

Appendix A: Feynman rules

In this section we summarize the Feynman rules for the EFT defined in Eq. (10), which describes a light particle worldline coupled to fluctuating gravitons in a Schwarzschild background—with the addition of the recoil operator. Interpreted as a background-field action, Eq. (10) has a corresponding set of background-field Feynman diagrams that can be used to compute the 1SF radial action, e.g. as depicted in Fig. 1. To compute in the PM expansion, however, it is natural to further expand these background-field Feynman diagrams order by order in Newton's constant. In this picture the fundamental perturbation theory is in flat space, and the difference of the Schwarzschild metric and particle geodesics from flat space and straight lines, respectively, are considered PM corrections.

To begin, let us define the background-field effective action governing the light particle worldline and the fluctuation graviton in a curved background,

$$S_{\text{BF}}[\bar{g}, \delta g, \bar{x}_L, \delta x_L] + S_{\text{GF}} = S[\bar{g}, \bar{x}_L] \quad (\text{A1})$$

$$+ \int_x \sqrt{-\bar{g}} \left[\delta g_{\mu\nu} \bar{T}_L^{\mu\nu} + \frac{1}{16\pi G} \left(-\frac{1}{4} \delta g_{\mu\nu} \bar{\nabla}^2 \delta g^{\mu\nu} + \frac{1}{8} \delta g \bar{\nabla}^2 \delta g \right. \right.$$

$$- \frac{1}{2} \delta g_{\mu\nu} \delta g_{\rho\sigma} \bar{R}^{\mu\rho\nu\sigma} - \frac{1}{2} (\delta g_{\mu\rho} \delta g_{\nu}{}^\rho - \delta g_{\mu\nu} \delta g) \bar{R}^{\mu\nu}$$

$$\left. + \frac{1}{4} (\delta g_{\mu\nu} \delta g^{\mu\nu} - \frac{1}{2} \delta g^2) \bar{R} \right] + \dots,$$

where we have added a Lorenz gauge fixing term $S_{\text{GF}} = \frac{1}{32\pi G} \int_x \sqrt{-\bar{g}} F_\mu F^\mu$ with $F_\mu = \bar{\nabla}^\nu \delta g_{\mu\nu} - \frac{1}{2} \bar{\nabla}_\mu \delta g$. In the first line of Eq. (A1), the quantity $S[\bar{g}, \bar{x}_L]$ is just the probe radial action, which as usual is computed by plugging in the background metric and light geodesic in Eq. (4). Starting at the second line of Eq. (A1), we show the terms needed to compute 1SF corrections, where the ellipses denote the higher order corrections.

Here $\bar{T}_L^{\mu\nu}$ is the stress-energy tensor for the geodesic trajectory of the light particle, corresponding to a source term which implies the momentum-space Feynman rule,

$$\textcircled{L} \text{---} \text{---} = \sqrt{-\bar{g}} \bar{T}_L^{\mu\nu}(p) = \lambda m_H \int d\tau e^{-ip\bar{x}_L} \dot{\bar{x}}_L^\mu \dot{\bar{x}}_L^\nu \quad (\text{A2})$$

$$= \lambda m_H e^{-ipb} (u_L^\mu u_L^\nu \delta(u_L p) + 2i \mathcal{O}_{L\alpha}^{\mu\nu} \bar{x}_1^\alpha(u_L p) + \dots),$$

where we have defined

$$\mathcal{O}_L^{\alpha\mu\nu} = \frac{1}{2} ((u_L^\mu \eta^{\nu\alpha} + u_L^\nu \eta^{\mu\alpha})(u_L p) - u_L^\mu u_L^\nu p^\alpha). \quad (\text{A3})$$

together with the frequency-domain trajectory, $\bar{x}_i^\mu(\omega) = \int d\tau e^{-i\omega\tau} \bar{x}_i^\mu(\tau)$, and we have also explicitly expanded to subleading order. Case in point, we can trivially recast the trajectories in Eq. (9) into the form of perturbative Feynman diagrams. Concretely, using identities such as

$$\text{arcsinh}\left(\frac{v\tau}{b}\right) = \frac{1}{\partial_\tau} \left(\frac{v}{(b^2 + v^2\tau^2)^{1/2}} \right), \quad (\text{A4})$$

we can write the 1PM time-domain trajectory as

$$\bar{x}_1^\mu = -\frac{r_S}{2v^2 b^2} (2v^2 + 1) b^\mu (b^2 + v^2 \tau^2)^{1/2}$$

$$+ \frac{r_S}{2v^2} (\sigma(2v^2 - 1) u_H^\mu + u_L^\mu) \frac{1}{\partial_\tau} \frac{1}{(b^2 + v^2 \tau^2)^{1/2}}. \quad (\text{A5})$$

