments will appear in Eqs. (53) and (54), and the shear stress and heat flux quantities at a location (\vec{R}_0, t_0) are influenced by the behavior of these moments in a region of space and time defined by $|R - R_0| \sim \bar{c} \tau_f = \lambda$, $|t - t_0| \sim \tau_f$.

For steady plane Couette flow, Eq. (83) states that $p_{xy} = \mu(\partial u/\partial y)$ over the whole range of gas density from free-molecule flow to the Navier-Stokes limit. Of course this result is a direct consequence of the restriction to low Mach number isothermal flow, and the choice of the two-stream Maxwellian. In general the two-stream Maxwellian does not yield the Navier-Stokes relation even for this simple geometry. According to Eq. (79), $p_{xy} = \text{const.}$, and therefore the velocity profile is given by

$$u = (p_{xy}/\mu) y, \quad (84)$$

for all gas densities. Identical results were obtained by Yang²⁷ and the present author by applying the linearized Grad equations to this problem, because the terms proportional to p_{xy} and \dot{q}_y in Grad's distribution function (Section III. 2.) make no contribution to the integral in Eq. (80) in this case, and all the other terms vanish.

In order to simplify the boundary conditions only the case of completely diffuse remission is considered here. At the upper plate in Couette flow (for example) $u_2 = U/2$. By Eq. (76b), $u_1 = 2u - (U/2)$ at $y = b/2$, so that Eq. (78) becomes

$$p_{xy} = p_0 \sqrt{(2RT_0)/\pi} \left[(U/2) - u \right] \quad (85)$$

By symmetry $u(-b/2) = -u(b/2)$ and the boundary condition at the lower plate is also satisfied.* Similarly, in Rayleigh's problem,

$$p_{xy}(0, t) = p_0 \sqrt{(2RT_0)/\pi} \left[u(0, t) - U \right], \quad (86)$$

and p_{xy} and $u \rightarrow 0$ as $y \rightarrow \infty$. The condition $v = 0$ at the plate surface is already satisfied [Eq. (75)], because $v \equiv 0$.

IV. 2. Steady, Plane Couette Flow

The differential equations and boundary conditions obtained by the present transport integral method are identical with those derived in Reference 27 from the linearized Grad equations (Section IV. 1.), and the expressions for shear stress and velocity are naturally also identical. In fact, as shown in that paper, the same results are obtained by employing the Maxwell "velocity slip" relation and the Navier-Stokes equations, as suggested originally by R. A. Millikan³⁸. The main difference is that velocity slip is not introduced as an ad hoc assumption at the boundaries, as in the usual treatment, but follows directly from the statistical averaging process [Eqs. (76a) and (76b)]. In this simple case velocity slip is obtained regardless of the nature of the solid boundary; only the magnitude is influenced by the reemission process.

It is instructive to examine the behavior of the functions $u_1(y)$ and $u_2(y)$ and the two-stream Maxwellian in this simple case. By employing Eqs. (84) and (85) at $y = b/2$, one obtains

$$1/C_D M = \sqrt{(\pi\gamma)/2} + \frac{1}{2} (Re/M), \quad (87)$$

where

$$C_D = p_{xy}/(\frac{1}{2}p_0 U^2), \quad Re = (\rho_0 U b)/(\mu_0), \quad M = U/\sqrt{\gamma RT_0}$$

* It should be emphasized that this particular relation between p_{xy} and u applies only at the plate surfaces.

Thus the drag coefficient has the correct free-molecule flow value in the limit $Re/M \rightarrow 0$, and the correct Navier-Stokes behavior ($C_D = 2/Re$) in the limit $Re/M \rightarrow \infty$. By Eq. (85),

$$u(y)/(\bar{U}/2) = 1/(1+\beta) \cdot y/(b/2) \quad (88a)$$

where

$$\beta^{-1} = 1/\sqrt{2\pi\gamma} \cdot Re/M \quad (88b)$$

By utilizing Eqs. (76b) and (78), one finds that

$$\left. \begin{array}{l} u_1(y) \\ \frac{u_1(y)}{\bar{U}/2} \\ u_2(y) \\ \frac{u_2(y)}{\bar{U}/2} \end{array} \right\} = 1/(1+\beta) \left[y/(b/2) \mp \beta \right] \quad (89)$$

The mean velocity $u(y)$ and the two functions $u_1(y)$ and $u_2(y)$ are sketched in Figure 5. Dr. Hans W. Liepmann pointed out to the author that the mean profile in this case can be regarded as being composed of one velocity distribution that exhibits "velocity slip" at the lower plate but not at the upper plate, and a second distribution symmetrically displaced that "slips" at the upper plate but not at the lower plate. In the free-molecule flow

$$\text{limit } (Re/M \rightarrow 0) \frac{u_1(y)}{\bar{U}/2} \rightarrow -1, \frac{u_2(y)}{\bar{U}/2} \rightarrow +1, \text{ and } u(y) \rightarrow 0;$$

of course f_1 and f_2 are precisely the free-molecular velocity distribution functions in this limiting case.

In order to bring out the character of the solution in the Navier-Stokes limit ($Re/M \gg 1$) more clearly, we replace Re/M by the ratio of mean free path, λ_0 , to plate spacing b . According to the relation

$$p_0/\mu_0 = (\pi/4)(1/\tau_f) = (\pi/4)(\bar{c}_0/\lambda_0),$$

the viscosity is given by $\mu_0 = \frac{1}{2} p_0 \bar{c}_0 \lambda_0 = \lambda p_0 \sqrt{(2/\pi) R T_0}$,

where $\bar{c}_0 = \sqrt{(8/\pi) R T_0}$. Thus the quantity β appearing in Eqs. (88a) and (89) is equal to $\lambda/(b/2)$, and the expression for C_D Eq. (87) can be rewritten in the form

$$\frac{C_D M}{(C_D M)_{F-M}} = p_{xy}/(p_{xy})_{F-M} = \frac{1}{1 + (b/2\lambda)} \quad (90)$$

where the subscript F-M denotes free-molecule flow values. From these results one sees that

$$\left. \begin{array}{l} \frac{u_1(y)}{U/2} \\ \frac{u_2(y)}{U/2} \end{array} \right\} = u \mp \lambda (\partial u / \partial y) .$$

In the Navier-Stokes regime $\lambda \ll b/2$ and $\lambda (\partial u / \partial y) \ll U/2$, so that f_1 and f_2 can be regarded as the local Maxwellians for particles that cross the plane at y from below and above, respectively, and retain the properties of their original layers located a distance of one mean free path away from this plane. The boundary condition at the plate surface [Eq. (85)] is exactly

$$U/2 - u(b/2) = \lambda (\partial u / \partial y)_{y=b/2} ,$$

as proposed by Maxwell³¹. Although formally correct for all values of λ/b , this description is meaningless (and unnecessary) when $\lambda/b = 0(1)$, or larger. A comparison between these simple results and the solutions obtained by other methods is presented in Section V.1.

IV.3. Rayleigh's Problem

By eliminating the flow velocity between Eqs. (79) and (83) one obtains

$$(\partial^2 p_{xy}) / (\partial y^2) - (1/a_I^2) \cdot (\partial^2 p_{xy}) / \partial t^2 = (1/\rho_0) \cdot (\partial p_{xy} / \partial t) \quad (91)$$

— the Telegrapher's Equation. [Here a_I is the isothermal speed of sound given by $a_I = \sqrt{RT_0}$. According to this simple model, the impulsive motion of the plate generates a shear stress-velocity wave of decaying amplitude propagating into the gas. Some distance behind the wave front, and especially near the plate surface, the motion eventually takes on the diffusive character of Rayleigh's classical solution. Yang²⁸ and the present author obtained a qualitatively similar behavior by applying the linearized Grad equations to this problem, but the equations were much more complicated because of the involved character of Grad's distribution function.

By simple momentum considerations the discontinuities in p_{xy} and u across the initial wave front (Figure 6) are connected by the relation

$$\Delta p_{xy} = -\rho_0 a_I \Delta u . \quad (92)$$

Since $p_{xy} = u = 0$ ahead of the front,

$$p_{xy} = -\rho_0 a_I u \quad , \quad (93a)$$

at any point located immediately behind the wave, so that

$$d p_{xy} = -\rho_0 a_I du \quad (93b)$$

along the front. On the other hand, the method of characteristics applied to Eq. (91) [or to Eqs. (79) and (83)] yields the following relation along the wave

$$d p_{xy} - \rho_0 a_I du + (p_0/\mu_0) p_{xy} dt = 0 \quad . \quad (94)$$

By combining Eqs. (93b) and (94) and integrating, one finds that the decay of p_{xy} and u along the wave front is given by

$$f(t) = f(0) e^{-\frac{1}{2}(p_0/\mu_0)t} = f(0) e^{-(\pi/8)(t/\tau_f)} \quad ,$$

where $f(t)$ is either shear stress or velocity. Evidently p_{xy} and u decrease only slightly from their initial values up to $t/\tau_f = 1$, but the decay is quite rapid once t exceeds τ_f , and is much faster than the diffusive $1/\sqrt{t}$ behavior for p_{xy} expected near the plate surface when $t/\tau_f \gg 1$.

At time $t = 0$ the wave has just started propagating outward from the plate surface, so that $p_{xy}(0)$ and $u(0) = -[p_{xy}(0)]/(\rho_0 a_I)$ must satisfy the boundary condition given by Eq. (86) (Section IV. 1.) . By utilizing this condition one obtains

$$p_{xy}(0) = -\rho_0 U \sqrt{R T_0 / 2\pi} \left(\frac{2}{1 + \sqrt{2/\pi}} \right) \quad ,$$

or

$$M c_f(0) = -0.685 \quad \text{for } \gamma = 5/3 \quad , \quad \text{where } c_f = p_{xy}/(\frac{1}{2} \rho_0 U^2) \quad ; \quad *$$

$$u(0)/U = \frac{1}{1 + \sqrt{\pi/2}} = 0.45 \quad . \quad *$$

Thus along the wave front $M c_f(t) = -0.685 e^{-0.39(t/\tau_f)}$ and

$$u(t)/U = 0.45 e^{-0.39(t/\tau_f)} \quad .$$

Once the behavior of p_{xy} and u along the wave front is known, the entire flow field can be calculated by the method of characteristics

* These values for $M c_f(0)$ and $u(0)/U$ differ slightly from the free-molecule flow values of 0.615 and $\frac{1}{2}$, respectively, for a plate at ambient temperature. (See Section V. 2.)

(Figure 6). In non-dimensional form the differential relations along characteristics are as follows:

$$d(c_f M) + (2/\sqrt{\gamma}) dw + (\pi/2)(c_f M) dT = 0 \quad (95)$$

along $dy = \frac{1}{2} dT$, where $w = u/U$, $Y = y/a_I \tau_f$, and $T = t/\tau_f$. The boundary condition along $Y = 0$ is $c_f M = -1.24(1 - w)$, for $\gamma = 5/3$. The calculated time history of the skin friction at the plate surface is shown in Figure 7; according to the boundary condition at the plate surface the velocity slip

is given by $\frac{U - u(0, t)}{U} = 1 - w = 0.81 |c_f M|$. Some typical shear stress and velocity "profiles" between the plate surface and the wave front are plotted in Figures 8 and 9. In Figure 10 we show the ratio

$$\left[\frac{\mu(\partial u/\partial y)}{p_{xy}} \right]_{y=0} = 8/(\pi\sqrt{\gamma}) \left[\frac{(\partial w/\partial Y)}{c_f M} \right]_{Y=0}$$

as a function of t/τ_f ; the difference between this ratio and unity is a measure of the numerical departure from the Navier-Stokes relation.

In order to examine the behavior of the solutions near the wave front and "far" from the front more carefully, it is helpful to derive analytical asymptotic forms for shear stress and velocity. By applying the Laplace transformation to both sides of Eq. (91), and to the boundary condition at the plate surface, one obtains the following expression for the Laplace transform of p_{xy} (for example):

$$\bar{p}_{xy}(y, s) = \frac{-p_o U \sqrt{RT_o}}{\left[s \sqrt{\pi/2} + \sqrt{s^2 + (p_o/\mu_o)} s \right]} \exp \left\{ -f(s) y \right\} \quad (96)$$

where

$$f(s) = 1/\sqrt{RT} \cdot \sqrt{s^2 + (p_o/\mu_o)} s \quad (97)$$

[Here $\bar{Q}(y, s) = \int_0^\infty e^{-st} Q(y, t) dt$]. Near the wave front

($\tilde{t} = \frac{t - (y/a_I)}{\tau_f} \ll 1$) s is large, and by retaining terms up to $1/s^2$

in the series expansion of $\sqrt{1 + (p_o/\mu)(1/s)}$, and inverting the transforms, one obtains

$$M c_f(y, t) = -0.685 \exp\{-0.39 Y - 0.087 \tilde{t}\} [\cosh(0.204 \tilde{t}) + (0.38 Y - 0.426) \sinh(0.204 \tilde{t})] \quad (98)$$

The shear stress and velocity are analytic functions of t/τ_f in this region; the skin friction at the plate surface given by Eq. (98) with $Y = 0$ is shown as the dashed line labelled $t/\tau_f < 1$ in Figure 7.

For $\tilde{t} \gg 1$ the transform of $c_f M$ is approximated by the expression

$$M c_f(y, s) \approx - \frac{1.24}{\sqrt{s} (\sqrt{s} + \sqrt{2/\pi} \sqrt{p_o/\mu_o})} \exp\left\{-\sqrt{(p_o/\mu_o)} (y/a_I) \sqrt{s}\right\}, \quad (99)$$

so that

$$M c_f(y, t) \approx -1.24 \exp\left\{(T/2) \left(1 + \frac{y}{At}\right)\right\} \operatorname{erfc}\left[\sqrt{T/2} \left(1 + \frac{y}{\bar{c}_o t}\right)\right], \quad (100)$$

where $T = t/\tau_f$, $A = a_I \sqrt{\pi/2}$, and $\bar{c}_o = \sqrt{(8/\pi) R T_o}$. The skin friction at the plate surface given by Eq. (100) with $y = 0$ is shown as the dashed line labelled $t/\tau_f \gg 1$ in Figure 7. [Evidently the two approximate expressions are adequate for $t/\tau_f < 1$ and $t/\tau_f > 10$.] Here the analytical behavior is a little more complicated, but the "correction" to the classical Rayleigh solution is analytic in τ_f/t "far" from the wave front. For example,

$$M c_f(0, T) = - \frac{-0.99}{\sqrt{T}} \left[1 - (1/T) + (3/T^2) - (15/T^3) + \dots\right] \quad (101)$$

By employing the relations for $M c_f$ given by Eqs. (98) and (100), we obtain approximate expressions for the ratio

$$\left[\frac{\mu (\partial u / \partial y)}{p_{xy}} \right]_{y=0} = R(T),$$

as follows:

For $T < 1$:

$$R(T) \approx 1 - (4/\pi) \left[\frac{0.174 \cosh(0.204 T) - 0.241 \sinh(0.204 T)}{\cosh(0.204 T) - 0.426 \sinh(0.204 T)} \right] \quad (102)$$

and $R(0) = 0.78$.

For $T \gg 1$:

$$R(T) \approx 1 + (2/\pi) \left[1 - (\sqrt{2/\pi}) (1/\sqrt{T}) \frac{1}{\exp(T/2) \operatorname{erfc}(\sqrt{T}/2)} \right], \quad (103a)$$

or

$$[R(T)] \approx 1 - (2/\pi) \left[(1/T) - (2/T^2) + \dots \right]. \quad (103b)$$

Thus for $T \gg 1$ the departure from the Navier-Stokes relation is of order

$1/T$ in first approximation, while the velocity slip at the plate surface [Eq. (85)] is of order $1/\sqrt{T}$.

The approximate expressions given by Eqs. (102) and (103a) are shown as the dashed lines in Figure 10.

V. DISCUSSION AND FUTURE WORK

V.1. Steady, Plane Couette Flow

The simple results for steady plane Couette flow obtained in Section IV. 2. for the limiting case $M^2 \ll 1$, $\Delta T/T \ll 1$ are all the more interesting when compared with the complicated behavior of the solutions generated by methods based on the linearized Maxwell-Boltzmann equation

$$\mathcal{F}_y (d\phi/dy) = J(\phi) \quad . \quad (104)$$

[Here $f = f_0 (1 + \phi)$, and $J(\phi)$ is the linearized collision operator^{9, 22.}] For example, Willis' integration scheme [Reference 15 and Section III. 1.] yields a velocity profile for near free-molecule flow of the form*

$$u(\xi)/(U/2) = \xi \left[-1.13 \alpha \log_e \alpha + 0.15 \alpha \right] + 0.564 \alpha \left[\left(\frac{1}{2} - \xi \right) \log_e \left(\frac{1}{2} - \xi \right) - \left(\frac{1}{2} + \xi \right) \log_e \left(\frac{1}{2} + \xi \right) \right] , \quad (105)$$

where $\xi = y/b$ and $\alpha = b/\lambda$. The shear stress is given by

$$p_{xy}/(p_{xy})_{F-M} = 1 - \sqrt{\pi} (\alpha/2) - (\alpha^2/2) \log_e \alpha + \dots \quad . \quad (106)$$

Thus the shear stress and velocity are non-analytic functions of the parameter (b/λ) (or Re/M) in the free-molecule limit. This non-analytic behavior is produced by Willis' velocity distribution function, which contains terms of the form

$$\exp \left\{ - \frac{\alpha f(\xi)}{\mathcal{F}_y} \right\} ,$$

and is therefore non-analytic at $\mathcal{F}_y = 0$.

For the parallel plate geometry most methods based on the linearized Maxwell-Boltzmann equation greatly overemphasize the role of the particles travelling nearly in the mean flow direction. It is by no means certain that similar difficulties do not arise even for the non-linear Maxwell-Boltzmann equation, and even for steady three-dimensional flows, unless special care is exercised. It is almost as if one tried to integrate the ordinary hydrodynamic momentum equation by writing it in the form

* A second approximation gives additional terms of order $\alpha^2 \log_e \alpha$, α^3 and α^4 [Eq. (4.19), page 25, Reference 15].

$\vec{W} \cdot \text{grad } \vec{W} = f(x_1, t)$. Evidently certain difficulties would arise at a stagnation point, but these are spurious difficulties introduced by the method of integration. To take a simple example, consider the ordinary differential equation $x(dy/dx) = y f(x)$, which admits of the solution $y = C \exp \left[\int \frac{f(x)}{x} dx \right]$. Clearly, integration schemes based on approximations to $f(x)$ may not be suitable in this case. One of the important advantages of Maxwell's transport integral approach is that such difficulties are smoothed over by the integration process. By analogy, the simple differential equation above is converted to

$$\int x^{m+1} (dy/dx) dx = \int x^m y f(x) dx ,$$

which is a more suitable form for approximate or iterative methods.

By applying half-range (Method B) or full-range (Method A) particle velocity moment equations to the equation for ϕ [Eq. (104)], Gross^{9, 23}, Jackson and Ziering avoid the singular behavior in the limit $b/\lambda \rightarrow 0$. But when the "second approximation",

$$\phi^{\pm} = a_0^{\pm}(y) \xi_x + a_1^{\pm}(y) \xi_x \xi_y ,$$

is utilized, both methods yield expressions for flow velocity and ϕ^{\pm} containing terms like $e^{\pm ab/\lambda}$ and $e^{\pm ay/\lambda}$, while the shear stress contains terms like $e^{\pm ab/\lambda}$. Thus these quantities approach the Navier-Stokes solution ($b/\lambda \rightarrow \infty$) in a non-analytic manner²³. The velocity profile exhibits a curved portion within a "Knudsen layer" close to the plate surface, but outside of this layer the velocity is very nearly linear in y .

This emphasis on "boundary effects" led Gross, et al, to reject as "too crude" the much simpler analytic solution obtained with $\phi^{\pm} = a_0^{\pm}(y) \xi_x$. However, in a recent report³⁹ they reexamine this "first approximation". For the special case of low Mach number isothermal flow this approach with Method A is much like the present scheme. In fact, the two-stream Maxwellian [Eqs. (35a) and (35b)] takes the form

$$f_{1,2} = f_0 \left[1 + (1/RT_0) \xi_x u_{1,2}(y) + \dots \right] ,$$

so that $\xi_x u_{1,2}(y) \sim \phi^{\pm}$, and the results obtained by the first approximation in Method A and the present method are

identical.* While admitting the attractive properties of this simple solution, Gross³⁹ and Ziering express a lack of confidence in it because the so-called "correct" behavior of p_{xy} in the free-molecule flow limit is not obtained. The normal behavior is supposed to be given by the value of the slope

$$\left[\frac{\partial [p_{xy}/(p_{xy})_{F-M}]}{\partial (b/\lambda)} \right]_{b/\lambda \rightarrow 0}$$

obtained by Wang Chang and Uhlenbeck by the technique of "Knudsen iteration". According to this method¹⁸,

$$\lim_{b/\lambda \rightarrow 0} \frac{\partial [p_{xy}/(p_{xy})_{F-M}]}{\partial (b/\lambda)} \approx -1.0059,$$

for hard-sphere molecules, or -1.242 for Maxwell molecules. Method B (second approximation) gives a slope for hard-sphere molecules very close to the Wang-Chang Uhlenbeck value, while the simple solution [Eq. (92)] yields the slope -0.50. However, the Knudsen iteration technique is known to be non-convergent and analytically inadmissible for this problem^{15, 23}. Wang Chang and Uhlenbeck¹⁸ recognized that a second approximation to the velocity profile was unobtainable by this method. Once an expansion procedure of this type is applied to the Maxwell-Boltzmann equation we are dealing with a singular perturbation problem, in the language of Lagerstrom⁴⁰ and Kaplun, and there is little reason for trusting the limiting value of the slope

$$\frac{\partial p_{xy}/(p_{xy})_{F-M}}{\partial (b/\lambda)}$$

in the limit $b/\lambda \rightarrow 0$.

In Figure 11 the ratio $(p_{xy})_{F-M}/p_{xy}$ is shown as a function of

* Unfortunately there is a numerical error in the expression for μ near the middle of page 6, Reference 39. If μ is defined by the relation

$$\mu = \frac{p_{xy}}{(du/dy)} = \frac{p_{xz}}{(dq_z/dx)}$$

in the notation of that paper, then one obtains $\mu = \rho_0 \bar{c} \lambda \left(\frac{\pi^{3/2}}{4 I_2 \lambda} \right)$.

By substituting the expression for $I_2/\pi^{3/2}$ obtained from this relation into the relations for p_{xz} and $q_z \equiv u$ given on pages 5 and 6, one verifies that the results obtained by the two methods are in fact identical.