Powers of the spatial distance $(b^2 + v^2 \tau^2)^{1/2}$ can be rewritten as simple Fourier integrals. In the frequency domain, the trajectory then has a simple form

$$\bar{x}_1^\mu(\omega) = \frac{i(2\pi)^3 r_S}{v^2} \int \frac{d^4 k}{(2\pi)^4} e^{-ik\bar{x}_0} \delta(\omega - u_L k) \delta(u_H k)$$

$$\times \left(\frac{8(2v^2 + 1) \Pi^{\mu\nu}}{b^2} \frac{k_\nu}{(k^2)^3} - \frac{(\sigma(2v^2 - 1) u_H^\mu + u_L^\mu)}{k^2 (u_L k)} \right), \quad (\text{A6})$$

where $\Pi^{\mu\nu} = \eta^{\mu\nu} - v^{-2} (\sigma u_H^\mu - u_L^\mu) u_L^\nu - v^{-2} (\sigma u_L^\mu - u_H^\mu) u_H^\nu$ projects onto directions orthogonal to the four-velocities. Note that factor of ω^{-2} which one would expect from perturbatively solving the geodesic equation has been reduced in power to ω^{-1} in one term, and entirely eliminated in the other.

The full background-field propagator can be expanded perturbatively around flat space as

$$\text{---} \text{---} \text{---} = \text{---} \text{---} \text{---} + \text{---} \textcircled{\times} \text{---} \text{---}$$

$$+ \text{---} \textcircled{\times} \text{---} \textcircled{\times} \text{---} + \dots \quad (\text{A7})$$

where the circles denote background field insertions and the leading term is the flat-space graviton propagator,

$$\text{---} \text{---} \text{---} = \frac{16\pi G i}{p^2} (\eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\mu\nu} \eta_{\rho\sigma}), \quad (\text{A8})$$

which is in de Donder gauge on account of our choosing Lorenz gauge the original background-field action defined in Eq. (A1).

Of course, the true background is the Schwarzschild metric, but in the PM expansion we can treat these effects as order by order corrections to the flat space graviton two-point function. These contributions are obtained by taking the difference of the isotropic gauge Schwarzschild metric from the flat space metric and expanding in PM, which is simply the momentum-space version of Eq. (8),

$$\bar{\gamma}_{\mu\nu}(p) = -\frac{8\pi G m_H}{p^2} (\eta_{\mu\nu} - 2u_{H\mu} u_{H\nu}) \delta(u_H p) \quad (\text{A9})$$

$$- \frac{8\pi^2 G^2 m_H^2}{\sqrt{-p^2}} (3\eta_{\mu\nu} + u_{H\mu} u_{H\nu}) \delta(u_H p) + \dots$$

The corresponding insertion is just the three-point vertex, from standard graviton perturbation theory in flat space, connecting two graviton lines to a linearized background metric. Note the appearance of non-zero curvatures in Eq. (A1), which also appear as insertions. These arise because the metric is not a vacuum solution but is sourced by the heavy particle.

At low PM orders, we only need the background and geodesics to linear order and hence the background field method is not more efficient than performing a flat space perturbative calculation. However, at higher orders one sees considerable simplification, since in isotropic gauge, the background metric insertions are simple powers in the radius r , whose Fourier transforms yield very simple dependencies on the momentum transfer q induced by the insertion. In particular, the resulting Feynman rules are the same as for simple loop integrands with numerator structures that depend solely on $\eta_{\mu\nu}$ and $u_{H\mu}u_{H\nu}$ and are thus effectively scalar. Hence, the background

field method effectively performs tensor reduction on subdiagrams within multiloop Feynman diagrams.

Finally, as explained in the main text, the background field action must be supplemented by the recoil operator in Eq. (2). It is trivial to compute the corresponding two-point vertex, which is

$$\begin{aligned} & \text{Diagram: a circle with 'H' inside, connected to two wavy lines} \\ & = \frac{im_H}{2} \frac{\delta(u_H p_1 + u_H p_2)}{(u_H p_1)(u_H p_2)} \mathcal{O}^{\alpha\mu_1\nu_1}(u_H, p_1) \mathcal{O}_\alpha{}^{\mu_2\nu_2}(u_H, p_2). \end{aligned} \tag{A10}$$

The above Feynman rules are sufficient to compute the 1SF radial action for point-like compact bodies order by order in the PM expansion.