$b/\lambda = \sqrt{2/\pi\gamma} (Re/M)$ for Willis' solution¹⁵, the Gross-Ziering²³ second approximation (Method B) for hard-sphere molecules, and the present solution. [The second approximation with Method A³⁹ (full-range moments) would give an initial slope of about -0.70.] Evidently the slopes of the Willis and Gross-Ziering solutions rapidly approach the value given by the present simple theory as b/λ increases. The actual numerical differences in p_{xy} are small.

The present author believes that there is no justification for regarding half-range moment methods (Method B) as "superior" to full-range moment methods such as Method A. One of the advantages of utilizing Maxwell's inverse fifth power force law is just the fact that the collision integrals in a full-range moment method are completely independent of the form of the distribution function, and this advantage is lost in the half-range moment method (Method B).

A more serious question is the use of polynomials in ϕ^+ . It is not clear that terms of the form $\xi_x \xi_y$ play any useful role in a rarefied gas flow. In the simple case of linearized plane parallel flow or heat transfer these terms do not lead to any special difficulties, but in non-linear problems, or in problems with curved geometry, one finds the same sort of undesirable cross-coupling between normal stresses and heat flux as in Grad's equations. Of course as Gross³⁹ and Ziering point out, there is no "unique" integral method. Perhaps the question of the most suitable method can be settled only by obtaining "exact" solutions of the Maxwell-Boltzmann equation [Eq. (27)] in some simple cases, or by experiment.

Because of the experimental difficulties involved in producing a plane Couette flow, we must rely at present on investigations of cylindrical Couette flow in which the width of the annulus is small compared to the radius of either cylinder. As shown by Yang²⁷ and the present author, if the drag on the stationary outer cylinder is written in the form

$$\frac{1}{C_D M} = A (r_2/r_1) + B (r_2/r_1) (Re/M) ,$$

the values of the slope $B(r_2/r_1)$ in the experiments of Kuhlthau⁴¹ and Bowyer⁴² and Talbot do not differ very much from the value of 0.50 obtained for plane Couette flow. [Here r_1 and r_2 are the radii of the inner and outer cylinders, respectively, and $Re = U(r_2 - r_1)/\nu$.] By analogy with the case of plane Couette flow, it was suggested²⁷ that this function is

identical with the expression obtained in the Navier-Stokes regime, i. e., $B(r_2/r_1) = \frac{1}{2} \left[1 + (r_2 - r_1)/(2r_1) \right]$. Recently D. K. Ai and C. Y. Liu of this laboratory solved the problem of linearized cylindrical Couette flow for small ratios of annulus width to cylinder radius, and verified that the function $B(r_2/r_1)$ is indeed given by the Navier-Stokes expression over the whole range of gas densities. However the value of $A(r_2/r_1)$ obtained differs somewhat from the correct free-molecule flow value because of the very simple distribution function employed.*

In Figure 12 the drag data obtained by Bowyer⁴² and Talbot are compared with the theoretical solution for isothermal cylindrical Couette flow with completely diffuse reemission. [Figure 12 also shows the Wang Chang-Uhlenbeck¹⁸ slopes in the limit $b/\lambda \rightarrow 0$.] In view of the experimental difficulties at low density the scatter in the experimental data is remarkably low. The slope of the experimental curve of $1/C_D M$ vs. Re/M follows the simple theory fairly well, not only in the low density regime shown in these figures, but over the whole range of densities. Nevertheless, by comparison with Figure 11, one sees that the experimental drag data is hardly able to settle the question of the most suitable integral method for this problem. It appears that measurements of some mean flow quantity such as velocity or temperature are required to provide conclusive evidence.

V. 2. Rayleigh's Problem

At first the wave-like character of the solutions obtained for Rayleigh's problem in Section IV. 3. may seem surprising, but collisions between particles in the gas are relatively infrequent when $t/\tau_f < 1$, and thus diffusive effects are secondary. The over-simplified version of the two-stream Maxwellian employed in Section IV. 3. introduces a certain averaging process over the particle velocities in the direction normal to the plate; the result is a single characteristic propagation velocity equal

* D. K. Ai utilized the linearized Grad equations, while C. Y. Liu employed the present method. For the shear flow Grad's equations offer no difficulties, but for heat transfer they lead to a physically incorrect result in the free-molecule flow limit. The present method yields consistent results over the whole range of gas densities. These results will appear in forthcoming reports. We are also investigating the problem for arbitrary ratios of annulus width to cylinder radius.

to the isothermal sound speed. This "defect" is easily removed by employing a slightly more sophisticated two-stream Maxwellian that differentiates between "fast" and "slow" particles. This new distribution function is the natural generalization of the function utilized by Yang⁴³ and the author for Rayleigh's problem in free-molecule flow with arbitrary plate Mach number and plate temperature. In free-molecule flow, particles crossing the horizontal plane at y from below at time t are aware of the plate motion only if $\xi_y > (y/t)$, where t is the time elapsed since the start of the impulsive motion. Thus, a more consistent two-stream Maxwellian is defined as follows: (Figure 3)

$$\begin{aligned} f &= f_1, & \xi_y > y/t \\ f &= f_2, & -\infty \leq \xi_y \leq y/t \end{aligned} \quad (104)$$

For low speed isothermal flow $n_1 = n_2 = \text{const.}$, and $T_1 = T_2 = \text{const.}$, as before. Of course the momentum equation [Eq. (79)] is also unchanged, but the transport integral for $Q = m \int \xi_x \xi_y$ [Eq. (53)] now yields the equation

$$-\frac{\partial p_{xy}}{\partial t} + p_o (\partial u / \partial y) - (\partial / \partial y) (p_{xy} \frac{y}{t}) = (p_o / \mu_o) p_{xy} \quad (105)$$

The equations are still hyperbolic, but the characteristics are defined by

$$(dy)/(a_I dt) = dY/dT = \frac{1}{2} (Y/T) \pm \sqrt{[(Y^2)/(4T^2)] + 1} \quad (106)$$

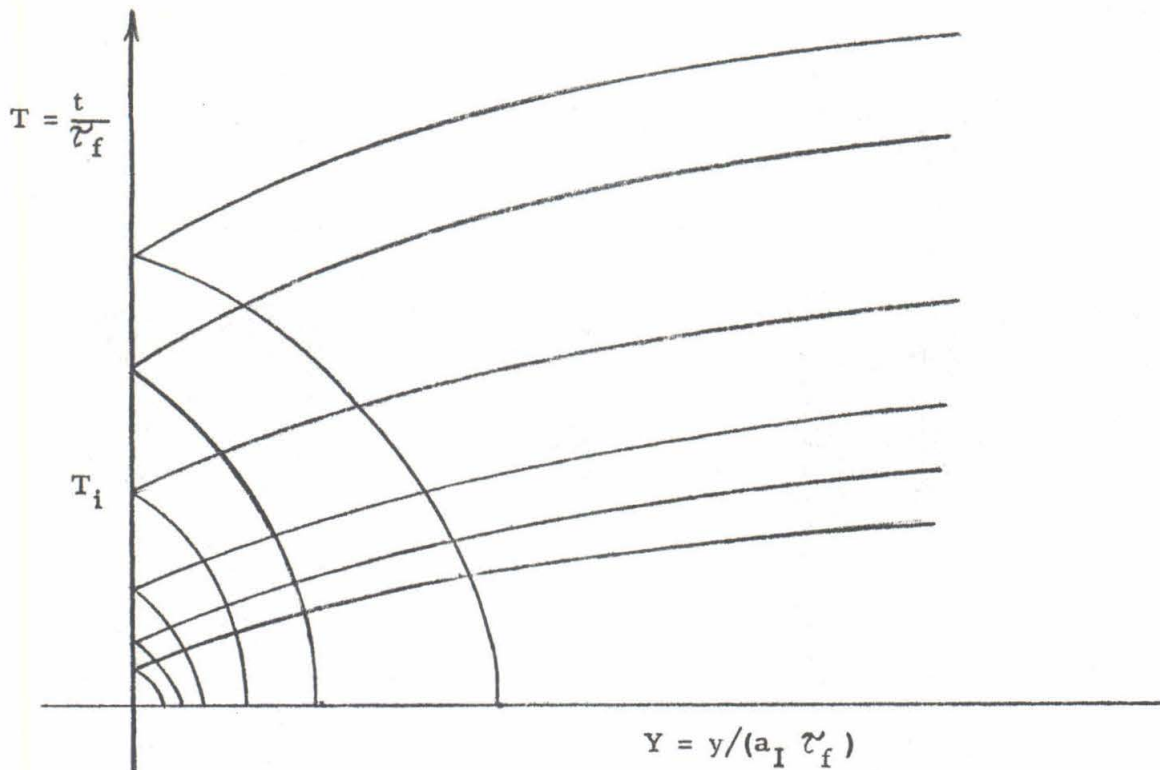
or

$$\log_e (T/T_i) = 1/4 (Y^2/T^2) \pm \left[\frac{1}{2} (Y/T) \sqrt{1 + (Y^2/4T^2)} + \frac{1}{2} \log \left(\frac{1}{2} \frac{Y}{T} + \sqrt{1 + \frac{1}{4} \frac{Y^2}{T^2}} \right) \right] \quad (107)$$

where T_i is the intercept along $Y = 0$ (see sketch on page 54). Near the plate surface the characteristic signal speeds are $\pm a_I$, as before (Section IV.3.), but far from the plate surface, or more precisely, when $Y/T \gg 1$, the characteristic speeds are 0 and $y/t \gg a_I$. Along $T = 0$ the characteristic speed is infinite. Without going into details, we state that the solutions for the shear stress and flow velocity for $T \ll 1$ are

$$M c_f(y, t) = 0.62 e^{-(y^2)/(2RT_o t^2)}$$

and



$$u/U(y, t) = \frac{1}{2} \operatorname{erfc} \left(\frac{y}{\sqrt{2RT_0 t}} \right),$$

and these solutions are identical with the free-molecule flow solutions^{24, 43}. For large times ($t/\tau_f \gg 1$), $p_{xy} \rightarrow \mu(\partial u/\partial y)$, as before, but

$$\lim_{t \rightarrow 0} \left(\frac{p_{xy}}{\mu(\partial u/\partial y)} \right)_{y=0} = 0.$$

By employing the method of characteristics we obtain the entire time history of the smooth transition from free-molecule flow to the classical Rayleigh solution.

V.3. Future Work

When the present method based on Maxwell's integral equations of transfer is applied to linearized flows, reasonable results are obtained fairly simply, and most of the essential features of the problem are uncovered. But low Mach number flows with small temperature differences are too simple to provide a serious test of any method that purports to be

general. Recently C. Y. Liu and the author applied this method to steady plane compressible Couette flow, in order to study the effects of large temperature differences and dissipation.* Similar techniques are now being investigated for the flow of an unionized, monatomic gas through a normal shock wave of arbitrary strength. We are also studying the steady flow over a cylinder or sphere.

No integral method such as the present one is capable of resolving old problems such as flow separation and the details of wake formation behind bluff bodies in the Navier-Stokes regime. For this reason it seems desirable at first to restrict the studies of flow over closed bodies to the two limiting cases of $M \gg 1$ and $M \ll 1$. At hypersonic speeds, the flow over the rear portion of the body does not influence the flow over the front portion very much. On the other hand, when $M \ll 1$, a wide range of the parameter Re/M can be investigated from free-molecule flow up to the Stokes or Oseen limit $[Re = O(1)]$, without penetrating the regime in which vortex shedding occurs.

So far the work is restricted to the study of a pure, unionized, monatomic gas, but in principle the method can be extended to include gas mixtures and charged particles.

* Report to appear soon.

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APPENDIX A

MAXWELL'S INTEGRAL EQUATIONS OF TRANSFER
IN AN ORTHOGONAL CURVILINEAR COORDINATE SYSTEM

Maxwell's integral equations of transfer of a quantity Q that is an arbitrary function of the velocity components of a particle must include the rate of change of Q produced by particle acceleration (Section III. 2.). This contribution is expressed as follows:

$$\int f \left(\sum_{i=1}^3 \frac{\partial Q}{\partial \xi_i} \frac{\partial \xi_i}{\partial t} \right) d\vec{\xi} = \int f \left(\frac{\partial \vec{\xi}}{\partial t} \cdot \vec{\nabla}_{\vec{\xi}} \right) Q d\vec{\xi} , \quad (\text{A-1})$$

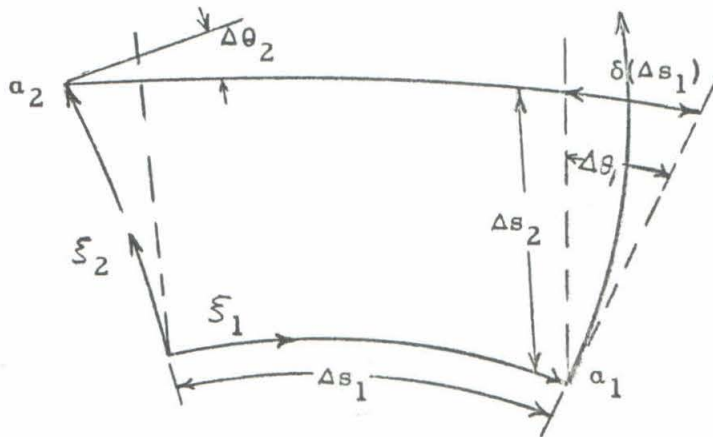
In a curvilinear coordinate system

$$\frac{\partial \vec{\xi}}{\partial t} + \vec{\Omega} \times \vec{\xi} = \vec{F}/m , \quad (\text{A-2})$$

where \vec{F} is the "external" force vector acting on a single particle, and $\vec{\Omega}$ is the local instantaneous angular velocity introduced by the curvature of the coordinate system. Thus

$$\int f \left(\frac{\partial \vec{\xi}}{\partial t} \cdot \vec{\nabla}_{\vec{\xi}} \right) Q d\vec{\xi} = \int f \left\{ \frac{\vec{F}}{m} - (\vec{\Omega} \times \vec{\xi}) \right\} \cdot \vec{\nabla}_{\vec{\xi}} Q d\vec{\xi} . \quad (\text{A-3})$$

The term $\vec{\Omega} \times \vec{\xi}$ is evaluated with the aid of the relations for orthogonal coordinate curvature given by elementary differential geometry. For simplicity, consider the two-dimensional case first (see sketch):



Suppose that the differential arc lengths along the curves $a_2 = \text{const.}$ and $a_1 = \text{const.}$ are given by

$$\Delta s_1 = h_1 \Delta a_1 \quad \text{and} \quad \Delta s_2 = h_2 \Delta a_2 \quad (\text{A-4})$$

respectively, where a_1 and a_2 are the orthogonal curvilinear coordinates, and h_1 and h_2 are known functions of a_1 and a_2 . Then the curvature in the a_1 direction is responsible for an apparent acceleration in that direction

given by $\xi_2 \frac{\partial \theta_1}{\partial t} = \xi_2 \xi_1 \frac{\partial \theta_1}{\partial s_1}$. Similarly, the curvature in the a_2

direction produces an acceleration in the a_1 direction given by

$-\xi_2^2 \frac{\partial \theta_2}{\partial t} = -\xi_2^2 \frac{\partial \theta_2}{\partial s_2}$, so that the total contribution in the a_1 direction is

$$\xi_2 \xi_1 \frac{\partial \theta_1}{\partial s_1} - \xi_2^2 \frac{\partial \theta_2}{\partial s_2} = \frac{\xi_2 \xi_1}{h_1} \frac{\partial \theta_1}{\partial a_1} - \frac{\xi_2^2}{h_2} \frac{\partial \theta_2}{\partial a_2} \quad (\text{A-5})$$

Now $\delta(\Delta s_1) = \Delta s_2 \cdot \Delta \theta_1$ (see sketch). But $\delta(\Delta s_1) = \partial h_1 \Delta a_1$, according to Eq. (A-4); therefore,

$$\frac{\partial \theta_1}{\partial a_1} = \frac{\partial h_1}{\partial s_2} = \frac{1}{h_2} \frac{\partial h_1}{\partial a_2} \quad (\text{A-6})$$

Similarly,

$$\frac{\partial \theta_2}{\partial a_2} = 1/h_1 (\partial h_2 / \partial a_1) \quad (\text{A-7})$$

By Eqs. (A-5), (A-6), and (A-7), the contribution to the component of acceleration in the a_1 direction made by coordinate curvature is given by

$$1/(h_1 h_2) \left[\xi_1 \xi_2 (\partial h_1 / \partial a_2) - \xi_2^2 (\partial h_2 / \partial a_1) \right] \quad (\text{A-8})$$

The contribution made by the component of velocity $\vec{\xi}_3$ and the coordinate curvatures in the a_3 and a_1 directions is obtained from Eq. (A-8) by replacing the subscript 2 by 3.

By generalizing this result to the three-dimensional case one obtains

$$\begin{aligned}
\int f \left(\frac{\partial \vec{\xi}}{\partial t} \cdot \vec{\nabla}_{\vec{\xi}} \right) Q d\vec{\xi} &= \int f \sum_{i=1}^3 (F_i/m) (\partial Q / \partial \xi_i) d\vec{\xi} \\
+ \int f \sum_{i=1}^3 &\left[\frac{1}{h_i h_j} \left(\xi_j^2 \frac{\partial h_j}{\partial a_i} - \xi_i \xi_j \frac{\partial h_i}{\partial a_j} \right) \right. \\
+ \left. \frac{1}{(h_i h_k)} \left(\xi_k^2 \frac{\partial h_k}{\partial a_i} - \xi_i \xi_k \frac{\partial h_i}{\partial a_k} \right) \right] &(\partial Q / \partial a_i) d\vec{\xi},
\end{aligned} \tag{A-9}$$

where the cyclical order of permutation of the indices i, j, k must be followed. Finally, Maxwell's integral equations of transfer in orthogonal curvilinear coordinates are as follows:

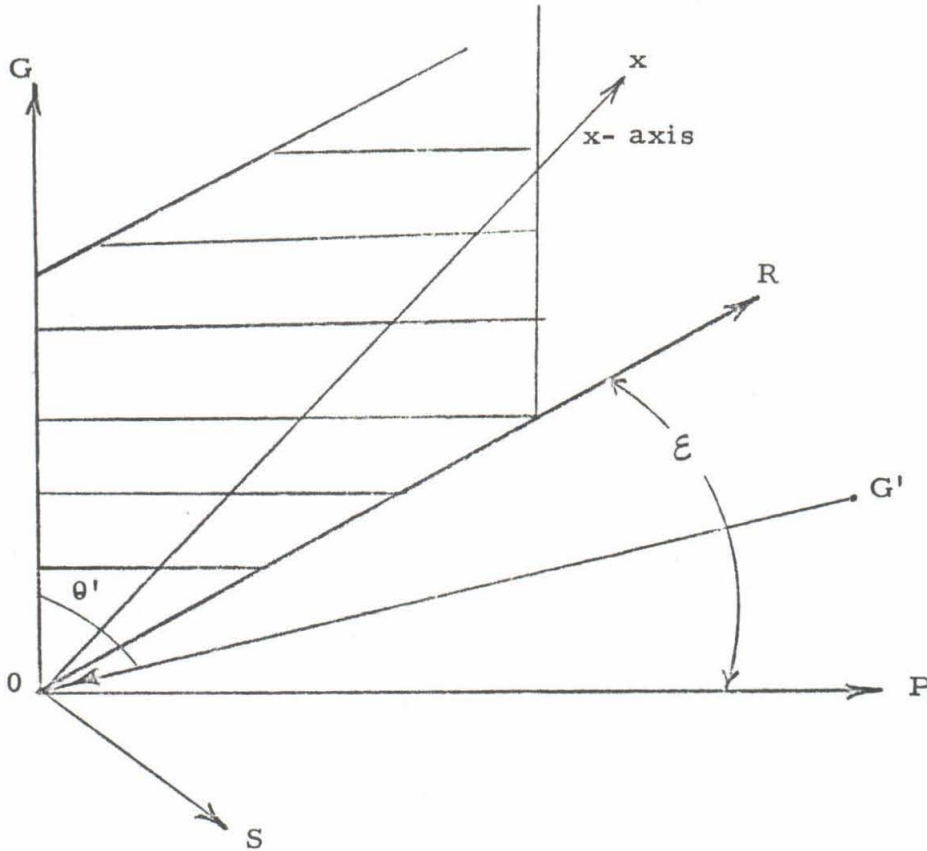
$$\begin{aligned}
\partial / \partial t \int f Q d\vec{\xi} + (1/h_1 h_2 h_3) \sum_{i=1}^3 (\partial / \partial a_i) &\left[h_j h_k \int f \xi_i Q d\vec{\xi} \right] \\
= \int f \left(\frac{\partial \vec{\xi}}{\partial t} \cdot \vec{\nabla}_{\vec{\xi}} \right) Q d\vec{\xi} + \Delta Q, &
\end{aligned} \tag{A-10}$$

where the first integral on the right-hand side of Eq. (A-10) is given by Eq. (A-9), and the collision integral ΔQ is given by Eq. (32) of the text. Of course ΔQ is evaluated locally and is independent of coordinate curvature. The evaluation of ΔQ for Maxwell particles is discussed in Section III. 3.

APPENDIX B

CHANGES IN COMPONENTS OF RELATIVE VELOCITY
DURING A BINARY COLLISION

This appendix contains a simple vector geometry derivation of the relations written down by Maxwell^{4b} and Jeans^{2c}. For clarity suppose Figure 4 is rotated so that OG plays the role of one coordinate axis:



As in Figure 4, OG, OR, and the x- axis lie in one plane, and OG, OG', and OP lie in another plane (the plane of the orbit) making a dihedral angle ϵ with the first plane. Now the x- component of the relative velocity after collision is given by

$$\cos (G', x) = \frac{(\xi' - \xi_1')_x}{V} = \vec{i}_{G'} \cdot \vec{i} \quad , \quad (B-1)$$

where $\vec{i}_{G'}$ and \vec{i} are unit vectors along OG' and the x- axis, respectively.

But $\vec{i}_{G'} = \vec{i}_G \cos \theta' + \vec{i}_P \sin \theta'$ (B-2)

and $\vec{i}_P = \vec{i}_R \cos \xi + \vec{i}_S \sin \xi$, (B-3)

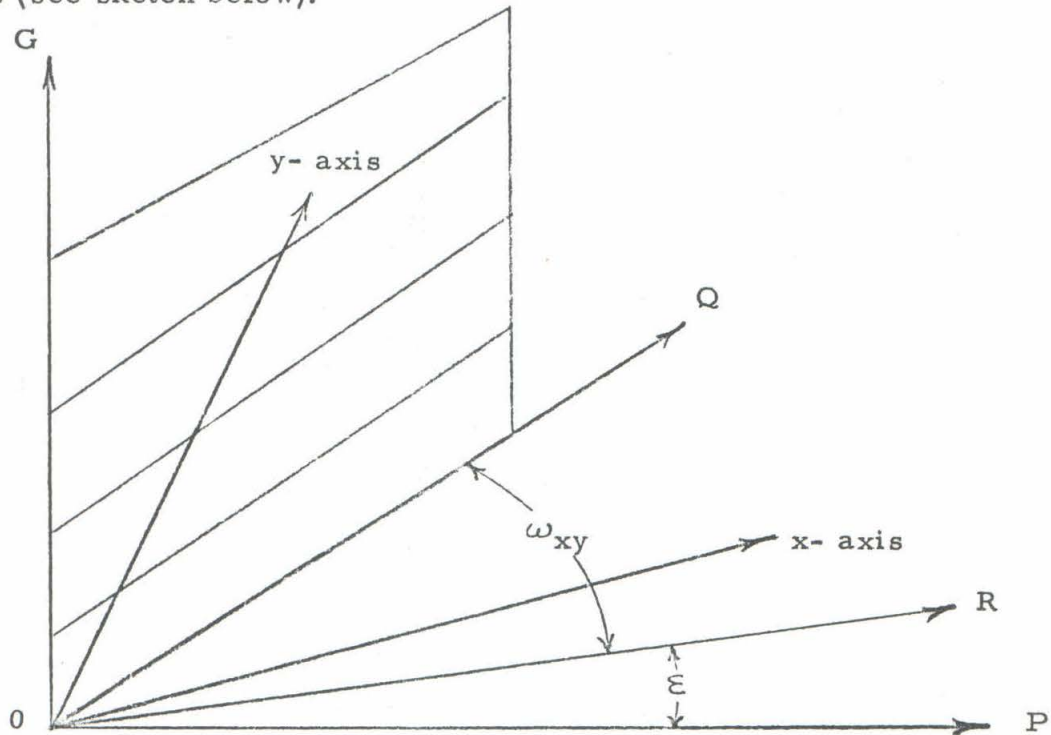
where OS is perpendicular to OR and perpendicular to the plane containing OG, OR, and the x- axis. Therefore,

$$\begin{aligned} \vec{i}_{G'} \cdot \vec{i} &= (\vec{i}_G \cdot \vec{i}) \cos \theta' + \sin \theta' [\vec{i}_R \cdot \vec{i} \cos \xi + \vec{i}_S \cdot \vec{i} \sin \xi] \\ &= \cos (G, x) \cos \theta' + \sin \theta' [\sin (G, x) \cos \xi + 0] . \end{aligned} \quad (B-4)$$

But, $\cos (G, x) = \frac{(\xi_1 - \xi)_x}{V}$, so that [Eqs. (B-1) and (B-4)] ,

$$(\xi' - \xi_1')_x = (\xi_1 - \xi)_x \cos \theta' + \sqrt{V^2 - (\xi_1 - \xi)_x^2} \sin \theta' \cos \xi. \quad (B-5)$$

Denote by ω_{yx} the dihedral angle between the plane containing OG, OQ, and the y- axis, and the plane containing OG, OR, and the x- axis (see sketch below).



OQ, OR, and OP all lie in a second plane perpendicular to OG. Now

$$\cos (G', y) = \frac{(\xi' - \xi_1')_y}{V} = \vec{i}_{G'} \cdot \vec{j} \quad , \quad (B-6)$$

where \vec{j} is the unit vector along the y- axis. But $\vec{i}_{G'}$ is given by Eq. (B-2), and the unit vector \vec{i}_P can be written as

$$\vec{i}_P = \vec{i}_Q \cos (\xi + \omega_{yx}) + \vec{i}_T \sin (\xi + \omega_{yx}) \quad , \quad (B-7)$$

where the direction OT is perpendicular to OQ and perpendicular to the plane containing OG and \vec{j} . Therefore,

$$\vec{i}_{G'} \cdot \vec{j} = \cos (G, y) \cos \theta' + \sin \theta' [\sin (G, y) \cos (\xi + \omega_{yx}) + 0] \quad (B-8)$$

and [Eqs. (B-6) and (B-8)] ,

$$(\xi' - \xi_1')_y = (\xi_1 - \xi)_y \cos \theta' + \sqrt{V^2 - (\xi_1 - \xi)_y^2} \sin \theta' \cos (\xi + \omega_{yx}) \quad , \quad (B-9)$$

which is the result quoted in Eq. (48) of the text, with y replaced by k.

The angle ω_{yx} is determined by expressing \vec{i} and \vec{j} as follows

$$\begin{aligned} \vec{i} &= \vec{i}_R \sin (G, x) + \vec{i}_G \cos (G, x) \\ \vec{j} &= \vec{i}_Q \sin (G, y) + \vec{i}_G \cos (G, y) \quad , \end{aligned} \quad (B-10)$$

and utilizing the vector identity $\vec{i} \cdot \vec{j} = 0$. Since $\vec{i}_Q \cdot \vec{i}_R = \cos \omega_{yx}$, and \vec{i}_R and \vec{i}_Q are perpendicular to \vec{i}_G ,

$$\vec{i} \cdot \vec{j} = 0 = \cos \omega_{yx} \sin (G, y) \sin (G, x) + \cos (G, y) \cos (G, x) \quad , \quad (B-11)$$

from which one obtains

$$\cos \omega_{yx} = - \frac{[(\xi_1)_y - \xi_y][(\xi_1)_x - \xi_x]}{\sqrt{V^2 - (\xi_1 - \xi)_y^2} \sqrt{V^2 - (\xi_1 - \xi)_x^2}} \quad . \quad (B-12)$$

The expression for $\cos \omega_{zx}$ is obtained by replacing the subscript y by z [Eq. (49)] .

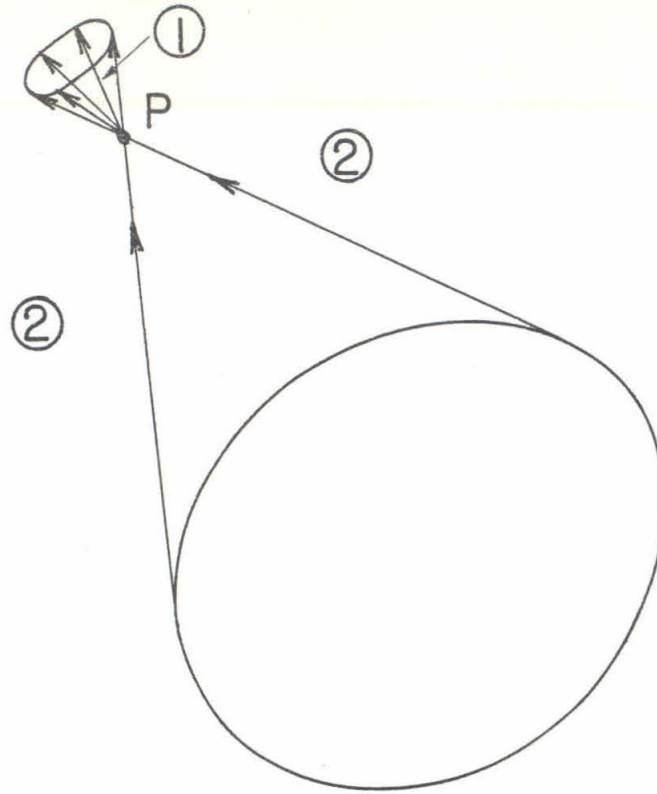


FIG. 2

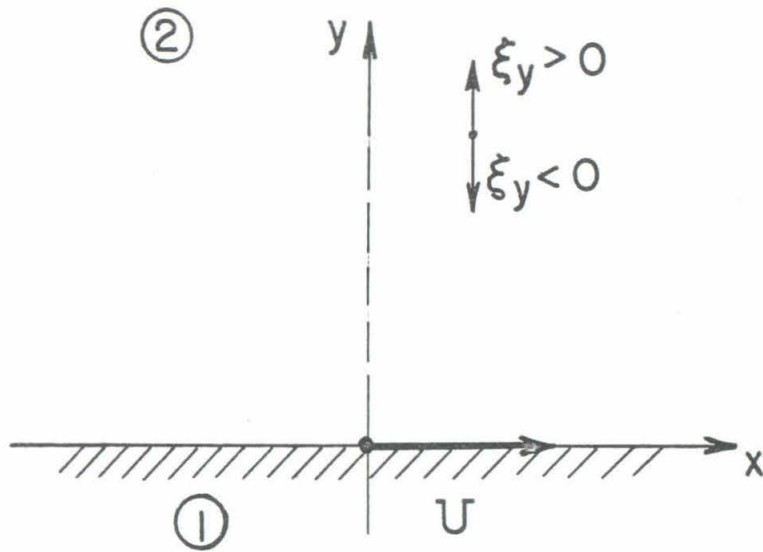
"TWO-STREAM"
MAXWELLIAN

FOR OUTWARDLY DIRECTED $\vec{\xi}$ IN REGION 1 :

$$f = f_1(\vec{\xi}, \vec{R}, t) = n_1(\vec{R}, t) \left(\frac{1}{2\pi \mathcal{R} T_1(\vec{R}, t)} \right)^{3/2} \exp. \left\{ - \frac{[\vec{\xi} - \vec{u}_1(\vec{R}, t)]^2}{2 \mathcal{R} T_1(\vec{R}, t)} \right\}$$

FOR ALL OTHER $\vec{\xi}$ (REGION 2) :

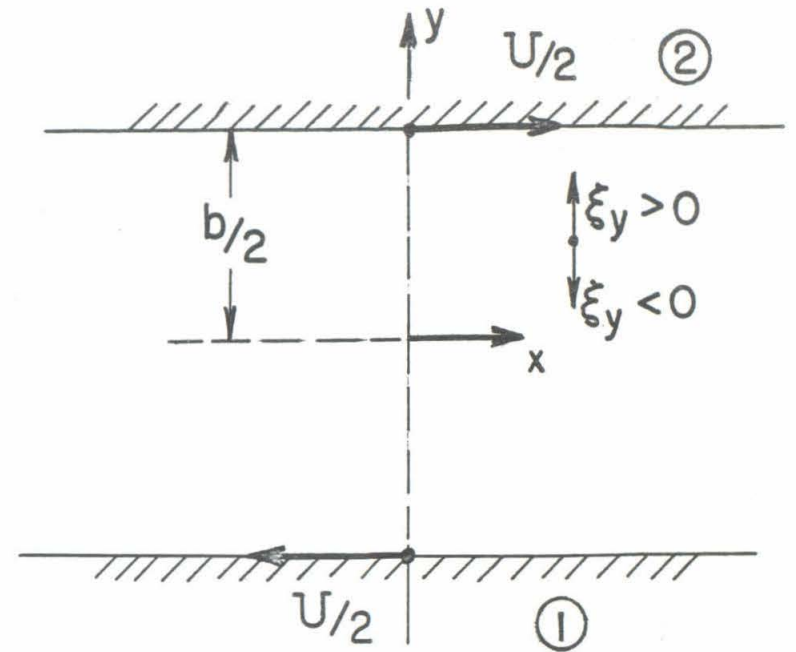
$$f = f_2(\vec{\xi}, \vec{R}, t) = \frac{n_2(\vec{R}, t)}{(2\pi \mathcal{R} T_2(\vec{R}, t))^{3/2}} \exp. \left\{ - \frac{[\vec{\xi} - \vec{u}_2(\vec{R}, t)]^2}{2 \mathcal{R} T_2(\vec{R}, t)} \right\}$$



RAYLEIGH'S PROBLEM

$$\xi_y > 0: f = f_1(y, \vec{\xi}, t) = \frac{n_1(y, t)}{[2\pi \mathcal{R} T_1(y, t)]^{3/2}}$$

$$\xi_y < 0: f = f_2(y, \vec{\xi}, t) = \frac{n_2(y, t)}{[2\pi \mathcal{R} T_2(y, t)]^{3/2}}$$



PLANE COUETTE FLOW

$$\exp. \left\{ -\frac{[(\xi_x - u_{x_1}(y, t))^2 + \xi_y^2 + \xi_z^2]}{2 \mathcal{R} T_1(y, t)} \right\}$$

$$\exp. \left\{ -\frac{[(\xi_x - u_{x_2}(y, t))^2 + \xi_y^2 + \xi_z^2]}{2 \mathcal{R} T_2(y, t)} \right\}$$

FIG. 3 TWO-STREAM MAXWELLIAN FOR PLANE PARALLEL FLOW

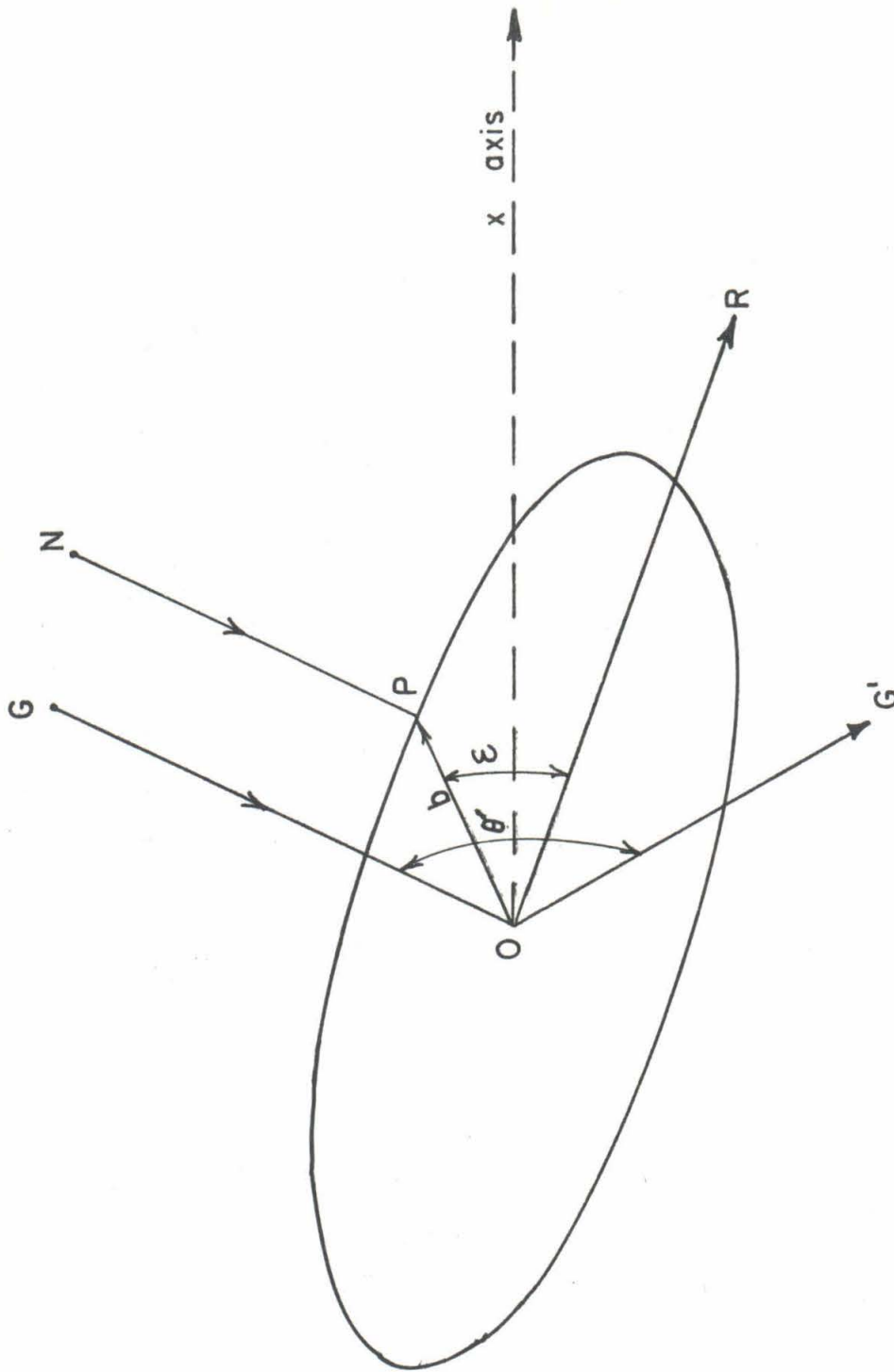


FIG. 4 CHANGE IN VELOCITY DURING A COLLISION

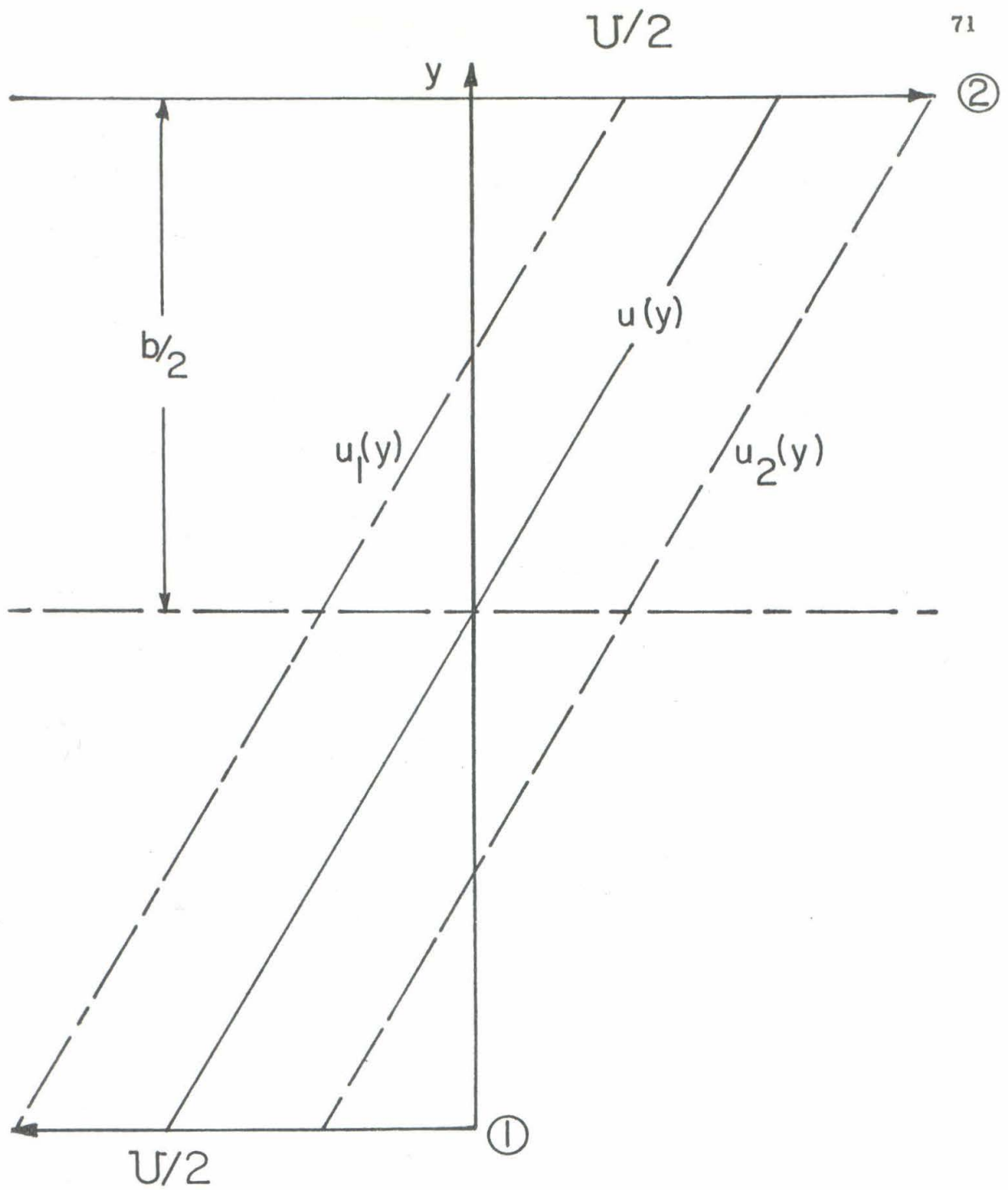


FIG.5 VELOCITY DISTRIBUTION FOR PLANE COUETTE FLOW

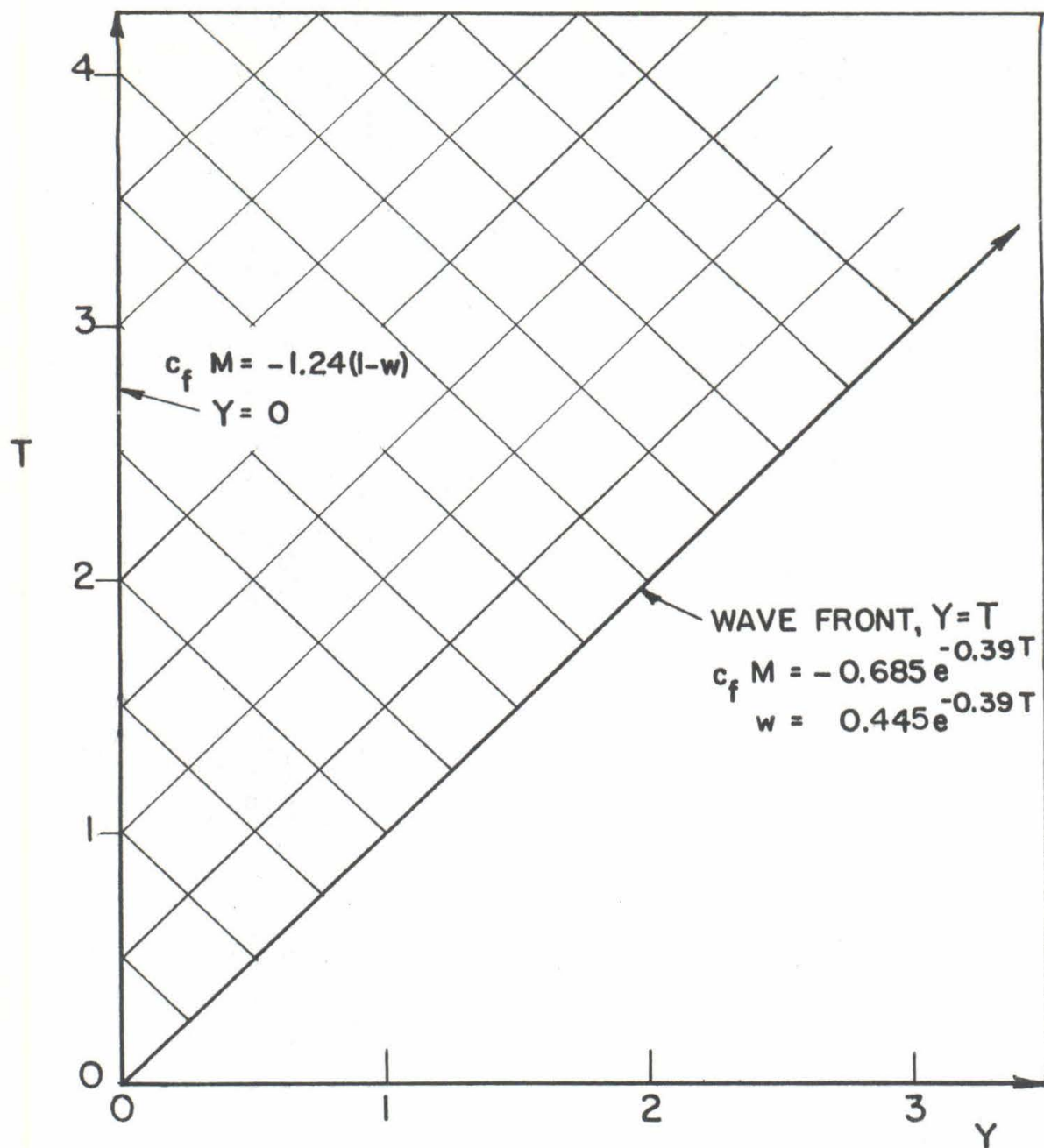


FIG.6 CHARACTERISTICS DIAGRAM FOR
RAYLEIGH'S PROBLEM

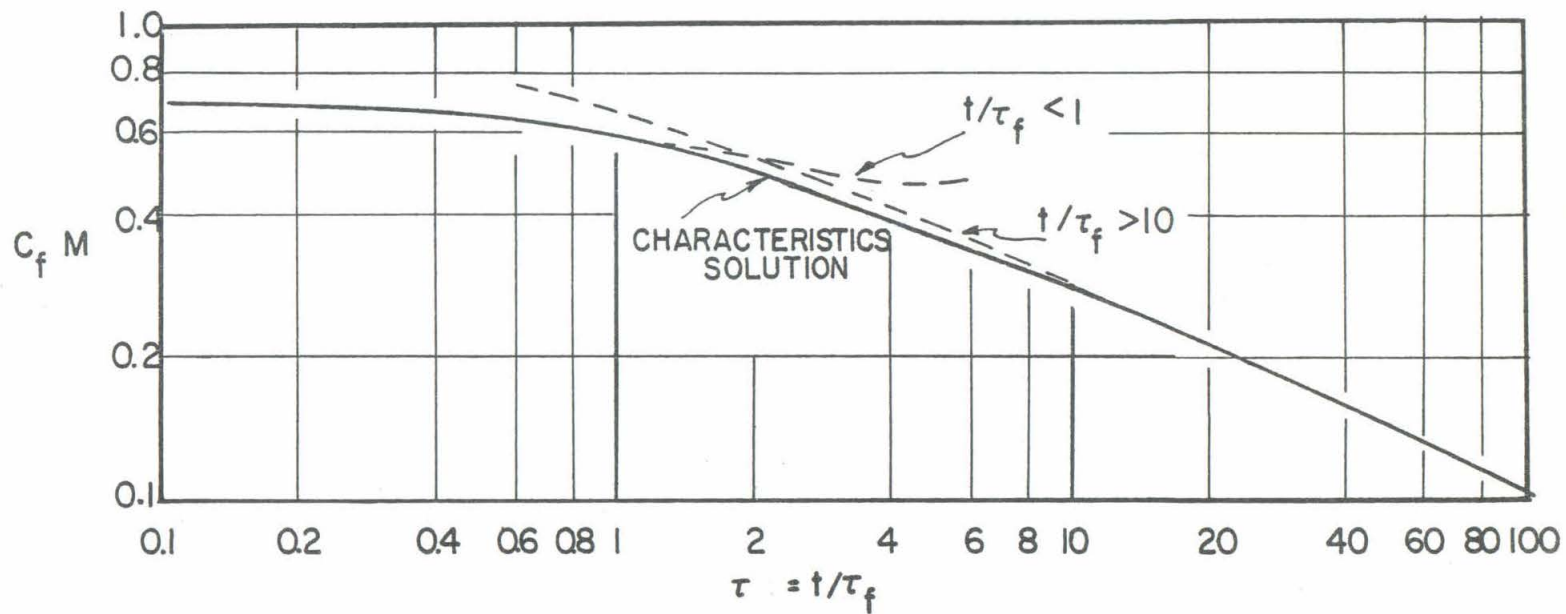


FIG.7 TIME HISTORY OF SKIN FRICTION AT PLATE SURFACE IN RAYLEIGH'S PROBLEM

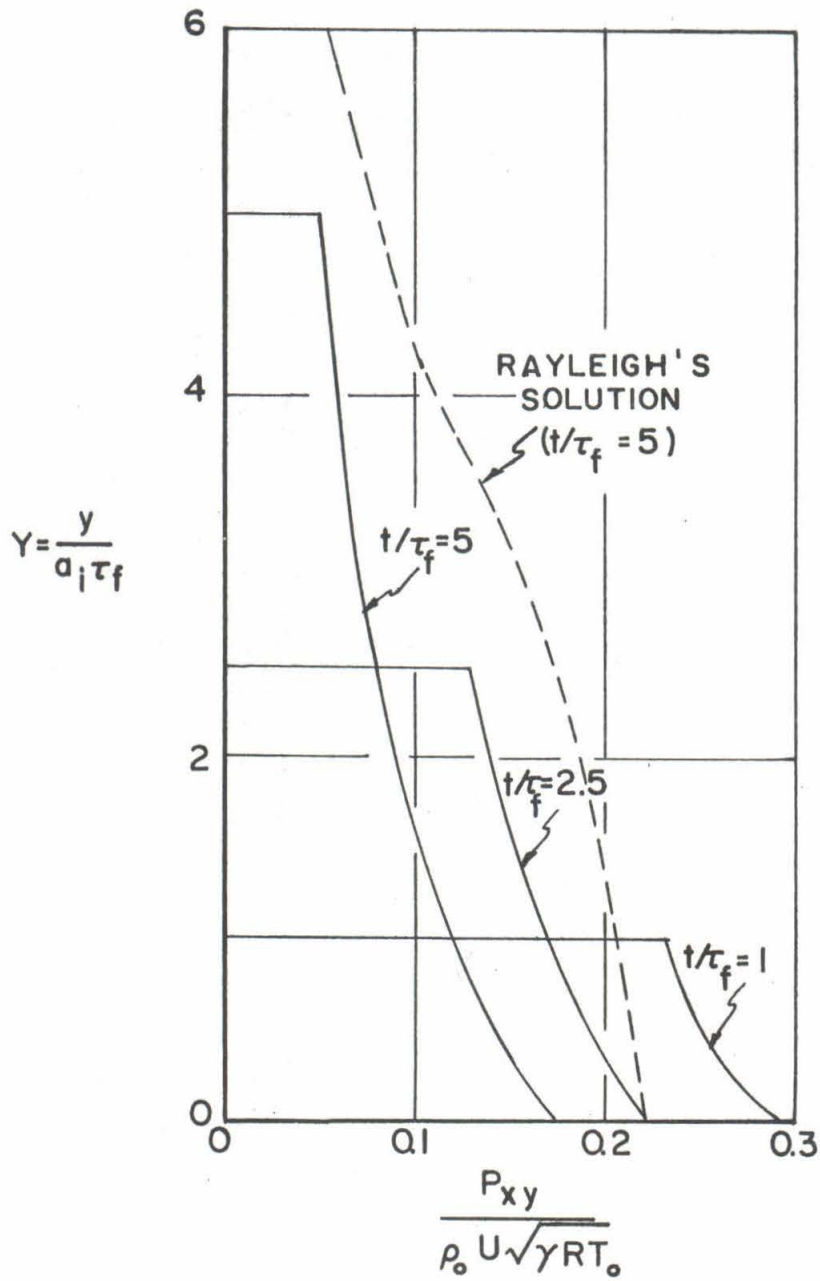


FIG.8 TYPICAL DISTRIBUTION OF SHEAR STRESS IN RAYLEIGH'S PROBLEM

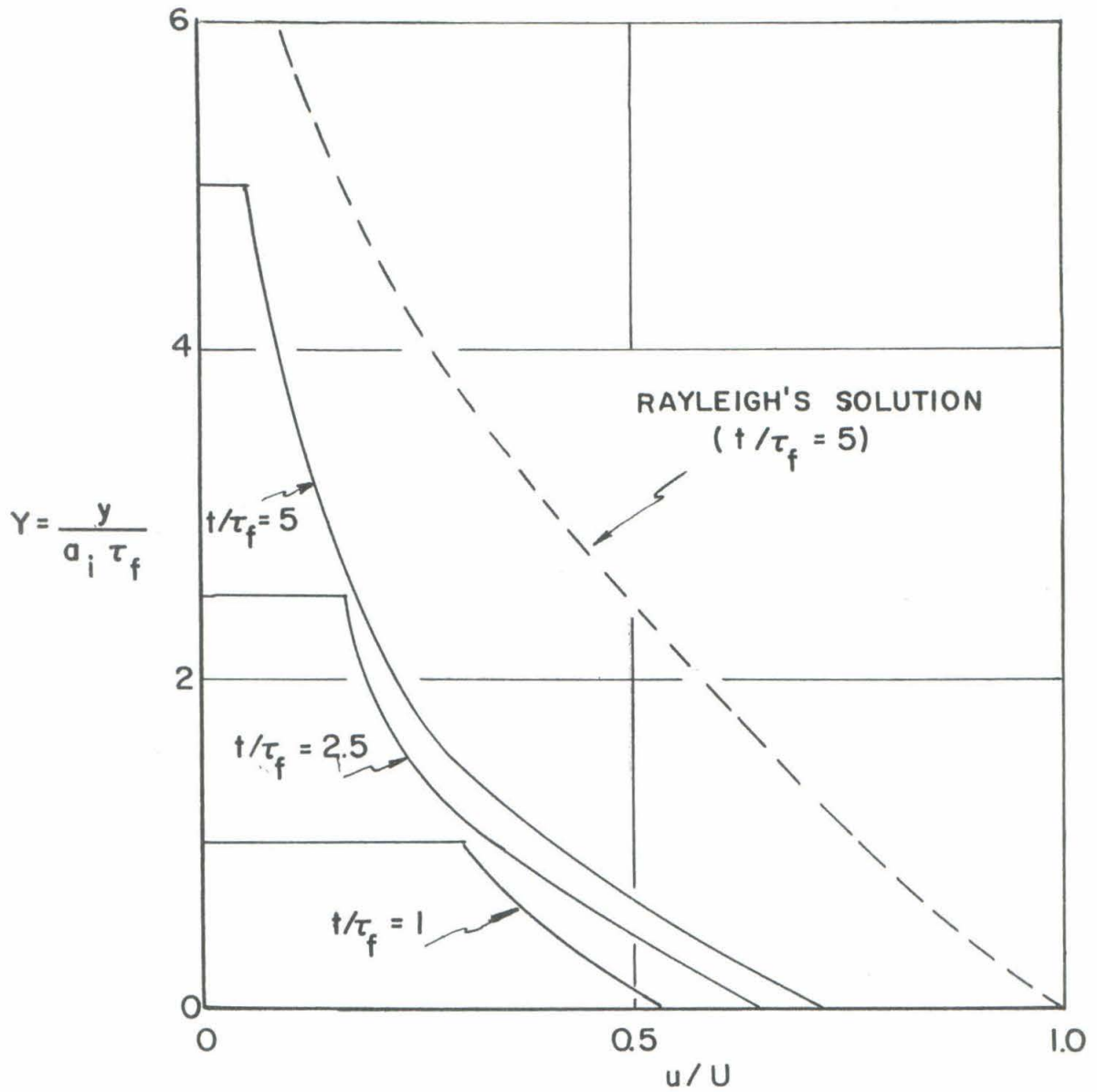


FIG.9 TYPICAL VELOCITY DISTRIBUTION
IN RAYLEIGH'S PROBLEM

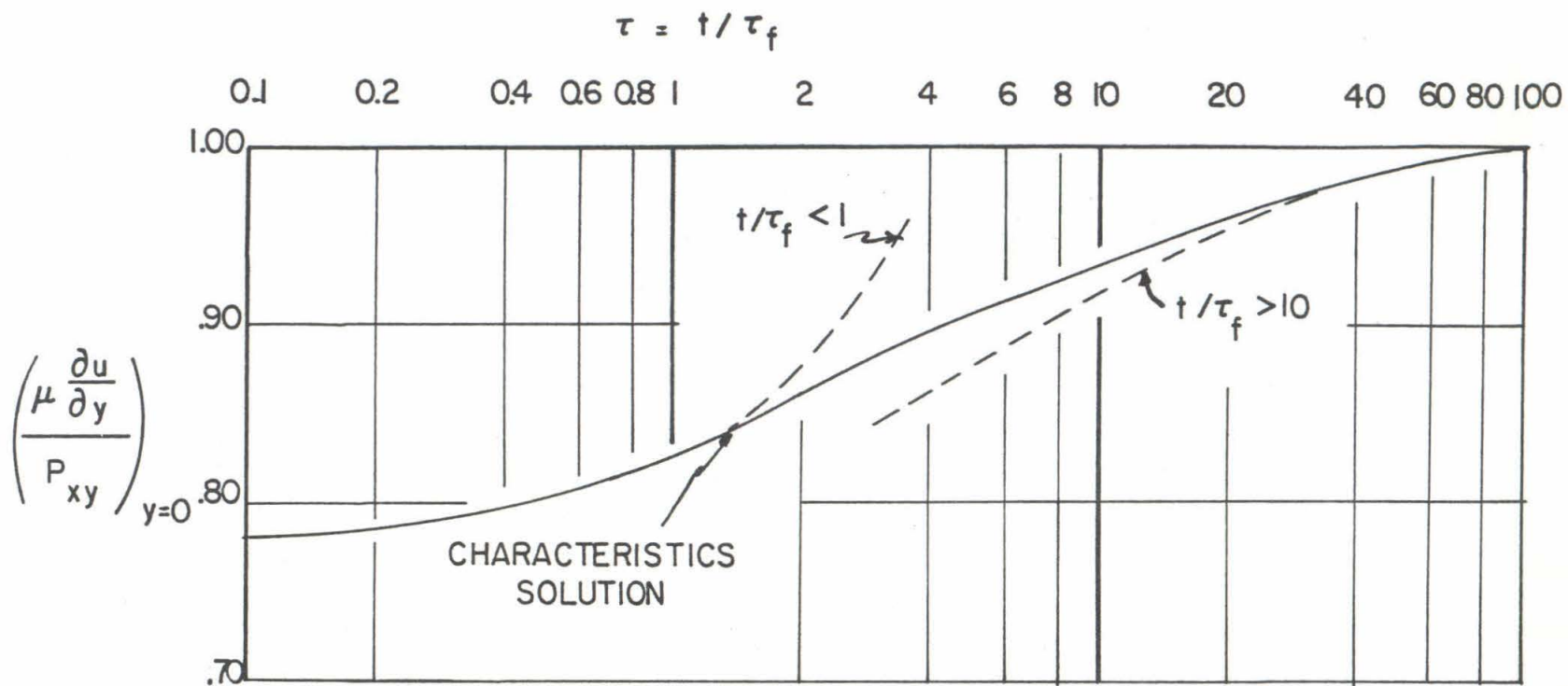


FIG.10 DEPARTURE FROM NAVIER-STOKES RELATION IN RAYLEIGH'S PROBLEM

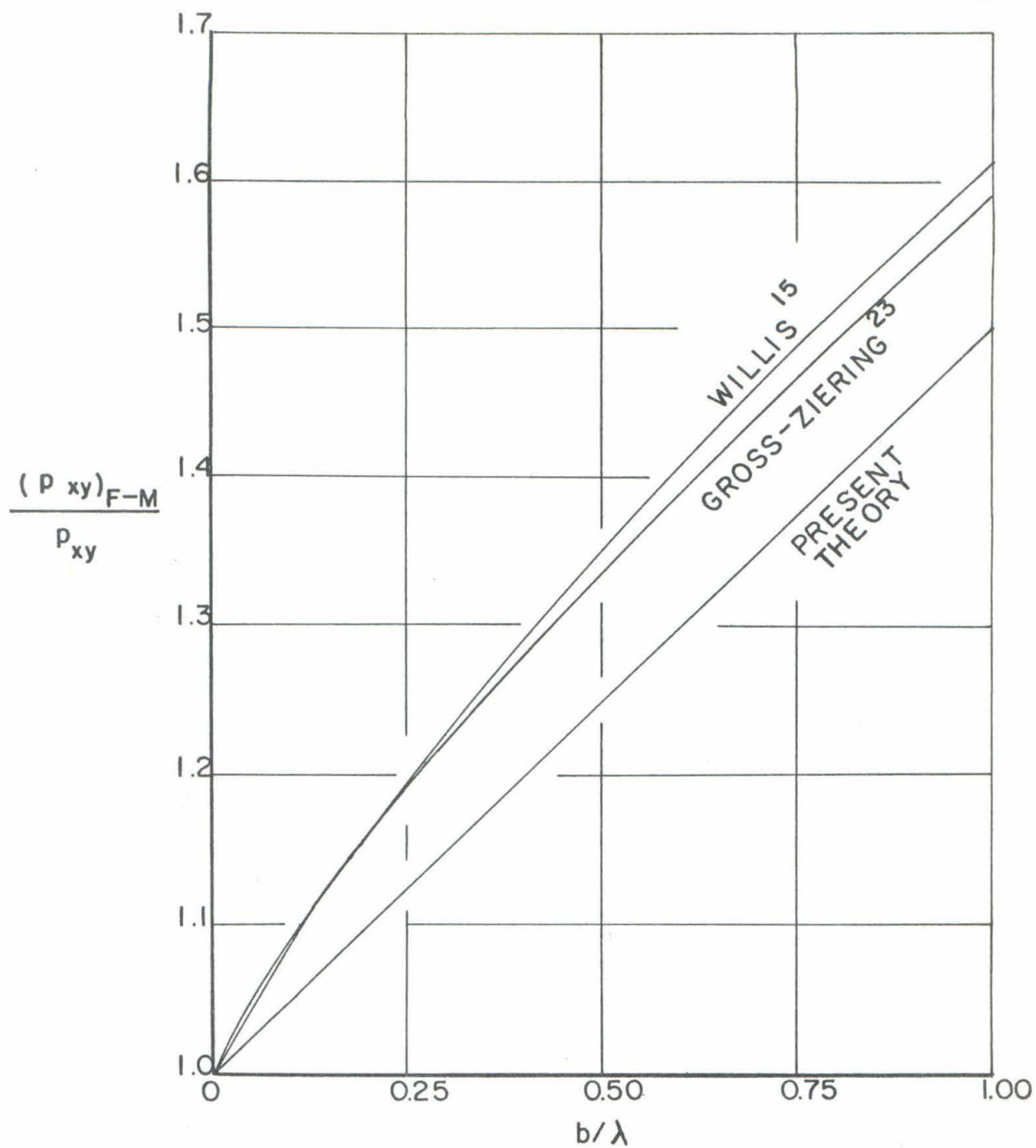


FIG.II SHEAR STRESS IN PLANE COUETTE FLOW

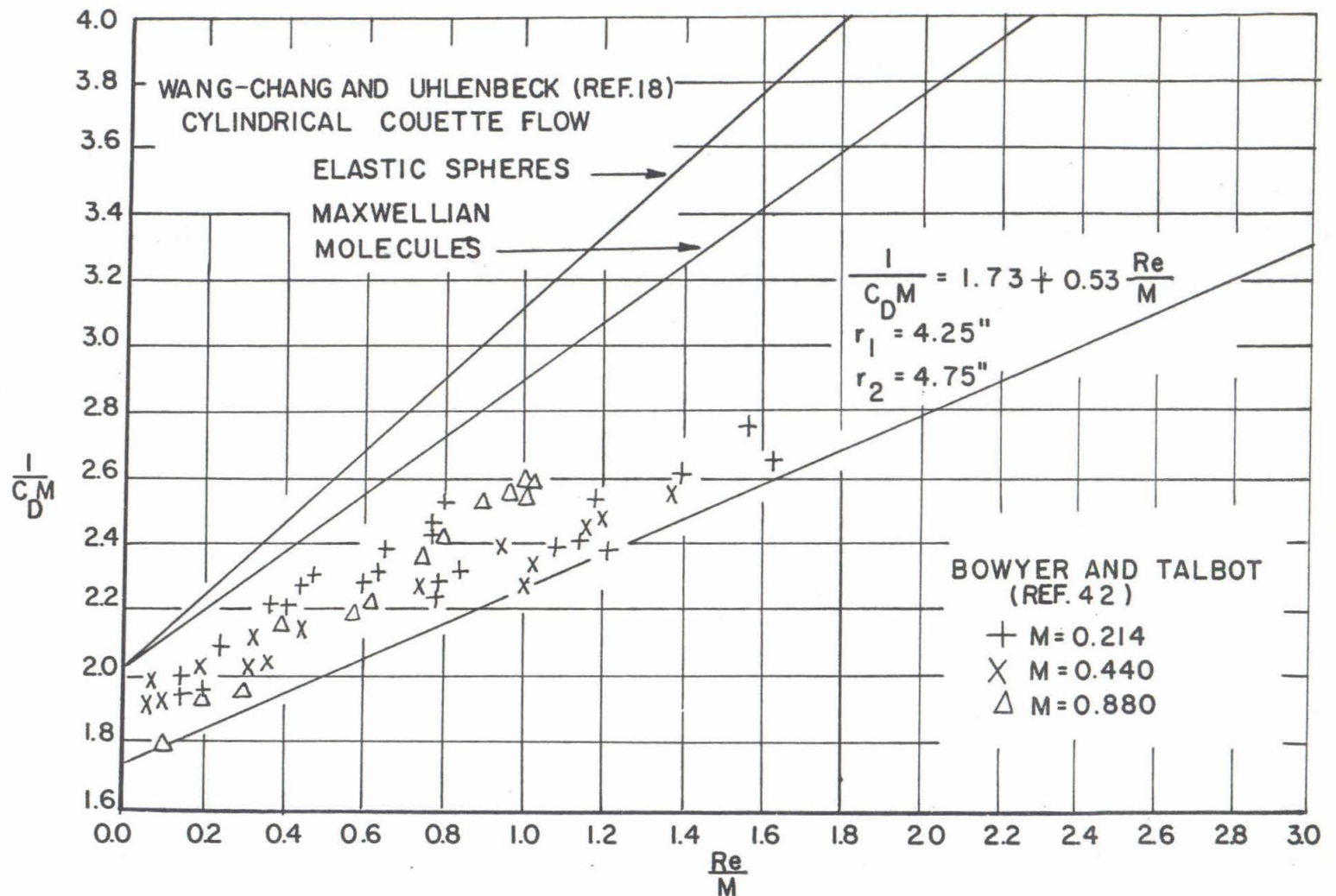


FIG.12 DRAG ON STATIONARY OUTER CYLINDER IN COUETTE FLOW — KRYPTON

1 February 1960

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